A numerical approach based on n-dimensional fractional Müntz-Legendre polynomials for solving fractional-order cohomological equations with variable coefficients

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Abstract

This research presents a numerical approach to obtain the approximate solution of the n-dimensional cohomological equations of fractional order in continuous-time dynamical systems. For this purpose, the n-dimensional fractional Müntz-Legendre polynomials (or n-DFMLPs) are introduced. The operational matrix of the fractional Riemann-Liouville derivative is constructed by employing n-DFMLPs. Our method transforms the cohomological equation of fractional order into a system of algebraic equations. Therefore, the solution of that system of algebraic equations is the solution of the associated cohomological equation. The error bound and convergence analysis of the applied method under the L^2-norm is discussed. Some examples are considered and discussed to confirm the efficiency and accuracy of our method.

Keywords. fractional-order cohomological equations, Riemann-Liouville fractional derivative, n-dimensional fractional Müntz-Legendre polynomials, fractional derivative operational matrix, Product operational matrix

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1 Introduction

In the process of solving some problems related to dynamical systems, we are faced with solving equations called cohomological equations. Some cohomological equations are obtained by verifying the triviality of the time changes of flows. These equations in verifying the smoothness of invariant measures, conjugacy, and rigidity of group actions are applicable. The other problems in differential dynamical systems that reduce to cohomological equations are Livšic theory [1, 2, 3], KAM theory [4], the existence of invariant volume forms [3], etc. Recently, researches have been done on cohomological equations [5, 6]. Despite a long 300-year history of fractional calculus, its applications in various fields have been considered in recent decades. For example, in biology, membranes of an organism cell possess fractional electrical conductivity [7]. Since the degree of freedom of fractional derivative is higher than integer derivative, then the use of fractional calculus leads to a more realistic model of some processes. For this reason, we are interested in modelling n-dimensional fractional cohomological equations (or n-DFCEs). Already, n-DFCEs modelled by Liouville fractional partial derivative [8]. But here, we prefer to use the Riemann-Liouville (or R-L) fractional partial derivatives of order $0 < \alpha < 1$.

In this research, we intend to solve the n-DFCEs by employing the operational matrix of R-L fractional derivative of n-dimensional fractional Müntz-Legendre polynomials. Most of the research that has been done so far focused on the existence of solution of cohomological equations, but we want to apply numerical analysis tools for solving n-DFCEs.

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One-dimensional and two-dimensional fractional Müntz-Legendre polynomials have already been introduced [9]. We have defined and applied n-DFMLPs to approximate the solutions of n-DFCEs. The numerical solutions of n-DFCEs are discussed for the first time in this work. Since all examples are presented for the first time, we have not comparison with other studies. Our goal is to obtain the approximate solution of $Xf = g$ such that $X = \sum_{i=1}^{n} a_i(x_1,\cdots,x_n)\frac{\partial^\alpha}{\partial x_i^\alpha}$, f and g are, respectively, the linear operator of fractional order $0 < \alpha < 1$, unknown and known functions. The coefficients of X, f and g are smooth functions.

In section 2, definitions of Riemann-Liouville fractional derivative and 1-DFMLPs are given. In section 3, orthogonal n-DFMLPs are introduced. In section 4, Riemann-Liouville fractional partial derivative operational matrix based on n-DFMLPs and product operational matrix are obtained. In section 5, the method of solution of n-DFCEs is explained. In section 6, the convergence of the proposed method and error bounds are investigated. In section 7, some examples are presented to prove the accuracy and efficiency of our method.

2 Preliminary concepts

In this section, the required concepts are defined. In subsection 2.1, we provide the definitions of the R-L fractional derivative of one-variable functions and the R-L fractional partial derivatives of multi-variable functions. In subsection 2.2, definitions of one-dimensional fractional Müntz-Legendre polynomials are given.

2.1 Fractional calculus

Today, fractional calculus has many applications in various fields of science and engineering. By using fractional differential equations, many phenomena are modelled. Fractional differential equations are solved both analytically and numerically. Many books have been written on fractional calculus, that we will refer the reader to some of the books [10, 11, 12].

