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RD-SYSTEMS WITH CHEMOTAXIS: GLOBAL EXISTENCE, BOUNDEDNESS AND BLOW-UP OF SOLUTIONS

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Abstract. We study the finite time blow-up of solutions to general RD-systems with chemotaxis for multi-species. Our result shows that the blow-up is equivalent to the blow-up of the L^r-norms of the solutions for r exceeding some critical value r_c. Under very loose conditions we give the estimation of r_c, relying on a variant of Gagliardo-Nirenberg inequality, and a kind of bootstrap method which is very similar to the Alikakos-Moser iteration procedure.

1. The chemotaxis RD-model and results

Our model has the form:

$$(1)\qquad \begin{cases} U_t = D_1\Delta U + g(U,V), & x\in\Omega,\ t>0,\\ V_t = D_2\Delta V + h(U,V) - CT(U,V), & x\in\Omega,\ t>0,\\ \frac{\partial U}{\partial \mathbf{n}} = 0 = \frac{\partial V}{\partial \mathbf{n}}, & x\in\partial\Omega,\ t>0,\\ U(x,0)=U_0(x)\ge 0,\ V(x,0)=V_0(x)\ge 0, & x\in\Omega, \end{cases}$$

where $U = (u_1, ..., u_m)$ (resp., $V = (v_1, v_2, ..., v_n)$) are the population densities of m *prey* (resp., n *predator*) species; Ω is a bounded domain in $\mathbb{R}^N$ $(N \ge 1)$ with a smooth boundary $\partial\Omega$; $\mathbf{n}$ is the unit outer normal, and no flux boundary condition (homogeneous Neumann boundary condition) is imposed. The diffusion matrices

$$(2)\qquad D_1 = \mathrm{diag}(d_1, ..., d_m), \quad D_2 = \mathrm{diag}(d'_1, ..., d'_n)$$

are assumed to be strictly positive, i.e., $d_i > 0$ and $d'_j > 0$ for all i, j. Finally, the chemotaxis term $CT(U,V)$ has the form

$$(3)\qquad CT(U,V)_i := \nabla\Big(\sum_{j=1}^{m} q_{ij}(U,V)\nabla u_j\Big) \quad (i = 1, ..., n).$$

Biologically, $g(U,V)$ (resp., $h(U,V)$) represents the growth rates of the preys (resp., predators). It is assumed that the predators V are attracted/repulsed by the preys U, so that they move in the direction proportional to the negative/positive gradients ($q_{ij} > 0$ or $q_{ij} < 0$) of the prey populations, and the movement is decided also by the predator's density. We model such chemotaxis effects by the terms $-CT(U,V)$ given as above, cf. [9, 8, 10]. It is valuable to mention that we do not assume that each of the function $q_{ij}(U,V)$ should keep only one sign. In fact, they are allowed to change their signs according to some realistic rules, for instance, they can change their signs if the densities of the preys/predators have been above some levels, cf. [10].

We impose the following conditions **(H1)-(H3)**:

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Date: 2018-Aug-20, 2019-Dec-05. HXRdMod.tex.

1991 *Mathematics Subject Classification.* 35K57, 35K59, 35B45, 92D25.

Key words and phrases. Reaction-diffusion-taxis, predator-prey, global existence and boundedness, blow-up.

(H1) Both mappings $g : \mathbb{R}_+^m \times \mathbb{R}_+^n \to \mathbb{R}_+^m$ and $h : \mathbb{R}_+^m \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ are of C^1 class, and satisfy

$$\begin{aligned} g(U,V)_i \geq 0 \quad \forall U,V \geq 0 \quad \text{with} \quad u_i = 0 \ (i = 1,...,m), \\ h(U,V)_j \geq 0 \quad \forall U,V \geq 0 \quad \text{with} \quad v_j = 0 \ (j = 1,...,n). \end{aligned} \tag{4}$$

Moreover, there exists a strictly positive constant vector $K_0 \in \mathbb{R}_+^m, K_0 > 0$ with the following property: For all $(U,V) \in \mathbb{R}_+^m \times \mathbb{R}_+^n$ there holds

$$g(U,V)_i \leq 0 \quad \text{if} \quad u_i \geq (K_0)_i \ (i = 1,...,m). \tag{5}$$

(H2) Each $q_{ij} : \mathbb{R}_+^m \times \mathbb{R}_+^n \to \mathbb{R}$ is a C^1 function satisfying $q_{ij}(U,0) = 0$ for all $0 \leq U \in \mathbb{R}_+^m$, and there exist a positive constant $C_q > 0$ and non-negative constants $\{\alpha_i, 1 \leq i \leq n\}$ such that

$$\sum_{j=1}^m |q_{ij}(U,V)| \leq C_q(1 + v_i^{\alpha_i}) \quad \forall (U,V) \in \mathbb{R}_+^m \times \mathbb{R}_+^n \ (i = 1,...,n). \tag{6}$$

Moreover, there exist non-negative constants $\{\beta_i, 1 \leq i \leq n\}$ and a continuous positive function $\varrho_0 : \mathbb{R}_+^m \to \mathbb{R}_+$ such that

$$\sum_{i=1}^m |g(U,V)_i| \leq \varrho_0(U) \times (1 + \sum_{j=1}^n v_j^{\beta_j}) \quad \forall (U,V) \in \mathbb{R}_+^m \times \mathbb{R}_+^n. \tag{7}$$

(H3) There exist constants $\{\gamma_i, i = 1,...,n\}$ and a continuous positive function $\varrho_1 : \mathbb{R}_+^m \to \mathbb{R}_+$ such that for all $(U,V) \geq 0$ and each i, $1 \leq i \leq n$, there holds

$$\gamma_i \geq 1, \quad h(U,V)_i \leq \varrho_1(U)(1 + v_i^{\gamma_i}). \tag{8}$$

Note that we need for the components of h in (8) only the control of growth from above, and the control of growth from below is not required. Roughly speaking, conditions **(H2)-(H3)** are concerned with the control of the growth of the predators given by the constants $\{(\alpha_i, \beta_i, \gamma_i), 1 \leq i \leq n\}$:

(1) The α_i values control the chemotaxis effects of the predators.
(2) The β_i values control of the predator growths within the preys.
(3) The γ_i values control the intrinsic growth of the predators.
(4) All growths are at most power type.

