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RESEARCH ARTICLE

Global existence of solutions to parabolic two species predator-prey chemotaxis system†

Gnanasekaran Shanmugasundaram[1] | Gurusamy Arumugam*[2] | Nithyadevi Nagarajan[3] | He Yang[4]

[1]Department of Mathematics, Bharathiar University, Tamil Nadu, India
[2]Department of Mathematics, National Institute of Technology Calicut, Kerala, India
[3]Department of Mathematics, Bharathiar University, Tamil Nadu, India
[4]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, 730070, People's Republic of China

Correspondence
*Gurusamy Arumugam, Department of Mathematics, National Institute of Technolodgy Calicut, Kerala, India, Email: guru.poy@gmail.com

Present Address
Gurusamy Arumugam, Department of Mathematics, National Institute of Technolodgy Calicut, Kerala, India

Abstract

We consider the following two species parabolic predator-prey chemotaxis system

$$\begin{cases} \mathbf{u}_{1t} = \mu_1\Delta\mathbf{u}_1 + \chi\nabla\cdot\left(\mathbf{u}_1\nabla\mathbf{v}\right) + \sigma_1\mathbf{u}_1(1-\mathbf{u}_1-a_1\mathbf{u}_2), & x\in\Omega,\ t>0,\\ \mathbf{u}_{2t} = \mu_2\Delta\mathbf{u}_2 - \xi\nabla\cdot\left(\mathbf{u}_2\nabla\mathbf{v}\right) + \sigma_2\mathbf{u}_2(1+a_2\mathbf{u}_1-\mathbf{u}_2), & x\in\Omega,\ t>0,\\ \mathbf{v}_t = \mu_3\Delta\mathbf{v} + \alpha f(\mathbf{u}_1) + \beta f(\mathbf{u}_2) - \gamma\mathbf{v}, & x\in\Omega,\ t>0, \end{cases} \tag{1}$$

under the homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 1$ with smooth boundary $\partial\Omega$. $f(s)$ is a nonnegative function belongs to $\mathbf{C}^1([\mathbf{0},\infty])$ that refers the production of s which also satisfies $0\leq f(s)\leq Ks^l$ for all $s\geq 0$, $0<l<1$. We assume that the parameters μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive and the nonnegative initial data $(\mathbf{u}_{10},\mathbf{u}_{20},\mathbf{v}_0)\in\mathbf{C}^0(\overline{\Omega})\times\mathbf{C}^0(\overline{\Omega})\times\mathbf{W}^{1,q}(\overline{\Omega})$. Under some additional assumptions on the parameters, we can choose $p>1$ and $\mu_i\in\left(0,\frac{(\sigma_i+1)p+1}{p}\right)$, $i=1,2$, $\delta>0$ and a_1 and a_2 satisfy

$$a_1<\left(\frac{\sigma_2}{2\sigma_1C(p)}\right)^{\frac{1}{p+1}},\quad a_2<\left(\frac{\sigma_1}{2\sigma_2C(p)}\right)^{\frac{1}{p+1}},$$

such that, if

$$\min\left\{\left(\frac{\sigma_1}{2}-C(p)a_2^{p+1}\sigma_2\right),\left(\frac{\sigma_2}{2}-C(p)a_1^{p+1}\sigma_1\right)\right\}>\delta,$$

then the system (1) has a unique globally bounded classical solution.

KEYWORDS:
predator-prey system, chemotaxis, classical solution, global existence

1 | INTRODUCTION

In this paper, we prove the global existence of classical solutions to the following initial-boundary value problem which describing the dynamics of predator-prey chemotaxis system:

$$\begin{cases} \mathbf{u}_{1t} = \mu_1\Delta\mathbf{u}_1 + \chi\nabla\cdot\left(\mathbf{u}_1\nabla\mathbf{v}\right) + \sigma_1\mathbf{u}_1(1-\mathbf{u}_1-a_1\mathbf{u}_2), & x\in\Omega,\ t>0,\\ \mathbf{u}_{2t} = \mu_2\Delta\mathbf{u}_2 - \xi\nabla\cdot\left(\mathbf{u}_2\nabla\mathbf{v}\right) + \sigma_2\mathbf{u}_2(1+a_2\mathbf{u}_1-\mathbf{u}_2), & x\in\Omega,\ t>0,\\ \mathbf{v}_t = \mu_3\Delta\mathbf{v} + \alpha f(\mathbf{u}_1) + \beta f(\mathbf{u}_2) - \gamma\mathbf{v}, & x\in\Omega,\ t>0,\\ \frac{\partial\mathbf{u}_1}{\partial\nu} = \frac{\partial\mathbf{u}_2}{\partial\nu} = \frac{\partial\mathbf{v}}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ \mathbf{u}_1(\cdot,0)=\mathbf{u}_{10},\quad \mathbf{u}_2(\cdot,0)=\mathbf{u}_{20},\quad \mathbf{v}(\cdot,0)=\mathbf{v}_0, & x\in\Omega, \end{cases} \tag{2}$$

†Global existence of predator-prey chemotaxis system
[0]**Abbreviations:**Parabolic two species predator-prey chemotaxis system

in a smooth bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$ with boundary $\partial\Omega$. Here ν represents the unit outward normal to $\partial\Omega$. The functions $\mathbf{u}_1 = \mathbf{u}_1(x,t)$ and $\mathbf{u}_2 = \mathbf{u}_2(x,t)$ describe the densities of the populations of prey and predators, respectively, and $\mathbf{v} = \mathbf{v}(x,t)$ denotes the concentration of the common chemical attractant. Here μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive constants and the initial data $\mathbf{u}_{10}, \mathbf{u}_{20}$ and $\mathbf{v}_0$ are non-negative functions. The constants μ_1, μ_2 and μ_3 are denoted as diffusion coefficients which state the natural dispersive force of the movements of the prey, predator and attractant, respectively. χ and ξ are the chemotaxis sensitivities, besides the term $\chi\nabla \cdot (\mathbf{u}_1\nabla\mathbf{v})$ represents the chemorepulsion, that is, the directional prey movements away from a substance secreted by the predator and the term $-\xi\nabla \cdot (\mathbf{u}_2\nabla\mathbf{v})$ describes the chemoattraction, that is, the directional predator movements towards the substance secreted by the prey. σ_1 and σ_2 are indicating that the growth rates of prey and predator, respectively. a_1 and a_2 represent the interactions between two species. The parameters α and β are the production rate of the prey and predator, respectively, and γ is the decay rate of the chemical attractant.
In this paper, we assume that the function $f(s) \in \mathbf{C^1}([\mathbf{0}, \infty])$ satisfies the condition

$$0 \leq f(s) \leq Ks^l, \qquad \text{for} \quad K, l > 0, \quad \text{and} \quad s \geq 0. \tag{3}$$

In addition, we also assume that the initial values $\mathbf{u}_{10}$, $\mathbf{u}_{20}$ and $\mathbf{v}_0$ satisfy

$$\begin{aligned} &\mathbf{u}_{10} \in \mathbf{C^0}(\overline{\boldsymbol{\Omega}}), \quad \text{with} \quad \mathbf{u}_{10} \geq 0 \quad \text{in}\,\Omega, \\ &\mathbf{u}_{20} \in \mathbf{C^0}(\overline{\boldsymbol{\Omega}}), \quad \text{with} \quad \mathbf{u}_{20} \geq 0 \quad \text{in}\,\Omega, \\ &\mathbf{v}_0 \in \mathbf{W^{1,q}}(\overline{\boldsymbol{\Omega}}), \quad \text{for some } \mathbf{q} > \max\{2, n\}, \quad \text{with} \quad \mathbf{v}_0 \geq 0 \quad \text{in}\,\Omega. \end{aligned} \tag{4}$$

The chemotaxis is the directional movement of a micro-organism response to the chemical stimulus. In 1970, the classical chemotaxis system was the first introduced by Keller-Segel in [1]

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

where u and v denote the cell density and the concentration of chemical signals, respectively, and the author provided a detailed analysis of the aggregation process of the cells. The Keller-Segel system and modified Keller-Segel system were studied extensively by many researchers in the past decades. For the recent developments in this field one can refer the recent reviews by Lankeit and Winkler[2] and Gurusamy and Tyagi[3] and the references therein. This field has been attracted by many researchers due to its importance in the fields related to chemotaxis systems such as biology, medicine and other sciences.