Definition 2.1. *[10] The Riemann-Liouville fractional derivative of the function $f : J \subseteq (-\infty,\infty) \longrightarrow \mathbb{R}$ of order $k-1 < \alpha < k$ is defined as follows*

$$ {}_cD_t^\alpha = \frac{1}{\Gamma(k-\alpha)}\left(\frac{d}{dt}\right)^k \int_c^t \frac{f(s)ds}{(t-s)^{\alpha-k+1}}\cdot \tag{1}$$

For convenience, we put $c = 0$. Let $f : \Omega := \underbrace{[0,1]\times\cdots\times[0,1]}_{n-times} \longrightarrow \mathbb{R}$ be a function such that $n \in \mathbb{N}$, $n \geq 2$. The R-L fractional partial derivative of f of order $k-1 < \alpha < k$ with respect to x_i, $1 \leq i \leq n$, is defined as follows

$$\frac{\partial^\alpha f}{\partial x_i^\alpha} = \frac{1}{\Gamma(k-\alpha)}\left(\frac{\partial}{\partial x_i}\right)^k \int_0^{x_i} \frac{f(x_1,\cdots,x_{i-1},s,x_{i+1},\cdots,x_n)ds}{(x_i-s)^{\alpha-k+1}}\cdot \tag{2}$$

The R-L fractional derivative of the power function $f(t) = (t-c)^\lambda$ of order $0 < \alpha \in \mathbb{R}$ is

$${}_0D_t^\alpha((t-c)^\lambda) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\lambda)}(t-c)^{\lambda-\alpha}, \quad \lambda > -1\cdot \tag{3}$$

2.2 One-dimensional fractional Müntz-Legendre polynomials

Fractional Müntz-Legendre polynomials $L_i(s,\gamma)$ on the $[0,1]$ are given as follows

$$L_i(s,\gamma) = \sum_{l=0}^{i} E_{i,l}s^{l\gamma}, \quad \gamma \in (0,+\infty) := \mathbb{R}^+,$$

such that $E_{i,l} = \frac{(-1)^{i-l}}{\gamma^i l!(i-l)!}\prod_{j=0}^{i-1}((l+j)\gamma+1)$. Fractional polynomials $\{L_0(s,\gamma),\cdots,L_k(s,\gamma)\}$ form an orthogonal basis for $M_{k,\gamma} = span\{1,s^\gamma,\cdots,s^{k\gamma}\}$, where $2 \leq k \in \mathbb{N}$. The analytical form of $L_i(s,\gamma)$ is

$$L_i(s,\gamma) = \sum_{l=0}^{i} a_{li}s^{l\gamma},$$

where

$$a_{li} = \frac{(-1)^{i-l}\Gamma(\frac{1}{\alpha}+l+i)}{l!(i-l)!\Gamma(\frac{1}{\alpha}+l)}\cdot \tag{4}$$

If $P_i^{(\gamma,\beta)}$ be Jacobi polynomial with parameters $\gamma, \beta > -1$, then

$$L_i(s,\gamma) = P_i^{0,\frac{1}{\gamma}-1}(2s^\gamma - 1), \quad \gamma > 0\cdot$$

For more information, see [13, 14]. The orthogonality of 1-DFMLPs on $[0,1]$ is derived by

$$\int_0^1 L_i(s,\gamma)L_j(s,\gamma)ds = \frac{1}{2i\gamma+1}\delta_{ij},$$

here δ_{ij} is the Kronecker function. If $f(s)$ be an integrable function in $[0,1]$, then $f(s)$ is expanded as follows

$$f(s) = \sum_{i=0}^{\infty} c_i L_i(s,\gamma), \tag{5}$$

where for every $i = 0,1,\cdots\infty$,

$$c_i = (2i\gamma+1)\int_0^1 f(s)L_i(s,\gamma)ds\cdot \tag{6}$$

We can use the finite series of (6) to approximate $f(s)$ as follows

$$f(s) \simeq f_k(s) = \sum_{i=0}^{k} c_i L_i(s,\gamma)\cdot \tag{7}$$

If $C = [c_0, c_1, \cdots, c_k]^T$ and $\Phi(s;\gamma) = [L_0(s,\gamma), L_1(s,\gamma), \cdots, L_k(s,\gamma)]^T$, then matrix form of (7) is

$$f(s) \simeq C^T\Phi(s;\gamma) = \Phi(s;\gamma)^T C\cdot \tag{8}$$

3 n-dimensional fractional Müntz-Legendre polynomials

We intend to solve n-dimensional fractional cohomological equations. Therefore, we will need to define n-dimensional fractional Muñtz-Legendre polynomials. In this section, n-dimensional fractional Müntz-Legendre polynomials on $\Omega = \underbrace{[0,1]\times\cdots\times[0,1]}_{n-times}$ will be introduced.