We will consider the following condition **(H4)** which ensures the L^1-boundedness of the solutions.

(H4) There exist a strictly positive vector $B \in \mathbb{R}_+^m \times \mathbb{R}_+^n$ and positive constants $b_1, b_2, \alpha < 1$ such that

$$\langle B, (g,h)(U,V) \rangle \leq b_1 + b_1 \langle B, (U,V) \rangle^\alpha - b_2 \langle B, (U,V) \rangle \quad \forall (U,V) \in \mathbb{R}_+^m \times \mathbb{R}_+^n. \tag{9}$$

Equivalently, **(H4)** says that the function $0 \leq (U,V) \mapsto \langle B, (g,h)(U,V) \rangle$ grows at most sublinearly .

Our main result goes as follows.

Theorem 1.1. *Assume* **(H1)-(H3)**. *Let*

$$r_c := N \times \max_{1 \leq i \leq n} \max\{\beta_i + \alpha_i - 1, \ (\gamma_i - 1)/2\}. \tag{10}$$

Let $0 \leq (U_0, V_0) \in W^{1,p}(\Omega)^{m+n}$ for $p > N$. Then there exists $T_{\max} > 0$ (maximal existence time) such that (1) *has a unique non-negative classical solution (U,V) satisfying*

$$0 \leq U \in \mathcal{G}^m, \quad 0 \leq V \in \mathcal{G}^n; \quad \mathcal{G} := C([0, T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \tag{11}$$

and for each $i, 1 \leq i \leq m$,

$$0 \leq U(t)_i \leq \max\{(K_0)_i, \|U_{i0}\|_\infty\} \quad \forall 0 \leq t < T_{\max}. \tag{12}$$

Furthermore, we have the following assertions (i)-(ii).

(i) (**Global existence and boundedness**) *Assume that there exists $r > r_c$ such that If there holds*

$$L(T) := \sup_{0\le t\le T} \|V(t)\|_r < \infty \quad \forall T < T_{\max}, \tag{13}$$

then $T_{\max} = \infty$ and

$$\sup_{0\le t\le T} (\|U(t)\|_{1,\infty} + \|V(t)\|_\infty) < \infty \quad \forall T < \infty. \tag{14}$$

Moreover, if $L(T)$ is uniformly bounded for $T > 0$, then

$$\limsup_{t\to\infty} (\|U(t)\|_{1,\infty} + \|V(t)\|_\infty) < \infty. \tag{15}$$

(ii) (L^1-**boundedness and global existence**) *Assume* **(H4)**. *If $r_c < 1$, then $T_{\max} = \infty$, and* (14) *holds true. Moreover, if either $b_2 > 0$ or $b_1 = 0$, then* (15) *is valid.*

Remark 1.2. (a) Our result in Theorem 1.1 covers the most part of known results, cf. [10, 11] and references therein. It shows that the finite time blow-up of solutions to (1) is equivalent to the blow-up of the L^r-norms of the solutions for $r > r_c$. Certainly, the weakest norm condition for avoiding blow-up is the L^1-boundedness. The condition **(H4)** gives a such simple condition ensuring the L^1-boundeness of the solutions. (b) The asymptotic behavior of solutions to (1) remains unknown. We would like to tackle it in the future.

We organize the present article as follows. In §2 we give an application of our Theorem 1.1 to improve previous results concerning simple chemotaxis prey-predator systems, which has been studied by several authors, cf. [10, 11]. We will prove in §4 the global existence and boundedness of non-negative solutions of (1) under conditions **(H1)-(H3)**, by a bootstrap method. Our estimations are subtle and based on an inequality given in §3 which is itself interesting. It is valuable to mention that we do not need the boundedness of the density functions q_{ij}. This point is certainly useful for the practical applications.

2. An application to simple chemotaxis prey-predator systems

We consider the model:

$$\begin{cases} u_t = d_1\Delta u + g(u,v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v + h(u,v) - \nabla(\rho(u)q(v)\nabla u), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x)\ge 0,\ v(x,0)=v_0(x)\ge 0, & x\in\Omega, \end{cases} \tag{16}$$

where the density of prey and predator is given by u and v, respectively. As before, Ω is a bounded domain in $\mathbb{R}^N$ ($N \ge 1$) with a smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outer normal; d_1, d_2, c are positive constants. Moreover, the prey-taxis effect, given by $-\nabla(q(v)\nabla u)$, shows the tendency of predator moving toward the increasing prey gradient direction.

We assume that all functions g, h and ρ, q are continuously differentiable, and there exist positive constants $c_g, c_q, \alpha, \beta, \gamma$ and a continuous function $\rho_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g(u,0)\ge 0, \quad h(0,v)\ge 0, \quad |q(v)| \le c_q(1+v^\alpha), \tag{17}$$

$$|g(u,v)| \le \rho_0(u)(1+v^\beta), \quad h(u,v) \le \rho_0(u)(1+v^\gamma) \tag{18}$$

for all $u, v \ge 0$, and

$$g(u,v) \le 0 \quad \forall u \ge c_g, v \ge 0. \tag{19}$$

Under conditions (17)-(19), we see that assumptions **(H2)-(H3)** in §1 are satisfied with the following choices:

$$m = 1 = n, \quad \alpha_1 = \alpha, \quad \beta_1 = \beta, \quad \gamma_1 = \gamma.$$

The corresponding exponent r_c is give by

$$r_c := N \times \max\{\alpha + \beta - 1, (\gamma - 1)/2\}. \tag{20}$$

Therefore, an application of Theorem 1.1 yields that a solution (u, v) of (16) exists globally, if the norms $\|v(t)\|_r$ for some $r > r_c$ do not blow up in finite time.