To the better understanding of the system (2), the two-species chemotaxis system with competitive kinetics

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla \cdot (u\nabla w) + \mu_1 u(1 - u - a_1 v), \\ v_t = d_2\Delta v - \chi_2\nabla \cdot (u\nabla w) + \mu_2 v(1 - v - a_2 u), \\ \tau w_t = d_3\Delta w + \alpha u + \beta v - \gamma w, \end{cases} \tag{5}$$

proposed and discussed about the asymptotic stability of the system under some suitable conditions on the logistic source coefficients with $\tau = 0$ by Tello and Winkler[4]. Many research works has been done in the parabolic-parabolic-elliptic case among them existence and boundedness of classical solutions for the two dimensional case of the system (5) with $d_1 = d_2 = d_3 = d_4 = 0$ were investigated in[5] and for asymptotic behavior (see[6,7]). When $a_1 \in (0,1)$ and $a_2 \in (0,1)$, Lin et. al.[8] studied the global boundedness of the solutions under the assumption that the coefficients μ_1 and μ_2 are sufficiently large. Moreover, the solution approaches the steady-state solution as $t \to \infty$ exponentially. On the other hand, when $a_1 \geq 1$ and $a_2 \in (0,1)$ and if μ_2 is large then the solution of the system (5) converges toward (0,1,1) as $t \to \infty$ in algebraic. It is worth to notice that the proposed results of[6,7,4] were improved in[9]. Further, the existence and boundedness of classical solutions studied by Wang[10] and the author also established that the relation between the coefficients of system (5) which improved the results of[6,8,7].

For the fully parabolic case, the unique global uniformly bounded classical solution to (5) discussed in[11,12,13] under the suitable assumptions on the coefficients . In[14,15] the authors studied the existence of classical solutions to the system (5) and conducted the numerical simulations as well. Bai and Winkler[16] proved that the unique bounded classical solution to (5) in the space dimension $n \leq 2$. On the other hand, for $n \geq 1$, if $a_1 < 1$ and $a_2 < 1$ and both μ_1 and μ_2 are sufficiently large, the solution of the system (5) converges to a unique positive spatially homogeneous equilibrium of (5). Besides, if $a_1 \geq 1$ and $a_2 < 1$, and μ_2 is large, then the solution of the system approaches $\left(0, 1, \frac{\beta}{\gamma}\right)$ uniformly and later this condition improved by Mizukami[17] and Zhang and Niu[18]. It is to be noted that Mizukami[19] were improved the conditions for the asymptotic behavior in the case $a_1, a_2 \in (0,1)$ assumed in the above research works[16,17]. Htwe and Wang[20] studied the global boundedness of solutions to the system under some weaker conditions. Based on the maximal Sobolev regularity, the existence of a globally bounded classical solution to the system (5) in $n \geq 3$ established by Zhou and Yang in[21]. Zhang and Li[22] proved that the system possess a unique global bounded solution provided the domain is convex and μ_1 and μ_2 are sufficiently large. The authors also established that the existence of global weak solution for any $\mu_1 > 0$ and $\mu_2 > 0$. For the three dimensional case, the results of[5,16,23] were partially improved by Li and Wang[24]. The existence and boundedness of solutions for the two and three-dimensional cases were proved by Gao et al. in[25] under the assumption that chemotaxis coefficients are smaller than the diffusion coefficient.

The existence of solutions to predator-prey systems with chemotaxis term is under intensive investigation. Let us review some of the existing results in literature. Amorim and Telch[26] considered the chemotaxis predator-prey system with indirect pursuit-evasion and studied the well-posedness of the system using the de Giorgi method and presented the numerical simulations as well. Haskell and Bell[27] proved the existence of solutions for the predator-mediated co-existence system. Furthermore, the authors also obtained the pattern formations using bifurcation analysis and finally they validated the analytical results through the numerical solutions of the system. Negreanu[28] discussed about the global existence and boundedness of solutions to the chemotaxis prey-predator system and they proved the asymptotic behaviour of the system for the different ranges of parameters. Li et. al.[29] showed that the global boundedness of solutions of the predator-prey system with indirect pursuit-evasion in the space dimension $n \leq 3$. Moreover, the stability of the solution also provided if $b\lambda < \mu$ and the small assumptions on χ and ξ. On the other hand, if $b\lambda > \mu$ and the small condition on χ then the solution converges to $(\lambda, 0)$ as $t \to \infty$. Later, Ahn and Yoo[30] shown that the global solvability of the prey-predator system with indirect predator-taxis up to the space dimension two. Furthermore, the authors also obtained the stability of the bounded solution by using Lyapunov function.

In[31], Fu and Miao proposed the following two-species chemotaxis predator-prey system

$$\begin{cases} u_{1t} = d_1\Delta u_1 + \chi\nabla\cdot(u_1\nabla v) + \mu_1 u_1(1-u_1-e_1u_2), \\ u_{2t} = d_2\Delta u_2 - \xi\nabla\cdot(u_2\nabla v) + \mu_2 u_2(1-u_2+e_2u_1), \\ v_t = d_3\Delta v + \alpha u_1 + \beta u_2 - \gamma v, \end{cases} \tag{6}$$

and they showed that the global existence of classical solution to the system (6) under the suitable assumptions of the parameters μ_1, μ_2, e_1 and e_2 in th space dimension $n \leq 2$. Furthermore, they also proved that the unique positive equilibrium is globally asymptotically stable for $e_1 < 1$ if both $\frac{\mu_1}{\chi^2}$ and $\frac{\mu_2}{\xi^2}$ were sufficiently large. Nevertheless, if $e_1 \geq 1$ and $\frac{\mu_2}{\xi^2}$ are sufficiently large then the solution converges to the semi-trivial equilibrium point uniformly as $t \to \infty$ by using Lyapunov functions. Very recently, the chemotaxis predator-prey system (6) with $d_1 = d_2 = d_3 = 1$ in three-dimension discussed by Miao et. al. in[32]. The authors also derived that the global boundedness for the predator-prey densities in L^2 norm and the signal gradient in L^4 norm.

Motivated by the above mentioned research works, in this paper, we consider the system (2) to prove the global existence and boundedness of classical solution under suitable conditions on the parameters a_1 and a_2 and the non-negative initial data $(\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_0) \in \mathbf{C^0}(\overline{\mathbf{\Omega}}) \times \mathbf{C^0}(\overline{\mathbf{\Omega}}) \times \mathbf{W^{1,q}}(\overline{\mathbf{\Omega}})$ for $\Omega \subset \mathbb{R}^n, n \geq 1$.

Our article is organized as follows: In Section 2, we present some basic inequalities and key lemma and we prove the local existence of classical solution. Section 3 deals with boundedness and global existence of classical solution to the system (2). Finally, in Section 4, we formulate the conclusion.

Theorem 1. Let $n \geq 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that the parameters μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive and $q > \max\{2, n\}$. Moreover, for any $p > 1$, $0 < l < 1$ and $\delta > 0$ we choose $\mu_i \in \left(0, \frac{(\sigma_i+1)p+1}{p}\right), i = 1, 2$ satisfy

$$a_1 < \left(\frac{\sigma_2}{2\sigma_1 C(p)}\right)^{\frac{1}{p+1}}, \quad a_2 < \left(\frac{\sigma_1}{2\sigma_2 C(p)}\right)^{\frac{1}{p+1}},$$

such that, if

$$\min\left\{\left(\frac{\sigma_1}{2} - C(p)a_2^{p+1}\sigma_2\right), \left(\frac{\sigma_2}{2} - C(p)a_1^{p+1}\sigma_1\right)\right\} > \delta,$$

then for any initial data $(\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_0)$ satisfy (4), the system (2) possesses a unique classical solution $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$ which is uniformly bounded in the sense that

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{v}(\cdot,t)\|_{\mathbf{W}^{1,q}(\Omega)} \leq C, \quad \forall t > 0,$$

where the constants $C(p)$ and C are positive.

2 | PRELIMINARIES AND LOCAL EXISTENCE

We recall some useful inequalities and key lemma which we are going to use in the sequel. In this section, we begin with the local existence of solutions to system (2) which is standard and its proof is based on the ideas of[33].

Definition 1 (Cauchy's inequality with ϵ[34])**.**

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \qquad a, b > 0, \epsilon > 0.$$

Definition 2 (Young's inequality with ϵ[34])**.** Let $1 < p,\ q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \epsilon a^p + C(\epsilon) b^q, \qquad a, b > 0, \epsilon > 0,$$

for $C(\epsilon) = (\epsilon p)^{\frac{-q}{p}} q^{-1}$.