Definition 3.1. *The elements of $\{L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n) | i_1,\cdots,i_n = 0\cdots\infty\}$ are called n-dimensional fractional Müntz-Legendre polynomials (or n-DFMLPs) on Ω, where $0 < \alpha_1,\cdots,\alpha_n < \infty$.*

The vector space spanned by n-DFMLPs $\{L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n) | i_1,\cdots,i_n = 0\cdots k\}$ denoted by $\Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$.

Theorem 3.1. *n-DFMLPs are orthogonal on Ω.*

Proof. Let $L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n)$, $L_{j_1}(x_1,\alpha_1)\cdots L_{j_n}(x_n,\alpha_n) \in \Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$, then

$$\int_0^1\cdots\int_0^1 \Big(L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n)\Big)\Big(L_{j_1}(x_1,\alpha_1)\cdots L_{j_n}(x_n,\alpha_n)\Big)dx_1\cdots dx_n =$$
$$\Big(\int_0^1 L_{i_1}(x_1,\alpha_1)L_{j_1}(x_1,\alpha_1)dx_1\Big)\cdots\Big(\int_0^1 L_{i_n}(x_n,\alpha_n)L_{j_n}(x_n,\alpha_n)dx_n\Big) =$$
$$\begin{cases} \frac{1}{2i_1\alpha_1+1}\cdots\frac{1}{2i_n\alpha_n+1}, & \text{if } i_1 = j_1,\cdots,i_n = j_n \\ 0 & \text{otherwise}\cdot \end{cases}$$

□

According to the orthogonality of polynomials in $\Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$, every $f \in L^2(\Omega)$ is expanded as follows

$$f(x_1,\cdots,x_n) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} c_{i_1\cdots i_n} L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n),$$

such that

$$c_{i_1\cdots i_n} = (2i_1\alpha_1+1)\cdots(2i_n\alpha_n+1)\int_0^1\cdots\int_0^1 f(x_1,\cdots,x_n)L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n)dx_1\cdots dx_n. \tag{9}$$

Let

$$\begin{aligned}
\Phi(x_1;\alpha_1) &= [L_0(x_1,\alpha_1),L_1(x_1,\alpha_1),\cdots,L_{k_1}(x_1,\alpha_1)]^T,\\
\Phi(x_2;\alpha_2) &= [L_0(x_2,\alpha_2),L_1(x_2,\alpha_2),\cdots,L_{k_2}(x_2,\alpha_2)]^T,\\
&\vdots\\
\Phi(x_n;\alpha_n) &= [L_0(x_n,\alpha_n),L_1(x_n,\alpha_n),\cdots,L_{k_n}(x_n,\alpha_n)]^T.
\end{aligned}$$

Kronecker product of Matrices $\Phi(x_1,\alpha_1),\cdots,\Phi(x_n,\alpha_n)$ is denoted by

$$\Phi(x;\alpha) = \Phi(x_1;\alpha_1)\otimes\Phi(x_2;\alpha_2)\otimes\cdots\otimes\Phi(x_n;\alpha_n), \tag{10}$$

where $x=(x_1,\cdots,x_n)\in\Omega$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{R}^{+^n}$ and for every $i=1\cdots n$, $k_i\geq 2$. Any function $f\in L^2(\Omega)$ can be approximated as follows

$$\begin{aligned}
f(x_1,\cdots,x_n) &\simeq f_{k_1,\cdots,k_n}(x_1,\cdots,x_n) = \sum_{i_1=0}^{k_1}\cdots\sum_{i_n=0}^{k_n} c_{i_1\cdots i_n}L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n)\\
&= C^T\Phi(x;\alpha) = \Phi(c;\alpha)^TC.
\end{aligned} \tag{11}$$

Let $k=k_1=k_2=\cdots=k_n$. If $1\leq i\leq (k+1)^n$ such that $i=(i_n+1)+i_{n-1}(k+1)+i_{n-2}(k+1)^2+\cdots+i_1(k+1)^{n-1}$, then c_i is $c_{i_1\cdots i_n}$.