A special case of (16) is the following system

$$\begin{cases} u_t = d_1 \Delta u + f_1(u) - \phi_1(u, v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + f_2(v) + \phi_2(u, v) - \nabla(\rho(u) q(v) \nabla u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 = \frac{\partial v}{\partial \mathbf{n}}, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) \geq 0,\ v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{21}$$

corresponding to the choice of g and h in (16):

$$g(u, v) = f_1(u) - \phi_1(u, v), \quad h(u, v) = f_2(v) + \phi_2(u, v). \tag{22}$$

Here, the term $\phi(u, v)$ represents the moving rate of prey to the predator.

We assume that all of the functions $\{f_1, f_2, \rho, q\}$ are continuously differentiable functions on $\mathbb{R}_+$, and $f_1(0) = 0, f_2(0) = 0$. Moreover, there exist positive constants c_1, c_2, c_q and α such that

$$f_1(u) \leq 0 \quad \forall u \geq c_1, \quad f_2(v) \leq 0 \quad \forall v \geq c_2, \tag{23}$$

$$|q(v)| \leq c_q(1 + v^\alpha) \quad \forall v \geq 0. \tag{24}$$

We assume that both functions $\phi_, \phi_2$ is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}_+$ and there exist constants $\gamma > 0, 0 < \delta < 1, c$ and non-negative constants b_1, b_2, and a continuous positive function ρ_0 such that

$$\phi_1(0, v) = 0 \leq \phi_1(u, v) \leq \rho_0(u)(1 + v^\gamma) \quad \forall u, v \geq 0, \tag{25}$$

$$0 \leq \phi_2(u, v) \leq c\phi_1(u, v) + b_1(1 + u + v)^\delta - b_2(u + v) \quad \forall u, v \geq 0. \tag{26}$$

Under (23)-(25), we see that the conditions (17)-(19) are satisfied with the choices $\beta = \gamma$. Hence, r_c is given by

$$r_c := N \times (\alpha + \gamma - 1). \tag{27}$$

Set $B := (1, c)$. We have for all $u, v \geq 0$ that

$$\langle B, (g, h)(u, v) \rangle = [f_1(u) - \phi_1(u, v)] + c[f_2(v) + \phi_2(u, v)] \leq f_1(u) + cf_2(v) + b_1(1 + u + v)^\delta - b_2(u + v) \tag{28}$$

by (26). It is routine to use condition (23) to establish that both functions f_1 and f_2 are uniformly bounded from above. Hence, condition **(H4)** is satisfied, and Theorem 1.1 is applicable. In particular, if

$$\alpha + \gamma < 1 + 1/N, \tag{29}$$

then $r_c < 1$ and thus all non-negative solutions of (21) exists globally.

The system (21) is a very general prey-predator model for studying prey-taxis and has been studied by a lot of authors, cf. [11] and references therein. In particular, the result in [11] states that a solution to (21) exists globally if the term q is sufficiently small by comparing to certain constants as well as the $L^\infty-$norm of the initial values u_0. However, our requirements (23)-(25) and (29) involve only the growth conditions on the functions f_1, f_2, ϕ and q, which will be satisfied by many known models, see [11, 10] for more details. In fact, the crucial condition for the global existence of solutions is $\alpha + \gamma < 1 + 1/N$, which yields a balance between the growth for prey-taxis (giving by q) and the moving rate of prey to predator (giving by ϕ_1, ϕ_2). For the usual case where $\rho \equiv 1, q(v) = \chi v$, we have that $\alpha = 1$, and thus the corresponding growth restriction on ϕ_1 reads as $\gamma < 1/N$. Such a condition can be considered as the case that the moving of prey to predator is restricted by the dimension N of the underlying space Ω, a requirement coinciding very well with the practical uses [10].

3. An inequality

We recall the following result as a consequence of applying the classical inequality of Gagliardo-Nirenberg combined with Poincare's inequality, cf. [6, 7, 8, 4].

Lemma 3.1. (Gagliardo-Nirenberg inequality) *Let N be the dimension of Ω. There holds*

$$\|u\|_p \le C\cdot(\|\nabla u\|_q+\|u\|_r)^\lambda\cdot\|u\|_r^{1-\lambda}\quad\forall u\in L^p(\Omega)\cap W^{1,q}(\Omega)\tag{30}$$

for all $p>1, q\ge 1$ satisfying $(p-q)N<pq$ and all $r\in(0,p)$, where

$$\lambda=\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}-\frac{1}{q}+\frac{1}{N}}\in(0,1).\tag{31}$$

The following lemma, as a consequence of the above Gagliardo-Nirenberg inequality, is crucial for our proof of the main result (Theorem 1.1), and itself also interesting.