Definition 3 (Hölder's inequality[34])**.** Assume $1 \leq p,\ q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $\mathbf{u} \in \mathbf{L^p(\Omega)},\ \mathbf{v} \in \mathbf{L^q(\Omega)}$, we have

$$\int_\Omega |\mathbf{uv}| d\mathbf{x} \leq \|\mathbf{u}\|_{\mathbf{L^p(\Omega)}} \|\mathbf{v}\|_{\mathbf{L^q(\Omega)}}.$$

Definition 4 (Interpolation inequality[34])**.** Assume $1 \leq s \leq r \leq t \leq \infty$ and $\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}$, Suppose also $\mathbf{u} \in \mathbf{L^s(\Omega)} \cap \mathbf{L^t(\Omega)}$. Then $\mathbf{u} \in \mathbf{L^r(\Omega)}$ and

$$\|u\|_{L^r} \leq \|u\|_{L^s}^{\theta} \|u\|_{L^t}^{1-\theta}.$$

Definition 5 (The Gagliardo-Nirenberg inequality[34])**.** Fix $1 \leq q, r \leq \infty$ and a natural number m. Suppose also that a real number α and a natural number j are such that

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1-\alpha}{q} \qquad \text{and} \qquad \frac{j}{m} \leq \alpha \leq 1,$$

then

$$\|D^j u\|_{\mathbf{L^p(\Omega)}} \leq C_1 \|D^m u\|_{\mathbf{L^r(\Omega)}}^{\alpha} \|u\|_{\mathbf{L^q(\Omega)}}^{1-\alpha} + C_2 \|u\|_{\mathbf{L^s(\Omega)}},$$

where $s > 0$ is arbitrary.

Lemma 1 (Local Existence)**.** Let $n \geq 1$, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain with smooth boundary. Suppose that the parameters μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive and $q > \max\{2, n\}$. Then there exists $T_{\max} \in (0, \infty]$ and a uniquely determined triple $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$ of nonnegative functions

$$\begin{aligned}
\mathbf{u}_1 &\in \mathbf{C^0}\left(\overline{\Omega} \times \left[0, T_{\max}\right)\right) \cap \mathbf{C^{2,1}}\left(\overline{\Omega} \times \left(0, T_{\max}\right)\right),\\
\mathbf{u}_2 &\in \mathbf{C^0}\left(\overline{\Omega} \times \left[0, T_{\max}\right)\right) \cap \mathbf{C^{2,1}}\left(\overline{\Omega} \times \left(0, T_{\max}\right)\right),\\
\mathbf{v} &\in \mathbf{C^0}\left(\overline{\Omega} \times \left[0, T_{\max}\right)\right) \cap \mathbf{C^{2,1}}\left(\overline{\Omega} \times \left(0, T_{\max}\right)\right) \cap \mathbf{L^\infty_{loc}}\left(\left[0, T_{\max}\right); \mathbf{W^{1,q}}(\Omega)\right),
\end{aligned}$$

solving (2) classically in $\Omega \times (0, T_{\max})$. Furthermore, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \left(\|\mathbf{u}_1(\cdot, t)\|_{\mathbf{L^\infty(\Omega)}} + \|\mathbf{u}_2(\cdot, t)\|_{\mathbf{L^\infty(\Omega)}} + \|\mathbf{v}(\cdot, t)\|_{\mathbf{W^{1,q}(\Omega)}}\right) = \infty. \tag{7}$$

Moreover, the solution $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v})$ satisfies

$$\|\mathbf{u}_1(\cdot, t)\|_{\mathbf{L^1(\Omega)}} \leq m_1 = \max\left\{\|\mathbf{u}_{10}\|_{\mathbf{L^1}}, |\Omega|\right\}, \qquad t \in (0, T_{\max}), \tag{8}$$

$$\|\mathbf{u}_2(\cdot, t)\|_{\mathbf{L^1(\Omega)}} \leq m_2 = \frac{C_3}{\sigma_1 a_1}, \qquad t \in (0, T_{\max}), \tag{9}$$

where $C_3 = \max\left\{\sigma_2 a_2 \|\mathbf{u}_{10}\|_{\mathbf{L^1(\Omega)}} + \sigma_1 a_1 \|\mathbf{u}_{20}\|_{\mathbf{L^1(\Omega)}}, \dfrac{\sigma_1 \sigma_2 (a_1 + a_2)}{\min\{\sigma_1 \sigma_2\}} |\Omega|\right\}$.

Proof. Let $\mathbf{w} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}) \in \mathbb{R}^3$, then system (2) can be reformed as

$$\begin{cases} \mathbf{w}_t = \nabla \cdot \big(A(\mathbf{w})\nabla \mathbf{w}\big) + f(\mathbf{w}), & x \in \Omega,\ t > 0, \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0, & x \in \Omega,\ t > 0, \\ \mathbf{w}(x,0) = (\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_0), & x \in \partial\Omega, \end{cases}$$

where

$$A(\mathbf{w}) = \begin{pmatrix} \mu_1 & 0 & \chi \mathbf{u}_1 \\ 0 & \mu_2 & -\xi \mathbf{u}_2 \\ 0 & 0 & \mu_3 \end{pmatrix}, \qquad f(\mathbf{w}) = \begin{pmatrix} \sigma_1 \mathbf{u}_1 (1 - \mathbf{u}_1 - a_1 \mathbf{u}_2) \\ \sigma_2 \mathbf{u}_2 (1 + a_2 \mathbf{u}_1 - \mathbf{u}_2) \\ \alpha f(\mathbf{u}_1) + \beta f(\mathbf{u}_2) - \gamma \mathbf{v} \end{pmatrix}.$$

Using theorem 14.4 of[33] we can ensure the existence of weak maximal solution. Also the solution is classical and satisfies (2) pointwise cf. theorem 14.6 of[33]. Further, (7) follows from theorem 15.5 of[33].
Next, from first equation of (2) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{u}_1 \le \sigma_1 \int_\Omega \mathbf{u}_1 - \sigma_1 \int_\Omega \mathbf{u}_1^2,$$

using Cauchy-Schwarz inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{u}_1 \le \sigma_1 \int_\Omega \mathbf{u}_1 - \frac{\sigma_1}{|\Omega|} \left(\int_\Omega \mathbf{u}_1 \right)^2,$$

by ODE comparison argument gives (8). Next, Integrating the sum of the $\sigma_2 a_2$ times the first equation in (2) and the $\sigma_1 a_1$ times the second equation in (2) on Ω by parts

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sigma_2 a_2 \mathbf{u}_1 + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sigma_1 a_1 \mathbf{u}_2 = \int_\Omega \sigma_1 \sigma_2 a_2 \mathbf{u}_1 - \int_\Omega \sigma_1 \sigma_2 a_2 \mathbf{u}_1^2 + \int_\Omega \sigma_1 \sigma_2 a_1 \mathbf{u}_2 - \int_\Omega \sigma_1 \sigma_2 a_1 \mathbf{u}_2^2,$$

using Cauchy Schawrz inequality and Young's inequality we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sigma_2 a_2 \mathbf{u}_1 + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sigma_1 a_1 \mathbf{u}_2 \le - \int_\Omega \sigma_1 \sigma_2 a_2 \mathbf{u}_1 - \int_\Omega \sigma_1 \sigma_2 a_1 \mathbf{u}_2 + \sigma_1 \sigma_2 |\Omega| (a_1 + a_2),$$

take $z(t) = \int_\Omega \sigma_2 a_2 \mathbf{u}_1 + \int_\Omega \sigma_1 a_1 \mathbf{u}_2$ by ODE argument, gives (9). □

Lemma 2. (See Cao[11] and Hieber[35]). Let $r \in (1, \infty)$. Consider the following evolution equation

$$\begin{cases} y_t = \Delta y - y + g, & x \in \Omega,\ t > 0, \\ \frac{\partial y}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{10}$$

For each $y_0 \in \mathbf{W^{2,r}}(\Omega)$ such that $\frac{\partial y}{\partial \nu} = 0$ on $\partial\Omega$ and any $g \in \mathbf{L^r}\Big((0,T); \mathbf{L^r}(\Omega)\Big)$, there exists a unique solution

$$y \in \mathbf{W^{1,r}}\Big((0,T); \mathbf{L^r}(\Omega)\Big) \cap \mathbf{L^r}\Big((0,T); \mathbf{W^{2,r}}(\Omega)\Big).$$

Moreover, for any $s_0 \in [0,T)$, there exists $C(r) > 0$ such that for all $s \in (s_0, T)$ we have

$$\int_{s_0}^{T} \int_\Omega y^r + \int_{s_0}^{T} \int_\Omega |y_t|^r + \int_{s_0}^{T} \int_\Omega |\Delta y|^r \le C_r \int_{s_0}^{T} \int_\Omega |g|^r + C_r \int_\Omega y_0^r + C_r \int_\Omega |\Delta y_0|^r \tag{11}$$

and

$$\int_{s_0}^{T} \int_\Omega e^{sr} |\Delta y|^r \le C_r \int_{s_0}^{T} \int_\Omega e^{sr} |g|^r + C_r \int_\Omega y_0^r + C_r \int_\Omega |\Delta y_0|^r. \tag{12}$$

Next we prove the main theorem of our system (2).