4 Operational matrices

The operational matrix method is one of the powerful tools for solving numerically differential equations. In this section, the product operational matrix and R-L fractional derivative operational matrix based on n-DFMLPs are obtained. We will use these operational matrices to solve the fractional cohomological equations.

4.1 Product operational matrix

Let $x\in\Omega$ and $\alpha\in\Lambda$. For simplicity, i-th element of vector $\Phi(x;\alpha)$, $L_{i_1}(x_1,\alpha_1)\cdots L_{i_n}(x_n,\alpha_n)$, is denoted by Φ_i or $\Phi_{i_1i_2\cdots i_n}$ if $i=(i_n+1)+i_{n-1}(k+1)+i_{n-2}(k+1)^2+\cdots+i_1(k+1)^{n-1}$.

Sometimes the process to solve differential equations faced with multiplication $\Phi(x;\alpha)\Phi(x;\alpha)^TC$ where C is a coefficient vector of order $(k+1)^n$ in Equation (11).

Theorem 4.1. *Let $\Phi(x;\alpha)$ and C be vectors defined in Equations (10) and (11), then*

$$\Phi(x;\alpha)\Phi(x;\alpha)^TC = \tilde{C}\Phi(x;\alpha),$$

where $\tilde{C}$ is product operational matrix of order $(k+1)^n\times(k+1)^n$. The entries of $\tilde{C}$ are computed as follows

$$\tilde{C}_{lr} = (2\alpha_1r_1+1)(2\alpha_2r_2+1)\cdots(2\alpha_nr_n+1)(\sum_{i=1}^{(k+1)^n} c_i\int_0^1\cdots\int_0^1 \Phi_l\Phi_i\Phi_r dx_1\cdots dx_n),$$

where c_i is i-th entry of vector C and $r=(r_n+1)+r_{n-1}(k+1)+r_{n-2}(k+1)^2+\cdots+r_1(k+1)^{n-1}$.

Proof. The l-th entry of vectors $\Phi(x;\alpha)\Phi(x;\alpha)^T C$ and $\tilde{C}\Phi(x;\alpha)$, respectively, are

$$\Phi_l\Phi_1 c_1 + \Phi_l\Phi_2 c_2 + \cdots + \Phi_l\Phi_{(k+1)^n} c_{(k+1)^n}, \quad \tilde{C}_{l1}\Phi_1 + \tilde{C}_{l2}\Phi_2 + \cdots + \tilde{C}_{l(k+1)^n}\Phi_{(k+1)^n}\cdot$$

If the polynomials $\Phi_l\Phi_1, \cdots, \Phi_l\Phi_{(k+1)^n}$ can be expanded as follows

$$\Phi_l\Phi_i = \sum_{j_1=0}^{k}\cdots\sum_{j_n=0}^{k}((2\alpha_1 j_1+1)\cdots(2\alpha_n j_n+1)\int_0^1\cdots\int_0^1 \Phi_l\Phi_i\Phi_j dx_1 dx_2\cdots dx_n)\Phi_j,$$

then,

$$\begin{aligned}
&\tilde{C}_{l1}\Phi_1 + \tilde{C}_{l2}\Phi_2 + \cdots + \tilde{C}_{l(k+1)^n}\Phi_{(k+1)^n} = \\
&\left(\sum_{j_1=0}^{k}\cdots\sum_{j_n=0}^{k}((2\alpha_1 j_1+1)\cdots(2\alpha_n j_n+1)\int_0^1\cdots\int_0^1 \Phi_l\Phi_1\Phi_j dx_1\cdots dx_n)\Phi_j\right)c_1 + \\
&\left(\sum_{j_1=0}^{k}\cdots\sum_{j_n=0}^{k}((2\alpha_1 j_1+1)\cdots(2\alpha_n j_n+1)\int_0^1\cdots\int_0^1 \Phi_l\Phi_2\Phi_j dx_1\cdots dx_n)\Phi_j\right)c_2 + \cdots + \\
&\left(\sum_{j_1=0}^{k}\cdots\sum_{j_n=0}^{k}((2\alpha_1 j_1+1)\cdots(2\alpha_n j_n+1)\int_0^1\cdots\int_0^1 \Phi_l\Phi_{(k+1)^n}\Phi_j dx_1\cdots dx_n)\Phi_j\right)c_{(k+1)^n},
\end{aligned}$$

and

$$\tilde{C}_{lr}\Phi_r \quad = \quad \sum_{s=1}^{(k+1)^n}((2\alpha_1 r_1+1)\cdots(2\alpha_n r_n+1)\int_0^1\cdots\int_0^1 \Phi_l\Phi_s\Phi_r dx_1\cdots dx_n)\Phi_r c_s\cdot$$