Lemma 3.2. *For each $k>1$, and $(r,p)>0$ satisfying*

$$p(1-2/N)<1<2p,\quad r<pk,\tag{32}$$

there exists a constant $c_1>0$, depending on k,p and Ω, such that

$$\|\nabla u^{k/2}\|_2^2\ge c_1\cdot\|u\|_r^{-c_0}\cdot\|u^{pk}\|_1^\delta-\|u\|_r^k\quad\forall u\in W^{1,2pk}(\Omega), u\ge 0,\tag{33}$$

where

$$\delta:=(k/r-1+2/N)/(pk/r-1),\quad c_0:=(p\delta-1)k.\tag{34}$$

Proof. We see that the pair $(2p,2)$ satisfies the condition $(2p-2)N<4p$ in Lemma 3.1 and $r/k<p$ whenever $p(N-2)<N$. Hence, we are in the position to apply Lemma 3.1 to the triplet $(2p,2,2r/k)$. It yields for any $u\in W^{1,2pk}(\Omega), u\ge 0$, that

$$\|u^{k/2}\|_{2p}\le C\cdot(\|\nabla u^{k/2}\|_2+\|u^{k/2}\|_{2r/k})^\lambda\cdot\|u^{k/2}\|_{2r/k}^{1-\lambda},\tag{35}$$

where

$$\lambda:=(k/r-1/p)/(k/r-1+2/N)\in(0,1).\tag{36}$$

Equivalently,

$$\|\nabla u^{k/2}\|_2^2\ge C\cdot\|u\|_r^{k(1-1/\lambda)}\cdot\|u^{pk}\|_1^\delta-\|u\|_r^k,\tag{37}$$

where $\delta:=1/(p\lambda)$ is given by (34). This completes the proof. □

4. The proof of Theorem 1.1

We begin some more preparations.

Lemma 4.1. (Divergence Theorem and Green's First Identity)

1. (Divergence Theorem) *For any $C^1(\bar\Omega)$ vector field $\mathbf{w}$ there holds*

$$\int_\Omega\nabla\cdot\mathbf{w}\,dx=\int_{\partial\Omega}\mathbf{w}\cdot\mathbf{n}\,dx.\tag{38}$$

2. (Green's First Identity) *Let $u\in W^{1,2}(\Omega),\ v\in W^{2,2}(\Omega)$. Then*

$$\int_\Omega u\Delta v\,dx=-\int_\Omega\nabla u\cdot\nabla v\,dx+\int_{\partial\Omega}v\frac{\partial u}{\partial\mathbf{n}}\,dx.\tag{39}$$

Particularly, if $\frac{\partial u}{\partial\mathbf{n}}\big|_{\partial\Omega}=0$, then

$$\int_\Omega u\Delta v\,dx=-\int_\Omega\nabla u\cdot\nabla v\,dx.\tag{40}$$

3. *Let* $u, g \in W^{1,2}(\Omega),\ v \in W^{2,2}(\Omega)$ *and* $\frac{\partial v}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0 = \frac{\partial g}{\partial \mathbf{n}}\big|_{\partial\Omega}$. *Then*

$$\int_\Omega u\nabla\cdot(g\nabla v)\,dx = -\int_\Omega g\nabla u\cdot\nabla v\,dx. \tag{41}$$

Proof. For the Divergence Theorem, one compares [7, p.13]. The Green First Identity is the result of applying the Divergence Theorem to the C^1 vector $\mathbf{w} := u\nabla v$. To prove (41), we note that

$$v\nabla\cdot(g\nabla u) = (vg)\Delta u + v\nabla g\cdot\nabla u, \qquad \frac{\partial(vg)}{\partial \mathbf{n}}\big|_{\partial\Omega} = \left(g\frac{\partial v}{\partial \mathbf{n}} + v\frac{\partial g}{\partial \mathbf{n}}\right)\big|_{\partial\Omega} = 0 \tag{42}$$

by assumptions. It follows that

$$\begin{aligned}\int_\Omega v\nabla\cdot(g\nabla u) &= \int_\Omega (vg)\Delta u + \int_\Omega v\nabla g\cdot\nabla u\\ &= \int_\Omega [v\nabla g\cdot\nabla u - \nabla(vg)\nabla u] = -\int_\Omega g\nabla v\cdot\nabla u, \quad \text{(by (40), (42))},\end{aligned}$$

completing the proof. $\square$

For $p \in (1,\infty)$ we define

$$Au := -\Delta u \quad \text{for} \quad u \in D(A) := \left\{ w \in W^{2,p}(\Omega) : \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}. \tag{43}$$

It is well-known (cf. [5]) that $-A$ generates a contractive C_0-semigroup $\{T(t) := e^{-tA} : t \geq 0\}$ of positive linear operators on each $L^p(\Omega)$ for $p \in [1,\infty)$. Moreover, $-A$ is symmetric and thus each $T(t)$ is a contraction on $L^\infty(\Omega)$. More precisely, there holds

$$\|T(t)f\|_p \leq \|f\|_p \quad \text{and} \quad f \geq 0 \implies T(t)f \geq 0$$

for all $t \geq 0$ and $f \in L^p(\Omega)$ for $p \in [1,\infty]$.

We use also the following estimates, cf. [8].

Lemma 4.2. *Assume that* $m \in \{0,1\}$, $p \in [1,\infty]$ *and* $q \in (1,\infty)$. *Then there exists a positive constant* C_1 *such that*

$$\|u\|_{m,p} \leq C_1\|(A+1)^\theta u\|_q \quad \forall u \in D((A+1)^\theta), \tag{44}$$

where $\theta \in (0,1)$ *satisfies*

$$2\theta > m - N\left(\frac{1}{p} - \frac{1}{q}\right).$$

If, in addition $q \geq p$, *then there exists constant* C_2 *and* $\gamma > 0$ *such that*

$$\|(A+1)^\theta e^{-t(A+1)}u\|_q \leq C_2 t^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\gamma t}\|u\|_p \quad \forall u \in L^p(\Omega),\, t > 0. \tag{45}$$

Moreover, for any $p \in (1,\infty)$ *and* $\varepsilon > 0$, *there exists a constant* C_3 *and* $\mu > 0$ *such that,*

$$\|(A+1)^\theta e^{-tA}\nabla\cdot u\|_p \leq C_3 t^{-\theta-\frac{1}{2}-\varepsilon}e^{-\mu t}\|u\|_p \quad \forall u \in L^p(\Omega),\, t > 0. \tag{46}$$