3 | GLOBAL EXISTENCE

In this section, we prove the global existence and boundedness of the solution to (2). First we drive $\mathbf{L^p}(\Omega)$ norm for $\mathbf{u}_1$ and $\mathbf{u}_2$. Given any $s_0 \in (0, T_{\max})$ such that $s_0 < 1$, from Lemma (1) gives $\mathbf{u}_1(\cdot, s_0), \mathbf{u}_2(\cdot, s_0), \mathbf{v}(\cdot, s_0) \in \mathbf{C^2}(\overline{\boldsymbol{\Omega}})$ with $\frac{\partial \mathbf{v}}{\partial \nu} = 0$, we pick $K > 0$ such that

$$\sup_{0\le s\le s_0} \|\mathbf{u}_1(\cdot, s)\|_{\mathbf{L}^\infty(\Omega)} \le K, \qquad \sup_{0\le s\le s_0} \|\mathbf{u}_2(\cdot, s)\|_{\mathbf{L}^\infty(\Omega)} \le K, \qquad \sup_{0\le s\le s_0} \|\mathbf{v}(\cdot, s)\|_{\mathbf{L}^\infty(\Omega)} \le K \quad \text{and} \quad \|\Delta\mathbf{v}(\cdot, s)\|_{\mathbf{L}^\infty(\Omega)} \le K. \tag{13}$$

Next, we drive boundedness in $t \in (s_0, T_{\max})$.

Lemma 3. Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$ be a bounded domain with smooth boundary $\partial\Omega$, the parameters μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive constants. For any $p > 1$, $0 < l < 1$ and there exists $\delta > 0$ choose $\mu_i \in \left(0, \dfrac{(\sigma_i + 1)p + 1}{p}\right)$ where $i = 1, 2$ satisfy

$$a_1 < \left(\frac{\sigma_2}{2\sigma_1 C(p)}\right)^{\frac{1}{p+1}}, \quad a_2 < \left(\frac{\sigma_1}{2\sigma_2 C(p)}\right)^{\frac{1}{p+1}},$$

such that, if

$$\min\left\{\left(\frac{\sigma_1}{2} - C(p)a_2^{p+1}\sigma_2\right), \left(\frac{\sigma_2}{2} - C(p)a_1^{p+1}\sigma_1\right)\right\} > \delta,$$

then

$$\|\mathbf{u}_1(\cdot, t)\|_{\mathbf{L^p}(\Omega)} + \|\mathbf{u}_2(\cdot, t)\|_{\mathbf{L^p}(\Omega)} \le C, \qquad \forall\, t \in (s_0, T_{\max}), \tag{14}$$

for some $C(p)$, $C(\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_0, \mu_1, \mu_2, \sigma_1, \sigma_2, \delta, \kappa, p, |\Omega|) > 0$.

Proof. Multiply with $\mathbf{u}_1^{p-1}$, $p > 1$, in the first equation of (2) and integrate with respect to Ω, we get

$$\int_\Omega \mathbf{u}_{1t}\mathbf{u}_1^{p-1} = \mu_1 \int_\Omega \mathbf{u}_1^{p-1}\Delta\mathbf{u}_1 + \chi \int_\Omega \mathbf{u}_1^{p-1}\nabla\cdot\left(\mathbf{u}_1\nabla\mathbf{v}\right) + \sigma_1 \int_\Omega \mathbf{u}_1^{p-1}\mathbf{u}_1(1 - \mathbf{u}_1 - a_1\mathbf{u}_2),$$

Applying integration by parts, we get

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u}_1^p = -\mu_1(p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 - \chi(p-1)\int_\Omega \mathbf{u}_1^{p-1}\nabla\mathbf{u}_1\nabla\mathbf{v} + \sigma_1\int_\Omega \mathbf{u}_1^p - \sigma_1\int_\Omega \mathbf{u}_1^{p+1} - a_1\sigma_1\int_\Omega \mathbf{u}_1^p\mathbf{u}_2,$$

Again use of integration by parts to the second term in R.H.S of the last equation leads to

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u}_1^p =& -\mu_1(p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 + \chi\frac{(p-1)}{p}\int_\Omega \mathbf{u}_1^p\Delta\mathbf{v} + \sigma_1\int_\Omega \mathbf{u}_1^p - \sigma_1\int_\Omega \mathbf{u}_1^{p+1} - a_1\sigma_1\int_\Omega \mathbf{u}_1^p\mathbf{u}_2, \\
=& -\frac{p+1}{p}\int_\Omega \mathbf{u}_1^p - \mu_1(p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 + \chi\frac{(p-1)}{p}\int_\Omega \mathbf{u}_1^p\Delta\mathbf{v} + \frac{p+1}{p}\int_\Omega \mathbf{u}_1^p + \sigma_1\int_\Omega \mathbf{u}_1^p - \sigma_1\int_\Omega \mathbf{u}_1^{p+1} \\
&- a_1\sigma_1\int_\Omega \mathbf{u}_1^p\mathbf{u}_2, \\
=& -\frac{p+1}{p}\int_\Omega \mathbf{u}_1^p - \sigma_1\int_\Omega \mathbf{u}_1^{p+1} - \mu_1(p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 + \chi\frac{(p-1)}{p}\int_\Omega \mathbf{u}_1^p\Delta\mathbf{v} + \left(\sigma_1 + \frac{p+1}{p}\right)\int_\Omega \mathbf{u}_1^p \\
&- a_1\sigma_1\int_\Omega \mathbf{u}_1^p\mathbf{u}_2, \\
=& -\frac{p+1}{p}\int_\Omega \mathbf{u}_1^p - \sigma_1\int_\Omega \mathbf{u}_1^{p+1} + I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{15}$$

Using Gagliardo-Nirenberg inequity and Young's inequality to the first term in R.H.S of (15), we get

$$
\begin{aligned}
\int_\Omega \mathbf{u}_1^p = \left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^2_{\mathbf{L^2}(\Omega)} &\le C\left(\left\|\nabla\mathbf{u}_1^{\frac{p}{2}}\right\|^{2a}_{\mathbf{L^2}(\Omega)}\left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^{2(1-a)}_{\mathbf{L^{\frac{2}{p}}}(\Omega)} + \left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^2_{\mathbf{L^{\frac{2}{p}}}(\Omega)}\right),\\
&\le \epsilon\left(\left\|\nabla\mathbf{u}_1^{\frac{p}{2}}\right\|^{2a}_{\mathbf{L^2}(\Omega)}\right)^{\frac{1}{a}} + C\,C(\epsilon)\left(\left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^{2(1-a)}_{\mathbf{L^{\frac{2}{p}}}(\Omega)}\right)^{\frac{1}{1-a}} + C\left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^2_{\mathbf{L^{\frac{2}{p}}}(\Omega)},\\
&\le \epsilon\left(\left\|\nabla\mathbf{u}_1^{\frac{p}{2}}\right\|^{2a}_{\mathbf{L^2}(\Omega)}\right)^{\frac{1}{a}} + C_0\left\|\mathbf{u}_1^{\frac{p}{2}}\right\|^2_{\mathbf{L^{\frac{2}{p}}}(\Omega)},\\
&\le \epsilon\left(\left\|\nabla\mathbf{u}_1^{\frac{p}{2}}\right\|^{2a}_{\mathbf{L^2}(\Omega)}\right)^{\frac{1}{a}} + C_0\|\mathbf{u}_1(\cdot,t)\|^p_{\mathbf{L^1}(\Omega)},\\
&\le \frac{4(p-1)}{p^2}\left\|\nabla\mathbf{u}_1^{\frac{p}{2}}\right\|^2_{\mathbf{L^2}(\Omega)} + C_1,\\
&\le \frac{4(p-1)}{p^2}\int_\Omega\left|\nabla\mathbf{u}_1^{\frac{p}{2}}\right|^2 + C_1,\\
&\le \frac{4(p-1)}{p^2}\int_\Omega\left(\frac{p}{2}\mathbf{u}_1^{\frac{p}{2}-1}\nabla\mathbf{u}_1\right)^2 + C_1,\\
&\le \frac{4(p-1)}{p^2}\frac{p^2}{4}\int_\Omega \mathbf{u}_1^{\left(\frac{p-2}{2}\right)^2}|\nabla\mathbf{u}_1|^2 + C_1,
\end{aligned}
$$

therefore, we get

$$
\int_\Omega \mathbf{u}_1^p \le (p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 + C_1, \tag{16}
$$

with $C_1 = C_0 m_1^p$, $C(\epsilon) = \left(\frac{4(p-1)}{p^2}\right)^{\frac{-a}{1-a}}\left(\frac{1}{a}\right)^{\frac{-a}{1-a}}(1-a)$, where $a = \dfrac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$. Now we estimate I_1 using (16) as follows:

$$
I_1 = -\mu_1(p-1)\int_\Omega \mathbf{u}_1^{p-2}|\nabla\mathbf{u}_1|^2 \le -\mu_1\int_\Omega \mathbf{u}_1^p + \mu_1 C_1.
$$