Consequently,

$$\tilde{C}_{lr} = (2\alpha_1 r_1+1)\cdots(2\alpha_n r_n+1)\left[\sum_{s=1}^{(k+1)^n} c_s\int_0^1\cdots\int_0^1 \Phi_l\Phi_s\Phi_r dx_1\cdots dx_n\right]\cdot$$

□

4.2 Riemann-Liouville fractional derivative operational matrix of n-DFMLPs

For our goal in this work, R-L fractional partial derivative operational matrix based on n-DFMLPs will be obtained. Let $\Phi(s;\alpha)$ be a vector that its entries are 1-DFMLPs where $s\in[0,1]$, $\alpha\in\mathbb{R}^+$. Riemann-Liouville fractional derivative of order $0<\beta<1$ of $\Phi(s;\alpha)$ is as follows

$${}_0D_s^{\beta}\Phi(s;\alpha) = [{}_0D_s^{\beta}L_0(s,\alpha), \cdots, {}_0D_s^{\beta}L_k(s,\alpha)]\cdot$$

Theorem 4.2. *R-L fractional derivative of order $0<\beta<1$ of $\Phi(s;\alpha)$ can be calculated as*

$${}_0D_s^{\beta}\Phi(s;\alpha) = D^{\alpha,\beta}\Phi(s;\alpha),$$

such that $D^{\alpha,\beta}$ is a $(k+1)\times(k+1)$ matrix. The entries of $D^{\alpha,\beta}$ are

$$D_{j+1,l+1}^{\alpha,\beta} = (2l\alpha+1)\sum_{r=0}^{j}\sum_{t=0}^{l} a_{tl}a_{rj}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}\frac{1}{(r+t)\alpha-\beta},$$

where a_{rj} is defined in Equation (4).

Proof. For every $0\le j\le k$,

$${}_0D_s^{\beta}L_j(s,\alpha) = \sum_{r=0}^{j} a_{rj}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}s^{r\alpha-\beta}.$$

Let ${}_0D_s^{\beta}L_j(s,\alpha)=\sum\limits_{l=0}^{k}d_lL_l(s,\alpha)$ such that

$$\begin{aligned}
d_l &\simeq (2\alpha l+1)\int_0^1 {}_0D_s^{\beta}L_j(s,\alpha)L_l(s,\alpha)ds\\
&= (2\alpha l+1)\int_0^1\sum_{r=0}^{j}a_{rj}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}x_i^{r\alpha-\beta}\sum_{t=0}^{l}a_{tl}s^{t\alpha}ds\\
&= (2\alpha l+1)\sum_{r=0}^{i}\sum_{t=0}^{l}a_{rj}a_{tl}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}\int_0^1 s^{(r+t)\alpha-\beta}ds\\
&= (2\alpha l+1)\sum_{r=0}^{j}\sum_{t=0}^{l}a_{rj}a_{tl}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}\frac{1}{(r+t)\alpha-\beta+1}\cdot
\end{aligned}$$

Consequently,

$$D_{j+1,l+1}^{\alpha,\beta}=(2\alpha l+1)\sum_{r=0}^{j}\sum_{t=0}^{l}a_{rj}a_{tl}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-\beta+1)}\frac{1}{(r+t)\alpha-\beta+1}\cdot$$

□

The partial fractional derivative of $\Phi(x;\alpha)$ with respect to x_i, $1\le i\le n$, of order $\beta_i\ge 0$ is denoted by $\frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}\Phi(x;\alpha)$. Therefore the operational matrix of partial fractional derivative $\frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}\Phi(x;\alpha)$ is as follows

$$\frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}\Phi(x;\alpha)=(I_{k+1}\otimes\cdots\otimes I_{k+1}\otimes\underbrace{D^{\alpha_i,\beta_i}}_{i-th\ component}\otimes I_{k+1}\otimes\cdots\otimes I_{k+1})\Phi(x;\alpha),$$

where $\otimes$ is Kronecker product.