We will use freely Young's inequality saying that

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \quad \forall a, b \geq 0, p, q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, we fix a constant $k > N$ and three constants $\{\theta, \theta_1, \theta_2\}$ such that

$$(1 + N/k)/2 < \theta < 1, \quad N/(2k) < \theta_1 < 1, \quad 1/2 + \theta_1 < \theta_2 < 1. \tag{47}$$

As direct consequences of Lemma 4.2, we have the following estimates:

$$\|u\|_{1,\infty} \leq C\cdot\|(A+1)^\theta u\|_k \quad \forall u \in D((A+1)^\theta), \tag{48}$$

$$\|u\|_\infty \le C \cdot \|(A+1)^{\theta_1} u\|_k \quad \forall u \in D((A+1)^{\theta_1}), \tag{49}$$

and

$$\|(A+1)^{\theta_1} e^{-t(A+1)} u\|_k + \|(A+1)^{\theta_1} e^{-tA} \nabla \cdot u\|_k \le C \cdot t^{-\theta_2} e^{-\gamma t} \|u\|_k \tag{50}$$

for all $t > 0, u \in L^k(\Omega)$. In the above, $C > 0, \gamma > 0$ are some constants.

Below we consider a non-negative classical local solution $0 \le (U, V)$ of (1), with the maximal existence time $T_{\max}$. In the sequel, we fix $\tau \in (0, T_{\max})$, and let

$$M(\tau) := \|K_0 + U_0\|_{1,\infty} + \|V(\tau)\|_\infty + \|(A+1)^\theta U(\tau)\|_k, \tag{51}$$

$$W_i(t) := \sup_{\tau \le s \le t} \|v_i(s)\|_k, \quad H(t) := \sup_{\tau \le s \le t} \|U(s)\|_{1,\infty} \quad \forall t \in [\tau, T_{\max}). \tag{52}$$

By definition, $H(\cdot)$ as well as $W_i(\cdot)$ are non-decreasing. This monotonicity will be used later. Denote

$$V_i(t) := \int_\Omega v_i(x,t)^k \, dx. \tag{53}$$

Our first result gives a way for estimating the bound of $\|V(\cdot)\|_\infty$ by virtue of $H(\cdot)$ combining with $k-$norms of V.

Lemma 4.3. *Assume* **(H3)**. *Then*

$$\|v_i(t)\|_\infty \le \|v_i(\tau)\|_\infty + C \cdot [1 + H(t) W_i(t)^{\max\{\alpha_i, \gamma_i\}}] \quad \forall t \in [\tau, T_{\max}), \tag{54}$$

where $C > 0$ is some constant.

Proof. Let

$$\phi(t) := d_i' v_i(t) + h(U,V)_i(t) \quad (t < T_{\max}). \tag{55}$$

By **(H3)** we can find a constant $C_1 \ge 0$ such that

$$\phi(t) \le C_1 + C_1 v_i(t)^{\gamma_i} \quad \forall t \in [\tau, T_{\max}). \tag{56}$$

Consider $t \in [\tau, T_{\max})$. Using the usual variation of constants formula to (1), we obtain that

$$0 \le v_i(t) = \hat{T}_i(t-\tau) v_i(\tau) + X_1(t) + X_2(t), \tag{57}$$

where $\hat{T}_i(s) := e^{-d_i'(A+1)s} = e^{-d_i' s} T(d_i' s)$, and

$$X_1(t) := -\int_\tau^t \hat{T}_i(t-s) \nabla (\sum_{j=1}^m q_{ij}(U,V) \nabla u_j(s)) \, ds, \quad X_2(t) := \int_\tau^t \hat{T}_i(t-s) \phi(s) \, ds. \tag{58}$$

Since each $\hat{T}_i(\cdot)$ is a contraction on $L^\infty(\Omega)$, we have that

$$\|\hat{T}_i(t-\tau) v_i(\tau)\|_\infty \le \|V_i(\tau)\|_\infty. \tag{59}$$

Moreover,

$$
\begin{aligned}
\|X_1(t)\|_\infty &\le C \cdot \|(A+1)^{\theta_1} X_1(t)\|_k \quad \text{(by (49))} \\
&\le C \cdot \int_\tau^t \|(A+1)^{\theta_1} \hat{T}_i(t-s)\big(\nabla \sum_{j=1}^m q_{ij}(U,V)\nabla u_j(s)\big)\|_k \, ds \\
&\le C \cdot \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} \Big\| \sum_{j=1}^m q_{ij}(U,V)\nabla u_j(s) \Big\|_k \, ds \quad \text{(by (50))} \\
&\le C \cdot \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} \left(\sum_{j=1}^m \|q_{ij}(U,V)\nabla u_j(s)\|_k \right) ds \qquad (60) \\
&\le C \cdot \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} H(s) \|1 + v_i(s)^{\alpha_i}\|_k \, ds \quad \text{(by } \mathbf{(H2)}) \\
&\le C \cdot (1 + H(t) W_i(t)^{\alpha_i}) \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} \, ds \quad \text{(by monotonicity of } H, W_i) \\
&\le C \cdot \Gamma(1-\theta_2) \cdot (1 + H(t) W_i(t)^{\alpha_i}),
\end{aligned}
$$

where $\Gamma(\cdot)$ is the usual Gamma function.