Next, for $p > 1$, $\left(1-\frac{1}{p}\right) < 1$, we have

$$
I_2 = \chi\frac{(p-1)}{p}\int_\Omega \mathbf{u}_1^p\Delta\mathbf{v} \le \chi\int_\Omega \mathbf{u}_1^p|\Delta\mathbf{v}|.
$$

Now applying Young's inequality with $\kappa > 0$, we get

$$
\begin{aligned}
I_2 &\le \kappa\int_\Omega\left(\mathbf{u}_1^p\right)^{(1+\frac{1}{p})} + C(\kappa)\int_\Omega(\chi|\Delta\mathbf{v}|)^{p+1},\\
&\le \kappa\int_\Omega \mathbf{u}_1^{p+1} + C_2\kappa^{-p}\chi^{p+1}\int_\Omega|\Delta\mathbf{v}|^{p+1}, \qquad (17)
\end{aligned}
$$

where $C_2 = \left(\frac{p+1}{p}\right)^{-p} \frac{1}{1+p}$. Again, using Young's inequality with $\eta > 0$, we obtain,

$$\begin{aligned} I_1 + I_3 &= \left(\sigma_1 + \frac{p+1}{p} - \mu_1\right) \int\limits_\Omega \mathbf{u}_1^p + \mu_1 C_1, \\ &\le \eta \int\limits_\Omega \left(\mathbf{u}_1^p\right)^{\frac{p+1}{p}} + \eta^{-p}\left(\frac{p+1}{p}\right)^{-p} \frac{1}{p+1} \int\limits_\Omega \left(\sigma_1 + \frac{p+1}{p} - \mu_1\right)^{p+1} + \mu_1 C_1, \\ &\le \eta \int\limits_\Omega \mathbf{u}_1^{p+1} + C_3, \end{aligned} \tag{18}$$

where $C_3 = \eta^{-p}\left(\frac{p+1}{p}\right)^{-p} \frac{1}{p+1}\left(\sigma_1 + \frac{p+1}{p} - \mu_1\right)^{p+1} |\Omega| + \mu_1 C_1$. Again, use of Young's inequality,

$$\begin{aligned} I_4 = -\int\limits_\Omega a_1\sigma_1 \mathbf{u}_1^p \mathbf{u}_2 &\le \int\limits_\Omega a_1 \sigma_1 \mathbf{u}_1^p \mathbf{u}_2 \le \frac{\sigma_1}{2} \int\limits_\Omega \left(\mathbf{u}_1^p\right)^{\frac{p+1}{p}} + C(\sigma_1) a_1^{p+1} \sigma_1^{p+1} \int\limits_\Omega \mathbf{u}_2^{p+1}, \\ &\le \frac{\sigma_1}{2} \int\limits_\Omega \mathbf{u}_1^{p+1} + C_4 \sigma_1^{-p} a_1^{p+1} \sigma_1^{p+1} \int\limits_\Omega \mathbf{u}_2^{p+1}, \\ &\le \frac{\sigma_1}{2} \int\limits_\Omega \mathbf{u}_1^{p+1} + C_4 a_1^{p+1} \sigma_1 \int\limits_\Omega \mathbf{u}_2^{p+1}, \end{aligned} \tag{19}$$

with $C_4 = \left(\frac{p+1}{p}\right)^{-p} \frac{1}{1+p}\left(\frac{1}{2}\right)^{-p}$. Substitute (17) - (19) in (15) we see that

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_\Omega \mathbf{u}_1^p \le &- \frac{p+1}{p} \int\limits_\Omega \mathbf{u}_1^p - \sigma_1 \int\limits_\Omega \mathbf{u}_1^{p+1} + \kappa \int\limits_\Omega \mathbf{u}_1^{p+1} + C_2 \kappa^{-p} \chi^{p+1} \int\limits_\Omega |\Delta \mathbf{v}|^{p+1} + \eta \int\limits_\Omega \mathbf{u}_1^{p+1} + C_3 + \frac{\sigma_1}{2} \int\limits_\Omega \mathbf{u}_1^{p+1} \\ &+ C_4 a_1^{p+1} \sigma_1 \int\limits_\Omega \mathbf{u}_2^{p+1}, \\ \le &- \frac{p+1}{p} \int\limits_\Omega \mathbf{u}_1^p - \left(\frac{\sigma_1}{2} - \kappa - \eta\right) \int\limits_\Omega \mathbf{u}_1^{p+1} + C_2 \kappa^{-p} \chi^{p+1} \int\limits_\Omega |\Delta \mathbf{v}|^{p+1} + C_4 a_1^{p+1} \sigma_1 \int\limits_\Omega \mathbf{u}_2^{p+1} + C_3, \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int\limits_\Omega \mathbf{u}_1^p\right) \le -(p+1)\left(\frac{1}{p}\int\limits_\Omega \mathbf{u}_1^p\right) - \left(\frac{\sigma_1}{2} - \kappa - \eta\right) \int\limits_\Omega \mathbf{u}_1^{p+1} + C_2 \kappa^{-p} \chi^{p+1} \int\limits_\Omega |\Delta \mathbf{v}|^{p+1} + C_4 a_1^{p+1} \sigma_1 \int\limits_\Omega \mathbf{u}_2^{p+1} + C_3. \tag{20}$$

Applying the variation-of-constants formula to (20), we obtain

$$\begin{aligned} \frac{1}{p}\int\limits_\Omega \mathbf{u}_1^p \le &-\left(\frac{\sigma_1}{2} - \kappa - \eta\right) \int\limits_{s_0}^{t} e^{-(p+1)(t-s)} \int\limits_\Omega \mathbf{u}_1^{p+1} + C_2 \kappa^{-p} \chi^{p+1} \int\limits_{s_0}^{t} e^{-(p+1)(t-s)} \int\limits_\Omega |\Delta \mathbf{v}|^{p+1} + C_4 a_1^{p+1} \sigma_1 \int\limits_{s_0}^{t} e^{-(p+1)(t-s)} \int\limits_\Omega \mathbf{u}_2^{p+1} \\ &+ C_5, \end{aligned} \tag{21}$$

where $C_5 = \frac{1}{p}\int\limits_\Omega \mathbf{u}_{10}^p + C_3 \frac{1}{p+1}$. According to Lemma (2), there exists $C_p > 0$ such that

$$\int\limits_{s_0}^{t} \int\limits_\Omega e^{(p+1)s} |\Delta \mathbf{v}|^{p+1} \le C_p \int\limits_{s_0}^{t} \int\limits_\Omega e^{(p+1)s} |\alpha \mathbf{u}_1^l + \beta \mathbf{u}_2^l|^{p+1} + C_p \int\limits_\Omega \mathbf{v}_0{}^{p+1} + C_p \int\limits_\Omega |\Delta \mathbf{v}_0|^{p+1}.$$

Thanks to the inequality $(a+b)^d \le 2^d(a^d+b^d)$ with $a, b \ge 0$ and $d \ge 1$, we have

$$\begin{aligned}\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}|\Delta \mathbf{v}|^{p+1} &\le C_p \int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}2^{p+1}|(\alpha \mathbf{u}_1^l)^{p+1} + (\beta \mathbf{u}_2^l)^{p+1}| + C_p\|\mathbf{v}_0\|^{p+1}_{\mathbf{W^{2,p+1}}},\\ &\le C_p \int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}2^{p+1}\left(\alpha^{p+1}\mathbf{u}_1^{l(p+1)} + \beta^{p+1}\mathbf{u}_2^{l(p+1)}\right) + C_p\|\mathbf{v}_0\|^{p+1}_{\mathbf{W^{2,p+1}}},\end{aligned} \tag{22}$$