5 Method of solution of FCEs

We intend to use of obtained operational matrices from previous sections for numerically solving of n-DFCEs. Consider the following n-DFCE

$$\sum_{i=1}^{n}a_i(x_1,\cdots,x_n)\frac{\partial^{\beta_i}f}{\partial x_i^{\beta_i}}(x_1,\cdots,x_n)=g(x_1,\cdots,x_n)\cdot \tag{12}$$

Assume the unknown function $f(x_1,\cdots,x_n)$, functions $g(x_1,\cdots,x_n)$, $a_i(x_1,\cdots,x_n)$, and partial fractional derivatives of unknown function $\frac{\partial^{\beta_i}f}{\partial x_i^{\beta_i}}(x_1,\cdots,x_n)$ can be approximated as follows

$$\begin{aligned}
f(x_1,\cdots,x_n) &\simeq F^T\Phi(x;\alpha)=\Phi(x;\alpha)^TF, && (13)\\
a_i(x_1,\cdots,x_n) &\simeq A_i^T\Phi(x;\alpha)=\Phi(x;\alpha)^TA_i,\quad 1\le i\le n && (14)\\
g(x_1,\cdots,x_n) &\simeq G^T\Phi(x;\alpha)=\Phi(x;\alpha)^TG, && (15)\\
\frac{\partial^{\beta_i}f}{\partial x_i^{\beta_i}}(x_1,\cdots,x_n) &\simeq \Phi(x;\alpha)^T\left(I_{k+1}\otimes\cdots\otimes I_{k+1}\otimes\underbrace{D^{\alpha_i,\beta_i}}_{i-th\ component}\otimes I_{k+1}\otimes\cdots\otimes I_{k+1}\right)^TF,\quad 1\le i\le n\cdot && (16)
\end{aligned}$$

A system of algebraic equations is obtained by substituting Equations (13, 14, 15, 16) in Equation (12) as follows

$$A_1^T\Phi(x;\alpha)\Phi(x;\alpha)^T(D_1^{\alpha_1,\beta_1})^TF+A_2^T\Phi(x;\alpha)\Phi(x;\alpha)^T(D_2^{\alpha_2,\beta_2})^TF+\cdots+A_n^T\Phi(x;\alpha)\Phi(x;\alpha)^T(D_n^{\alpha_n,\beta_n})^TF=$$
$$G\qquad {}^T\Phi(x;\alpha)=\Phi(x;\alpha)^TG,$$

where for every $1 \le i \le n$, $D_i^{\alpha_i,\beta_i} = (I_{k+1} \otimes \cdots \otimes I_{k+1} \otimes \underbrace{D^{\alpha_i,\beta_i}}_{i-th\ component} \otimes I_{k+1} \otimes \cdots \otimes I_{k+1})$. Since for every $1 \le i \le n$,

$$A_i^T \Phi(x;\alpha)\Phi(x;\alpha)^T (D_i^{\alpha_i,\beta_i})^T F = \Phi(x;\alpha)^T \tilde{A_i}^T (D_i^{\alpha_i,\beta_i})^T F,$$

then,

$$\Phi(x;\alpha)^T \left(\tilde{A_1}^T (D_1^{\alpha_1,\beta_1})^T + \tilde{A_2}^T (D_2^{\alpha_2,\beta_2})^T + \cdots + \tilde{A_n}^T (D_n^{\alpha_n,\beta_n})^T \right) F = \Phi(x;\alpha)^T G\cdot$$

Therefore to approximately solve FCEs of Equation (12), we solve the system of algebraic equations

$$\left(\tilde{A_1}^T (D_1^{\alpha_1,\beta_1})^T + \tilde{A_2}^T (D_2^{\alpha_2,\beta_2})^T + \cdots + \tilde{A_n}^T (D_n^{\alpha_n,\beta_n})^T \right) F = G,$$

where the entries of vector F are unknown variables.

6 The convergence of the proposed method

In this section, the convergence of proposed method will be shown. We suppose that the approximation in (8) is the best approximation for one-variable function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Now we want to estimate the accuracy of that approximation.

Taylor's Theorem for one variable functions with Lagrange form of the remainder is as follows [15, 16, 17, 18].