On the other hand, we have by (56) that

$$
X_2(t) \le C_1 X_3(t), \quad X_3(t) := \int_\tau^t \hat{T}_i(t-s)\tilde{\phi}(s)\, ds, \quad \tilde{\phi}(s) := 1 + v_i(s)^{\gamma_i}. \tag{61}
$$

Note that

$$
\begin{aligned}
\|X_3(t)\|_\infty &\le C \cdot \|(A+1)^{\theta_1} X_3(t)\|_k \quad \text{(by (49))} \\
&\le C \cdot \int_\tau^t \|(A+1)^{\theta_1} \hat{T}_i(t-s)\tilde{\phi}(s)\|_k \, ds \\
&\le C \cdot \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} \|\tilde{\phi}(t-s)\|_k \, ds \quad \text{(by (50))} \qquad (62) \\
&\le C \cdot \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} (1 + \|v_i(s)\|_k^{\gamma_i}) \, ds \quad \text{(by (56))} \\
&\le C \cdot (1 + W_i(t)^{\gamma_i}) \int_\tau^t (t-s)^{-\theta_2} e^{-\gamma(t-s)} \, ds \quad \text{(by monotonicity of } W_i) \\
&\le C \cdot \Gamma(1-\theta_2) \cdot (1 + W_i(t)^{\gamma_i}).
\end{aligned}
$$

By (57) and (61) we have that

$$
0 \le v_i(t) \le \hat{T}_i(t-\tau) v_i(\tau) + X_1(t) + C_1 X_3(t). \tag{63}
$$

Combining (59), (60) and (62), we find from (63) that

$$
\|v_i(t)\|_\infty \le \|v_i(\tau)\|_\infty + C \cdot (1 + H(t) W_i(t)^{\max\{\alpha_i, \gamma_i\}}),
$$

giving (54). This completes the proof. □

Our next result reveals that we can control H using the $k-$norms of V.

Lemma 4.4. *There holds*

$$
H(t) \le C \cdot [1 + \max_{1 \le i \le n} W_i(t)^{\beta_i}] \quad \forall t \in [\tau, T_{\max}). \tag{64}
$$

Proof. On the one hand, we have by using the variation of constants formula that

$$u_i(t) = T_i(t-\tau)u_i(\tau) + U_1(t), \quad U_1(t) := \int_\tau^t T_i(t-s)\varphi(s)\,ds \tag{65}$$

where

$$T_i(s) := e^{-d_i(A+1)s} = e^{-d_i s}T(d_i s), \quad \varphi(s) := d_i u_i(s) + g(U,V)_i(s). \tag{66}$$

For $s \geq \tau$ we have that

$$\begin{aligned} \|\varphi(s)\|_k &\leq d_i\|u_i(s)\|_k + \|g(U,V)_i(s)\|_k \\ &\leq C \cdot (M(\tau) + \max_{1\leq j\leq n} \|v_j(s)^{\beta_j}\|_k) \quad \text{(by (12) and } (\mathbf{H2})) \\ &\leq C \cdot (M(\tau) + \max_{1\leq i\leq n} W_i(s)^{\beta_i}). \end{aligned} \tag{67}$$

Therefore,

$$\begin{aligned} \|U_1(t)\|_{1,\infty} &\leq C \cdot \|(A+1)^\theta U_1(t)\|_k \quad \text{(by (48))} \\ &\leq C \cdot \int_\tau^t \|(A+1)^\theta T_i(t-s)\varphi(s)\|_k\,ds \\ &\leq C \cdot \int_\tau^t (d_i(t-s))^{-\theta} e^{-\gamma d_i(t-s)}\|\varphi(s)\|_k\,ds \quad \text{(by (50))} \\ &\leq C \cdot \int_\tau^t (d_i(t-s))^{-\theta} e^{-\gamma d_i(t-s)} \cdot (M(\tau) + \max_{1\leq i\leq n} W_i(s)^{\beta_i})\,ds \quad \text{(by (67))} \\ &\leq C \cdot (M(\tau) + \max_{1\leq i\leq n} W_i(t)^{\beta_i}) \cdot \int_0^\infty (d_i s)^{-\theta} e^{-\gamma d_i s}\,ds \\ &\qquad \text{(by monotonicity of each } W_i) \\ &\leq C \cdot \Gamma(1-\theta)(M(\tau) + \max_{1\leq i\leq n} W_i(t)^{\beta_i}), \end{aligned} \tag{68}$$

In the above, the constant C may change from line to line, but it depends only on k and $M(\tau)$.

On the other hand, we have that

$$\|T_i(t-\tau)u_i(\tau)\|_{1,\infty} \leq C \cdot \|T_i(t-\tau)(A+1)^\theta u_i(\tau)\|_k \leq C \cdot \|(A+1)^\theta u_i(\tau)\|_k. \tag{69}$$

For the last inequality we have used the fact that each $T_i(\cdot)$ is a contraction on $L^k(\Omega)$. Combining (68), (69) and (65), we obtain (64). ☐

For the proof of Theorem 1.1 we need also the following estimation result.

Lemma 4.5. *Fix $T < T_{\max}$ and an index i. Let*

$$\delta_i := \alpha_i - 1. \tag{70}$$

Assume $r > 0$ and $k > N$ to be such that

$$r/N > \{\delta_i, (\gamma_i - 1)/2\}, \quad (1 + 2r/(kN))(1 - 2/N) < 1, \tag{71}$$

and

$$\sup_{0\leq t\leq T} \|v_i(t)\|_r < \infty. \tag{72}$$

Then there holds the estimate:

$$\|v_i(t)\|_k \leq C \cdot \max\{1 + \|v_i(0)\|_k, H(t)^{\kappa_i}\} \quad \forall t \leq T, \tag{73}$$

where

$$\kappa_i := 1/(r/N - \delta_i) > 0. \tag{74}$$

Proof. We have that

$$\dot{V}_i(t)/k = \int_\Omega v_i^{k-1}(v_i)_t = E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 :=& d_i' \int_\Omega v_i^{k-1} \Delta v_i \, dx, \quad E_2 := - \int_\Omega v_i^{k-1} \nabla \cdot (\sum_{j=1}^m q_{ij}(U,V) \nabla u_j) \, dx, \\ E_3 :=& \int_\Omega v_i^{k-1} h(U,V)_i \, dx. \end{aligned}$$