Applying Young's inequality with $0 < l < 1$, we get

$$\begin{aligned}C_p\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}2^{p+1}\alpha^{p+1}\mathbf{u}_1^{l(p+1)} &\le C_p\zeta_1\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\left(\mathbf{u}_1^{l(p+1)}\right)^{\frac{1}{l}} + C_pC(\zeta_1)\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\left(2^{p+1}\alpha^{p+1}\right)^{\frac{1}{1-l}}\\ &\le C_p\zeta_1\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\mathbf{u}_1^{p+1} + C_pC(\zeta_1)\left(2^{p+1}\alpha^{p+1}\right)^{\frac{1}{1-l}}|\Omega|\int_{s_0}^{t} e^{(p+1)s}\mathrm{d}s\end{aligned} \tag{23}$$

where $C(\zeta_1) = \left(\zeta_1\frac{1}{l}\right)^{\frac{-l}{1-l}}\left(\frac{1}{1-l}\right)^{-1}$. Similarly, we obtain

$$\begin{aligned}C_p\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}2^{p+1}\beta^{p+1}\mathbf{u}_2^{l(p+1)} &\le C_p\zeta_2\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\left(\mathbf{u}_2^{l(p+1)}\right)^{\frac{1}{l}} + C_pC(\zeta_2)\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\left(2^{p+1}\beta^{p+1}\right)^{\frac{1}{1-l}}\\ &\le C_p\zeta_2\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\mathbf{u}_2^{p+1} + C_pC(\zeta_2)\left(2^{p+1}\beta^{p+1}\right)^{\frac{1}{1-l}}|\Omega|\int_{s_0}^{t} e^{(p+1)s}\mathrm{d}s\end{aligned} \tag{24}$$

with $C(\zeta_2) = \left(\zeta_2\frac{1}{l}\right)^{\frac{-l}{1-l}}\left(\frac{1}{1-l}\right)^{-1}$. Now we rewrite the inequality (22) using (23) and (24) as

$$\begin{aligned}C_2\kappa^{-p}\chi^{p+1}e^{-(p+1)t}\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}|\Delta\mathbf{v}|^{p+1} &\le C_2\kappa^{-p}\chi^{p+1}e^{-(p+1)t}C_p\zeta_1\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\mathbf{u}_1^{p+1} + C_2\kappa^{-p}\chi^{p+1}e^{-(p+1)t}C_p\zeta_2\int_{s_0}^{t}\int_{\Omega} e^{(p+1)s}\mathbf{u}_2^{p+1}\\ &\quad + C_6 + C_7 + C_8,\end{aligned} \tag{25}$$

with $C_6 = C_2\kappa^{-p}\chi^{p+1}C_pC(\zeta_1)\left(2^{p+1}\alpha^{p+1}\right)^{\frac{1}{1-l}}|\Omega|\frac{1}{p+1}$, $C_7 = C_2\kappa^{-p}\chi^{p+1}C_pC(\zeta_2)\left(2^{p+1}\beta^{p+1}\right)^{\frac{1}{1-l}}|\Omega|\frac{1}{p+1}$ and $C_8 = C_pC_2\kappa^{-p}\chi^{p+1}e^{-(p+1)t}\|\mathbf{v}_0\|^{p+1}_{\mathbf{W^{2,p+1}}}$. Substituting the last inequality (25) in to (21), we get

$$\begin{aligned}\frac{1}{p}\int_{\Omega}\mathbf{u}_1^p \le &-\left(\frac{\sigma_1}{2}-\kappa-\eta\right)\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_1^{p+1} + C_2C_p\zeta_1\kappa^{-p}\chi^{p+1}\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_1^{p+1}\\ &+ C_2C_p\zeta_2\kappa^{-p}\chi^{p+1}\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_2^{p+1} + C_4a_1^{p+1}\sigma_1\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_2^{p+1} + C_9,\end{aligned} \tag{26}$$

Combining the terms in the last inequality, we obtain

$$\begin{aligned}\frac{1}{p}\int_{\Omega}\mathbf{u}_1^p \le &-\left(\frac{\sigma_1}{2}-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\chi^{p+1}\right)\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_1^{p+1} + \left(C_2C_p\zeta_2\kappa^{-p}\chi^{p+1} + C_4a_1^{p+1}\sigma_1\right)\int_{s_0}^{t}\int_{\Omega} e^{-(p+1)(t-s)}\mathbf{u}_2^{p+1}\\ &+ C_9.\end{aligned} \tag{27}$$

Similarly, we estimate for $\mathbf{u}_2$ as

$$\frac{1}{p}\int_\Omega \mathbf{u}_2^p \le -\left(\frac{\sigma_2}{2}-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\xi^{p+1}\right)\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}\mathbf{u}_2^{p+1}+\left(C_2C_p\zeta_2\kappa^{-p}\xi^{p+1}+C_4a_2^{p+1}\sigma_2\right)\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}\mathbf{u}_1^{p+1} + C_{10}, \tag{28}$$

where the constant $C_{10} > 0$. Adding the inequalities (27) and (28), we obtain

$$\begin{aligned}\frac{1}{p}\left(\int_\Omega \mathbf{u}_1^p+\int_\Omega \mathbf{u}_2^p\right) \le & -\left(\frac{\sigma_1}{2}-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\chi^{p+1}-C_2C_p\zeta_2\kappa^{-p}\xi^{p+1}-C_4a_2^{p+1}\sigma_2\right)\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}\mathbf{u}_1^{p+1}\\ &-\left(\frac{\sigma_2}{2}-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\xi^{p+1}-C_2C_p\zeta_2\kappa^{-p}\chi^{p+1}-C_4a_1^{p+1}\sigma_1\right)\int_{s_0}^t\int_\Omega e^{-(p+1)(t-s)}\mathbf{u}_2^{p+1}\\ &+C_{11},\end{aligned} \tag{29}$$

where $C_{11} > 0$. Let

$$\delta = \max\left\{\kappa + C_2C_p\zeta_1\kappa^{-p}\chi^{p+1}+C_2C_p\zeta_2\kappa^{-p}\xi^{p+1},\quad \kappa + C_2C_p\zeta_1\kappa^{-p}\xi^{p+1}+C_2C_p\zeta_2\kappa^{-p}\chi^{p+1}\right\},$$

we may choose

$$\eta = \left(0,\, \min\left\{\left(\frac{\sigma_1}{2}-C_4a_2^{p+1}\sigma_2\right),\left(\frac{\sigma_2}{2}-C_4a_1^{p+1}\sigma_1\right)\right\}-\delta\right),$$

such that

$$\left(\left(\frac{\sigma_1}{2}-C_4a_2^{p+1}\sigma_2\right)-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\chi^{p+1}-C_2C_p\zeta_2\kappa^{-p}\xi^{p+1}\right)>0,$$

and

$$\left(\left(\frac{\sigma_2}{2}-C_4a_1^{p+1}\sigma_1\right)-\kappa-\eta-C_2C_p\zeta_1\kappa^{-p}\xi^{p+1}-C_2C_p\zeta_2\kappa^{-p}\chi^{p+1}\right)>0.$$

Hence, we deduce from (29), that

$$\frac{1}{p}\left(\int_\Omega \mathbf{u}_1^p+\int_\Omega \mathbf{u}_2^p\right)\le C_{11}, \qquad \forall\, t\in(s_0, T_{\max}), \tag{30}$$

where the constant $C_{11} = C(\mathbf{u}_{10}, \mathbf{u}_{20}, \mathbf{v}_0, \eta, \kappa, \zeta_1, \zeta_2, p, |\Omega|) > 0$. □

Lemma 4. Let the assumptions of Lemma 3 hold. Suppose that the initial data $\mathbf{u}_{10}$, $\mathbf{u}_{20}$ and $\mathbf{v}_0$ satisfy (4) and the parameters μ_1, μ_2, μ_3, χ, ξ, σ_1, σ_2, a_1, a_2, α, β and γ are positive constants. Let $q > \max\{2, n\}$, $p > 1$ and $0 < l < 1$. If

$$\sup_{t\in(s_0,T_{\max})}\left(\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L^p}(\Omega)}+\|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L^p}(\Omega)}\right)<\infty, \tag{31}$$

for some $p > \frac{n}{2}$, then we have

$$\sup_{t\in(s_0,T_{\max})}\left(\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L^\infty}(\Omega)}+\|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L^\infty}(\Omega)}+\|\mathbf{v}(\cdot,t)\|_{\mathbf{W^{1,q}}(\Omega)}\right)<\infty. \tag{32}$$

Proof. Let $q > \max\{2, n\}$ and for each fixed $p > \frac{n}{2}$ there holds

$$\frac{np}{(n-p)_+} = \begin{cases}\infty, & \text{if } \ p\ge n,\\ \frac{np}{n-p} > n, & \text{if } \ \frac{n}{2}<p<n,\end{cases} \tag{33}$$

and choose $q < \frac{np}{(n-p)_+}$ and $1 < r < q$ fulfilling $n < r < \frac{np}{(n-p)_+}$ which enables to choose $\rho > 1$, $n < \rho r < \frac{np}{(n-p)_+}$ and $\rho r \le q$. We fix arbitrary $t \in (s_0, T_{\max})$. Applying the variation of constants formula to the third equation of (2), we get

$$\mathbf{v}(\cdot,t) = e^{-\gamma(t-s_0)}e^{\mu_3(t-s_0)\Delta}\mathbf{v}_0+\int_{s_0}^t e^{-\gamma(t-s)}e^{\mu_3(t-s)\Delta}\Big(\alpha f\left(\mathbf{u}_1(\cdot,s)\right)+\beta f\left(\mathbf{u}_2(\cdot,s)\right)\Big)\mathrm{d}s.$$

Now,

$$\|\nabla \mathbf{v}(\cdot,t)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})} \leq e^{-\gamma(t-s_0)}\|\nabla e^{\mu_3(t-s_0)\Delta}\mathbf{v}_0\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})} + \int_{s_0}^{t} e^{-\gamma(t-s)}\left\|\nabla e^{\mu_3(t-s)\Delta}\Big(\alpha f\big(\mathbf{u}_1(\cdot,s)\big)+\beta f\big(\mathbf{u}_2(\cdot,s)\big)\Big)\right\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})}\mathrm{d}s.$$

By using the estimates for the Neumann heat semigroup (Winkler[36]) and $\rho r \leq q$, we obtain

$$\begin{aligned}\|\nabla \mathbf{v}(\cdot,t)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})} &\leq C_1 e^{-\gamma(t-s_0)}\|\mathbf{v}_0\|_{\mathbf{W}^{1,q}(\Omega)} \\ &+ C_2\int_{s_0}^{t} e^{-\gamma(t-s)}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{\rho r}\right)}\right)e^{-\lambda(t-s)}\left\|\alpha f\big(\mathbf{u}_1(\cdot,s)\big)+\beta f\big(\mathbf{u}_2(\cdot,s)\big)\right\|_{\mathbf{L}^{p}(\Omega)}\mathrm{d}s, \qquad (34)\end{aligned}$$

where C_1 and C_2 are positive constants. Because of our assumption $\rho r < \frac{np}{(n-p)}$ and $\rho r \leq q$, we can ensure that $\frac{1}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{\rho r}\right) < 1$,

$$\int_0^{\infty} x^{-n}e^{-\lambda x} = \lambda^{n-1}\,\Gamma(1-n), \qquad \text{for} \quad Re(n)<1, Re(\lambda)>0,$$

thus using Gamma function, we obtain

$$\int_0^{\infty} e^{-\gamma\zeta}\left(1+\zeta^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{\rho r}\right)}\right)e^{-\lambda\zeta} < \infty.$$

Since $0<l<1$, we use Young's inequality to get

$$\int_{\Omega} f(s)^p \leq \epsilon\|s\|^p_{\mathbf{L}^{p}(\Omega)} + C(\epsilon)K^{\frac{p}{1-l}}|\Omega|, \qquad (35)$$

where $c(\epsilon) = \left(\frac{\epsilon}{l}\right)^{\frac{-l}{1-l}}\left(\frac{1}{1-l}\right)^{-1}$. Substituting (35) in to (34), we conclude that

$$\|\nabla \mathbf{v}(\cdot,t)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})} \leq C_1\|\mathbf{v}_0\|_{\mathbf{W}^{1,q}(\Omega)} + C_3\left(\sup_{t\in(s_0,T_{\max})}\Big(\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^{p}(\Omega)}+\|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^{p}(\Omega)}\Big)\right)+C_4.$$

Where C_3 and C_4 are positive constants. Finally, we obtain

$$\|\nabla \mathbf{v}(\cdot,t)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})} \leq C_5, \qquad \forall\, t\in(s_0,T_{\max}). \qquad (36)$$

where $C_5>0$. Let $t_0=\max\{s_0,\ t-1\}$, and using variation of constants formula to the first equation of (2), we get

$$\mathbf{u}_1(\cdot,t) = e^{\mu_1(t-t_0)\Delta}\mathbf{u}_1(\cdot,t_0) + \chi\int_{t_0}^{t} e^{\mu_1(t-s)\Delta}\nabla\cdot\Big(\mathbf{u}_1(\cdot,s)\nabla\mathbf{v}(\cdot,s)\Big)\mathrm{d}s + \sigma_1\int_{t_0}^{t} e^{\mu_1(t-s)\Delta}\mathbf{u}_1(\cdot,s)\Big(1-\mathbf{u}_1(\cdot,s)-a_1\mathbf{u}_2(\cdot,s)\Big)\mathrm{d}s. \qquad (37)$$

Next, taking $\mathbf{L}^{\infty}(\Omega)$ on both sides of (37), we obtain

$$\begin{aligned}\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^{\infty}(\Omega)} &\leq \left\|e^{\mu_1(t-t_0)\Delta}\mathbf{u}_1(\cdot,t_0)\right\|_{\mathbf{L}^{\infty}(\Omega)} + \chi\int_{t_0}^{t}\left\|e^{\mu_1(t-s)\Delta}\nabla\cdot\Big(\mathbf{u}_1(\cdot,s)\nabla\mathbf{v}(\cdot,s)\Big)\right\|_{\mathbf{L}^{\infty}(\Omega)} \\ &+ \sigma_1\int_{t_0}^{t}\left\|e^{\mu_1(t-s)\Delta}\mathbf{u}_1(\cdot,s)\Big(1-\mathbf{u}_1(\cdot,s)-a_1\mathbf{u}_2(\cdot,s)\Big)\right\|_{\mathbf{L}^{\infty}(\Omega)}\mathrm{d}s\end{aligned}$$

for all $t\in(s_0,T_{\max})$. If $t\leq 1$, then $t_0=s_0$, we can use the maximum principle,

$$\|e^{\mu_1(t-t_0)\Delta}\mathbf{u}_1(\cdot,t_0)\|_{\mathbf{L}^{\infty}(\Omega)} = \|e^{\mu_1(t-s_0)\Delta}\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^{\infty}(\Omega)} \leq \|\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^{\infty}(\Omega)}.$$

If $t>1$, using Neumann heat semigroup (Winkler[36]) property, with $C>0$

$$\|e^{\mu_1(t-t_0)\Delta}\mathbf{u}_1(\cdot,t_0)\|_{\mathbf{L}^{\infty}(\Omega)} \leq C(t-t_0)^{-\frac{n}{2}}\|\mathbf{u}_1(\cdot,t_0)\|_{\mathbf{L}^{1}(\Omega)} \leq Cm_1,$$

because $t-t_0=1$. In view of the estimates for the Neumann heat semigroup (Winkler[36]), $C_6>0$ satisfying

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \le \max\{\|\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^\infty(\Omega)}, Cm_1\} + \chi C_6 \int_{t_0}^{t} \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{\infty}\right)}\right) e^{-\lambda(t-s)} \left\|\mathbf{u}_1(\cdot,s)\nabla\mathbf{v}(\cdot,s)\right\|_{\mathbf{L}^r(\boldsymbol{\Omega})} \mathrm{d}s$$

$$+\sigma_1 \int_{t_0}^{t} \left\|e^{\mu_1(t-s)\Delta}\mathbf{u}_1(\cdot,s)\Big(1-\mathbf{u}_1(\cdot,s)-a_1\mathbf{u}_2(\cdot,s)\Big)\right\|_{\mathbf{L}^\infty(\Omega)} \mathrm{d}s. \tag{38}$$

Using the Cauchy's inequality with ϵ, we have

$$\begin{aligned}\sigma_1\mathbf{u}_1(1-\mathbf{u}_1-a_1\mathbf{u}_2) &\le \sigma_1\mathbf{u}_1-\sigma_1\mathbf{u}_1^2,\\ &\le \sigma_1\mathbf{u}_1^2+\frac{1}{4\sigma_1}\sigma_1^2-\sigma_1\mathbf{u}_1^2,\\ &\le \sigma_1\mathbf{u}_1^2+\frac{\sigma_1}{4}-\sigma_1\mathbf{u}_1^2,\\ &\le \frac{\sigma_1}{4}.\end{aligned}$$