We have that

$$\begin{aligned} E_1 &= -d_i' \int_\Omega (\nabla v_i^{k-1}) \cdot \nabla v_i \, dx \qquad \text{(using (40) to the pair } (v_i^{k-1}, v_i)) \\ &= -4d_i'(k-1)k^{-2} \int_\Omega |\nabla v_i^{k/2}|^2, \end{aligned} \tag{75}$$

$$\begin{aligned} E_2 &= \int_\Omega (\sum_{j=1}^m q_{ij}(U,V) \nabla u_j) \cdot \nabla v_i^{k-1} \, dx \qquad \text{(using (41) to triples } (v_i^{k-1}, q_{ij}(U,V), u_j)) \\ &\le \int_\Omega (\sum_{j=1}^m |q_{ij}(U,V)| \cdot |\nabla u_j| \cdot |\nabla v_i^{k-1}|) \, dx \\ &\le C_q(k-1)H(t) \int_\Omega v_i^{k-2}(1 + v_i^{\alpha_i}) |\nabla v_i| \, dx \quad \text{(by } \mathbf{(H2)} \text{ and (52))}. \end{aligned} \tag{76}$$

Since

$$\begin{aligned} C_q H(t)(1 + v_i^{\alpha_i}) |\nabla v_i| &\le C_q^2 H(t)^2 (1 + v_i^{\alpha_i})^2 / d_i' + (d_i'/2) |\nabla v_i|^2 \\ &\le 2C_q^2 H(t)^2 (1 + v_i^{2\alpha_i}) / d_i' + (d_i'/2) |\nabla v_i|^2, \end{aligned}$$

we have that

$$\dot{V}_i(t)/k \le G_i + E_3 + Z_i, \tag{77}$$

where

$$G_i := \rho(t) \| v_i^{k-2} + v_i^{k-2+2\alpha_i} \|_1 \quad \text{with} \quad \rho(t) := [2(k-1)C_q^2 H(t)^2 / d_i'], \tag{78}$$

and

$$Z_i := -2d_i'(k-1)k^{-2} \int_\Omega |\nabla v_i^{k/2}|^2 \, dx. \tag{79}$$

To estimate E_3, we use **(H3)** to find constant $C_1 > 0$ such that $h(U,V)_i \le C_1 + C_1 v_i^{\gamma_i}$. It follows that

$$E_3 \le \int_\Omega v_i^{k-1}(C_1 + C_1 v_i^{\gamma_i}) \, dx. \tag{80}$$

On the other hand, we choose $p := 1 + 2r/(kN) > 1$, i.e., $r = kN(p-1)/2$. It follows that

$$\delta := (k/r - 1 + 2/N)/(pk/r - 1) = 1. \tag{81}$$

Moreover, we infer from condition (71) that $p(1 - 2/N) < 1$, and there holds

$$pk = k + 2r/N > k + \max\{2\delta_i, \gamma_i - 1\} \tag{82}$$

by (71) again. Hence, we obtain by Lemma 3.2 combining condition (72) that

$$Z_i \le c_1 - c_2 \cdot \|v_i^{pk}\|_1, \tag{83}$$

where $c_1 > 0, c_2 > 0$ are some constants (also depending on the value L).

Taking together (77), (80) and (83), we find that there exist two positive constants C_2, C_3 such that

$$\dot V_i(t) \le C_2 \cdot \|1 + v_i^{k-1} + v_i^{k-1+\gamma_i}\|_1 + \rho(t)\|v_i^{k-2} + v_i^{k-2+2\alpha_i}\|_1 - C_3 \cdot \|v_i^{pk}\|_1. \tag{84}$$

Under condition (82) we are able to use Young's inequality. For example, we obtain that

$$\rho(t)\|v_i^{k-2+2\alpha_i}\|_1 \le C_3' \cdot \rho(t)^{1-(k+2(\alpha_i-1))/(pk)} + (C_3 \cdot \|v_i^{pk}\|_1)/8$$

with some appropriate constant $C_3' > 0$. Finally, we obtain from (84) that

$$\dot V_i(t) \le C_4[1 + \rho(t)^{\sigma_i}] - C_5 \cdot \|v_i^{pk}\|_1, \tag{85}$$

where $C_4 > 0, C_5 > 0$ are two constants, and $\sigma_i = pk/[pk - (k + 2(\alpha_i - 1))] > 0$. Using Hölder's inequality we have that

$$V_i(t) = \|v_i^k\|_1 \le |\Omega|^{1-1/p} \cdot \|v_i^{pk}\|_1^{1/p}.$$

Therefore,

$$\dot V_i(t) \le C_4[1 + \rho(t)^{\sigma_i}] - C_6 \cdot V_i(t)^p, \tag{86}$$

with some constant $C_6 > 0$. The above implies that $V_i(t)$ is decreasing if $C_6 V_i(t)^p > C_4[1+\rho(t)^{\sigma_i}]$. Hence, it is routine to derive from (86) the following estimate:

$$\|v_i(t)\|_k = V_i(t)^{1/k} \le C \cdot \max\{1 + \|v_i(0)\|_k, \rho(t)^{\kappa_i/2}\}, \tag{87}$$

where $\kappa_i = 2\sigma_i/(pk) = 1/(r/N + 1 - \alpha_i)$ is given by (74). Since $\rho(t) = [2(k-1)C_q^2 H(t)^2/d_i']$ (see (78)), we find from (87) the desired result in (73). □

Proof of Theorem 1.1. The local existence of solutions results from an application of Theorem 14.6 in [2]. Fix i and let $w := 0, \bar w := \max\{K, \|u_{i0}\|_\infty\}$ and define two vectors W and $\bar U$ by $W_i = 0, \bar U_i := \bar w$ and $W_k = \bar U_k := U_k$ if $k \ne i$. We have that $g(\bar U, V)_i \le 0 \le g(W, V)_i$ by **(H1)** and thus

$$\partial_t \bar w - [d_i \Delta \bar w + g(\bar U, V)_i] \ge 0 \ge \partial_t w - [d_i \Delta w + g(W, V)_i].$$

It follows from the Comparison Principle [3] that $0 = w \le u_i \le \bar w$. This is just the estimate in (12). The non-negativity of each v_j results also from the Comparison Principle and **(H1)**.