Due to $t-t_0\le 1$ and the maximum principle, the last term in (38) can be written as

$$\sigma_1\int_{t_0}^{t}\left\|e^{\mu_1(t-s)\Delta}\mathbf{u}_1(\cdot,s)\Big(1-\mathbf{u}_1(\cdot,s)-a_1\mathbf{u}_2(\cdot,s)\Big)\right\|_{\mathbf{L}^\infty(\Omega)}\mathrm{d}s \le \sigma_1\int_{t_0}^{t}\left\|\mathbf{u}_1(\cdot,s)\Big(1-\mathbf{u}_1(\cdot,s)-a_1\mathbf{u}_2(\cdot,s)\Big)\right\|_{\mathbf{L}^\infty(\Omega)}\mathrm{d}s \le \frac{\sigma_1}{4}. \tag{39}$$

Therefore, inserting (39) in (38), we gte

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \le \max\left\{\|\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^\infty(\Omega)}, Cm_1\right\} + \chi C_6\int_{t_0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\right)e^{-\lambda(t-s)}\left\|\mathbf{u}_1(\cdot,s)\nabla\mathbf{v}(\cdot,s)\right\|_{\mathbf{L}^r(\boldsymbol{\Omega})}\mathrm{d}s+\frac{\sigma_1}{4}, \tag{40}$$

for all $t\in(s_0,T_{\max})$. Here by using the Holder inequality and Interpolation inequality and (8) and (36), we obtain

$$\begin{aligned}\|\mathbf{u}_1(\cdot,s)\nabla\mathbf{v}(\cdot,s)\|_{\mathbf{L}^r(\boldsymbol{\Omega})} &\le \|\mathbf{u}_1(\cdot,s)\|_{\mathbf{L}^{\hat{\rho}r}(\boldsymbol{\Omega})}\ \|\nabla\mathbf{v}(\cdot,s)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})},\\ &\le \|\mathbf{u}_1(\cdot,s)\|^k_{\mathbf{L}^\infty(\Omega)}\ \|\mathbf{u}_1(\cdot,s)\|^{1-k}_{\mathbf{L}^1(\Omega)}\ \|\nabla\mathbf{v}(\cdot,s)\|_{\mathbf{L}^{\rho r}(\boldsymbol{\Omega})},\\ &\le C_7\|\mathbf{u}_1(\cdot,s)\|^k_{\mathbf{L}^\infty(\Omega)},\end{aligned}$$

where $\hat{\rho}$ is the dual exponent of ρ and $k=1-\dfrac{1}{\hat{\rho}r}\in(0,1)$, $\forall s\in(t_0,t)$ and $C_7>0$. Inserting the last inequality in (40), it follows that

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \le \max\left\{\|\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^\infty(\Omega)}, Cm_1\right\} + C_6C_7\chi\int_{0}^{\infty}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2r}}\right)e^{-\lambda(t-s)}\|\mathbf{u}_1(\cdot,t)\|^k_{\mathbf{L}^\infty(\Omega)}\mathrm{d}s+\frac{\sigma_1}{4},$$

where $\frac{1}{2}+\frac{n}{2r}<1$ because of $r>n$. This gives

$$\int_0^\infty\left(1+\zeta^{-\frac{1}{2}-\frac{n}{2r}}\right)e^{-\lambda\zeta}<\infty.$$

Thus we obtain

$$\sup_{t\in(s_0,T)}\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \le \max\left\{\|\mathbf{u}_1(\cdot,s_0)\|_{\mathbf{L}^\infty(\Omega)}, Cm_1\right\} + C_8\sup_{t\in(s_0,T)}\|\mathbf{u}_1(\cdot,t)\|^k_{\mathbf{L}^\infty(\Omega)}+\frac{\sigma_1}{4}, \tag{41}$$

for all $T\in(s_0,T_{\max})$. Now we define, $M(T):=\sup_{t\in(s_0,T)}\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)}$. The inequality (41) can be rewritten as

$$M(T)\le C_9+C_8M^k(T),\qquad \forall T\in(s_0,T_{\max}). \tag{42}$$

Using Young's inequality with ϵ,

$$\begin{aligned} C_8 M(T)^k &\leq \epsilon \left(M^k(T)\right)^{\frac{1}{k}} + C(\epsilon)\left(C_8\right)^{\frac{1}{1-k}}, \\ &\leq \frac{1}{2}M(T) + C_{10}, \end{aligned}$$

where $C(\epsilon) = \left(\epsilon\frac{1}{k}\right)^{\frac{-k}{1-k}}\left(\frac{1}{1-k}\right)^{-1}$. Thus

$$M(T) \leq C_9 + \frac{1}{2}M(T) + C_{10} \leq C_{11}, \qquad \forall T \in (s_0, T_{\max}),$$

where $C_{11} > 0$. Finally, we conclude that

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq C_{11}, \qquad \forall t \in (s_0, T_{\max}). \tag{43}$$

Similarly, if we apply the above procedure, for $\mathbf{u}_2$, we can obtain

$$\mathbf{u}_2(\cdot,t) = e^{\mu_2(t-s)\Delta}\mathbf{u}_2(\cdot,t_0) - \xi\int_{t_0}^{t} e^{\mu_2(t-s)\Delta}\nabla\cdot\Big(\mathbf{u}_2(\cdot,s)\nabla\mathbf{v}(\cdot,s)\Big)\mathrm{d}s + \sigma_2\int_{t_0}^{t} e^{\mu_2(t-s)\Delta}\mathbf{u}_2(\cdot,s)\Big(1 + a_2\mathbf{u}_1(\cdot,s) - \mathbf{u}_2(\cdot,s)\Big)\mathrm{d}s,$$

Next,

$$\begin{aligned} \|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq \|e^{\mu_2(t-t_0)\Delta}\mathbf{u}_2(\cdot,t_0)\|_{\mathbf{L}^\infty(\Omega)} &+ \xi\int_{t_0}^{t}\left\|e^{\mu_2(t-s)\Delta}\nabla\cdot\Big(\mathbf{u}_2(\cdot,s)\nabla\mathbf{v}(\cdot,s)\Big)\right\|_{\mathbf{L}^\infty(\Omega)}\mathrm{d}s \\ &+ \sigma_2\int_{t_0}^{t}\left\|e^{\mu_2(t-s)\Delta}\mathbf{u}_2(\cdot,s)\Big(1 + a_2\mathbf{u}_1(\cdot,s) - \mathbf{u}_2(\cdot,s)\Big)\right\|_{\mathbf{L}^\infty(\Omega)}\mathrm{d}s. \end{aligned} \tag{44}$$

A use of Cauchy's inequality gives us that

$$\begin{aligned} \mathbf{u}_2(1 + a_2\mathbf{u}_1 - \mathbf{u}_2) &\leq \mathbf{u}_2(1 + a_2\mathbf{u}_1) - \mathbf{u}_2^2, \\ &\leq \mathbf{u}_2^2 + \frac{1}{4}(1 + a_2\mathbf{u}_1)^2 - \mathbf{u}_2^2, \\ &\leq \frac{(1 + a_2\mathbf{u}_1)^2}{4}. \end{aligned} \tag{45}$$

Therefore, substituting (45) in to (44), we get

$$\|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq \max\left\{\|\mathbf{u}_2(\cdot,s_0)\|_{\mathbf{L}^\infty(\Omega)}, Cm_2\right\} + \xi C_{12}\int_{t_0}^{t}\left(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2r}}\right)e^{-\lambda(t-s)}\left\|\mathbf{u}_2(\cdot,s)\nabla\mathbf{v}(\cdot,s)\right\|_{\mathbf{L}^r}\mathrm{d}s + \frac{\sigma_2(1 + a_2C_{11})^2}{4}. \tag{46}$$

Again, we deduce that

$$\|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq C_{13}, \qquad \forall t \in (s_0, T_{\max}), \tag{47}$$

where $C_{13} > 0$. □

Proof of the Theorem (1). Lemmas (3) and (4) imply that (43) and (47) and holds for $t \in (s_0, T_{\max})$. Using (13), we can conclude that

$$\|\mathbf{u}_1(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq C_{14}, \qquad \|\mathbf{u}_2(\cdot,t)\|_{\mathbf{L}^\infty(\Omega)} \leq C_{15} \qquad \text{and} \qquad \|\mathbf{v}(\cdot,t)\|_{\mathbf{W}^{1,q}(\Omega)} \leq C_{16}, \qquad \forall t \in (0, T_{\max}). \tag{48}$$

This completes the proof of Theorem (1).

4 | CONCLUSIONS

In this paper, we proved the local and global existence of classical solution to the two species predator-prey chemotaxis system (2).

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Conflict of interest

The authors declare no potential conflict of interests.

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