To prove Theorem 1.1-(i), we assume that there exists a constant $r > r_c$ such that

$$L(T) = \sup_{0\le t\le T} \|V(t)\|_r < \infty \quad \forall T < T_{\max}. \tag{88}$$

We take $k > N$ to be so large that $(1 + 2r/(kN))(1 - 2/N) < 1$. We want to prove the boundedness of $H(t)$. Our proof idea is based on the following bootstrap method, which is similar to the Alikakos-Moser iteration procedure [1]: First we estimate the $k-$norm of V by virtue of the function $H(\cdot)$. Then we estimate $H(\cdot)$ using the obtained estimation for all $k-$norms of V, and finally we derive the desired result using an elementary argument.

Fix $\tau \in (0, T_{\max})$, and $T, \tau < T < T_{\max}$. Let $t \in [\tau, T]$. First, we are in the position to Lemma 4.5 and it yields that

$$W_i(t) = \sup_{\tau\le s\le t} \|v_i(s)\|_k \le C \cdot (1 + H(t)^{\kappa_i}) \quad \forall t \le T, \tag{89}$$

where $\kappa_i = 1/(r/N + 1 - \alpha_i) > 0$. Second, we have by Lemma 4.4 combining with (89) that

$$H(t) \le C_T \cdot (1 + H(t)^{\mu}) \quad \forall t \le T, \tag{90}$$

where C_T is a constant which depends on $L(T)$, and

$$\mu := \max_{1\le i\le n}(\beta_i\kappa_i) = \max_{1\le i\le n}\beta_i/(r/N+1-\alpha_i) < 1, \tag{91}$$

since $r > r_c$. By considering both cases $H(t) \le 1$ and $H(t) > 1$ separably, we derive from (90) that

$$H(t) \le \max\{1, (2C_T)^{1/(1-\mu)}\} \quad \forall t \le T.$$

It follows, by using Lemma 4.5 again, that $W_i(t)$ is uniformly bounded, and so is $\|V(t)\|_\infty$ by Lemma 4.3 for all $t \le T$. The assertion $T_{\max} = +\infty$ follows from [3, Theorem 15.5], proving Theorem 1.1-(i).

To prove Theorem 1.1-(ii), we assume **(H4)**. By Theorem 1.1-(i) we need only to establish the boundedness of the L^1-norms of the $V-$components. To this end, we consider

$$X(t) := \int_\Omega f(t,x)\,dx, \quad f(t,x) := \langle B, (U,V)(t,x)\rangle \quad (x\in\Omega, t < T_{\max}). \tag{92}$$

Since the vector B is strictly positive, we can find a positive constant c_1 such that

$$\|V(t,x)\|_1 \le c_1 X(t) \quad \forall t < T_{\max}. \tag{93}$$

We will establish the boundedness of $X(t)$. A calculation using (40) (see the proof of Lemma 4.5) yields that

$$\dot X \le \int_\Omega \langle B, (g,h)(U(t,x), V(t,x))\rangle\,dx \tag{94}$$

We have $\langle B, (g,h)(U(t,x), V(t,x))\rangle \le b_1 + b_2 f(t,x)^\alpha - b_2 f(t,x)$ by (9) in **(H4)**. This implies that

$$\dot X(t) \le b_1|\Omega| + b_1\int_\Omega f(t,x)^\alpha\,dx - b_2X(t) \le b_3(1+X(t)^\alpha) - b_2X(t), \tag{95}$$

where $b_3 := b_1|\Omega| + b_1|\Omega|^{1-\alpha}$. For the last estimation we have used Hölder's inequality, since $\alpha < 1$. If $b_1 = 0$, then $b_3 = 0$ and thus $\dot X(t) \le 0$. This implies that $X(t) \le X(0)$ for all $t < T_{\max}$. Consider the case $b_2 > 0$. Let $y_0 > 0$ be such that

$$b_3(1+y^\alpha) - b_2y < 0 \quad \forall y > y_0.$$

It is routine to show that

$$X(t) \le X(0) + y_0 \quad \forall t < T_{\max}. \tag{96}$$

Therefore, we have shown that $X(t)$ is uniformly bounded if either $b_1 = 0$ or $b_2 > 0$.

Consider the rest case $b_2 = 0$. If $X(t) > 1$, then we find from (95) that $\dot X(t) \le 2b_3X(t)^\alpha$. It follows that

$$\frac{d}{dt}X(t)^{1-\alpha} = (1-\alpha)X(t)^{-\alpha}\dot X(t) < 2b_3$$

and thus there holds $X(t) \le X(0) + 1 + 2b_3t$ for all $t < T_{\max}$. This completes the proof of Theorem 1.1. □

Acknowledgements. The first author (Xu) is supported partially by the Natural Sciences Foundation of Heilongjiang Province of China (No.LH2020A002)and the Ph.D. Programs Foundation of Harbin University of China (No.HUDF2019101). The second author (Huang) would like to express his deep gratitude to Wumart Co. and Dr. Zhang Wenzhong and Prof. Wang Zhaojun for their kind helps in the establishment of the ZhiYing Research Center for Health Data of Nankai University. We thank Prof. R. Nagel, Prof. Zhang Zhimin and Prof. Wang Jinliang for valuable discussions.

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