Theorem 6.1. *Let $k \ge 1$ be an integer number and $a \in \mathbb{R}$. Suppose for $f : \mathbb{R} \longrightarrow \mathbb{R}$, all derivatives of order 1 to $k+1$ of f exist. Then*

$$\begin{aligned} f(x) &= f(a) + \frac{df}{dx}(a)(x-a) + \frac{d^2 f}{dx^2}(a)\frac{(x-a)^2}{2!} + \cdots + \frac{d^k f}{dx^k}(a)\frac{(x-a)^k}{k!} \\ &+ \frac{d^{k+1} f}{dx^{k+1}}(\xi)\frac{(x-a)^{k+1}}{(k+1)!}, \quad a < \xi < x\cdot \end{aligned}$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function that satisfies in Theorem 6.1 at x=0. Then f can be approximated as follows

$$f(x) \simeq \widehat{f_k}(x) = f(0) + \frac{df}{dx}(0)x + \frac{d^2 f}{dx^2}(0)\frac{x^2}{2!} + \cdots + \frac{d^k f}{dx^k}(0)\frac{x^k}{k!}\cdot$$

Lemma 6.1. *If we let the approximation by one-dimensional Müntz-Legendre polynomials of any function $f : [0,1] \longrightarrow \mathbb{R}$, $f_k(x)$, be best approximation and $\frac{d^{k+1} f}{dx^{k+1}}$ be bounded, then*

$$\|f(x) - f_k(x)\|_2 \le \frac{\lambda}{(k+1)!\sqrt{2k+3}},$$

$$\|{}_0D_x^{\beta} f(x) - {}_0D_x^{\beta} f_k(x)\|_2 \le \frac{\lambda}{\Gamma(k+2-\beta)\sqrt{2k+3-\beta}},$$

where $\lambda = \sup |\frac{d^{k+1} f}{dx^{k+1}}|\cdot$

Proof. Since $f_k(x)$ is the best approximation, then

$$\begin{aligned} \|f(x) - f_k(x)\|_2 &\le \|f(x) - \widehat{f_k}(x)\|_2 \\ &= \|\frac{d^{k+1} f}{dx^{k+1}}(\xi)\frac{x^{k+1}}{(k+1)!}\|_2 = \frac{\lambda}{(k+1)!\sqrt{2k+3}}\cdot \end{aligned}$$

And

$$\begin{aligned}\|{}_0D_x^{\beta} f(x) - {}_0D_x^{\beta} f_k(x)\|_2 &\leq \|\frac{d^{k+1}f}{dx^{k+1}}(\xi)\frac{\Gamma(k+2)}{\Gamma(k+2-\beta)}\frac{x^{k+1-\beta}}{(k+1)!}\|_2 \\ &= \frac{\lambda}{\Gamma(k+2-\beta)\sqrt{2k+3-\beta}}.\end{aligned}$$

□

The error bounds tends to zero if $k \longrightarrow \infty$.

Theorem 6.2. *[19] Suppose $U \subseteq \mathbb{R}^n$ be a convex open set and f be a smooth real-valued function on U. Let $m \geq 1$, $x \in U$, and $x_0 \in \mathbb{R}^n$ be small enough such that $x + x_0 \in U$, then*

$$\begin{aligned}f(x+x_0) &= \sum_{m=0}^{k}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{m!}\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(x_0)x_{i_1}\cdots x_{i_m} \\ &+ \sum_{i_1=1}^{n}\cdots\sum_{i_{k+1}=1}^{n}\int_0^1\frac{1}{k!}(1-t)^k\frac{\partial^{k+1}f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(x_0+tx)x_{i_1}\cdots x_{i_{k+1}}dt.\end{aligned}$$

Theorem 6.3. *Let f be a function on Ω, such that for $m = 0, 1, \cdots, k+1$, $\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}} \in C(\Omega)$. Let all $\frac{\partial^{k+1} f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}$ be bounded when $i_j = 1, \cdots, n$, $j = 1, \cdots, k+1$. If $f_{k,\cdots,k}$ be the best approximation of f in $\Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$, then*

$$\|f(x) - f_{k,\cdots,k}(x)\|_2 \leq \sqrt{(\frac{1}{3})^{k+1}\frac{Mn^{k+1}}{(k+1)!}}, \tag{17}$$

$$\|\frac{\partial^{\beta_j} f}{\partial x_j^{\beta_j}}(x) - \frac{\partial^{\beta_j} f_{k,\cdots,k}}{\partial x_j^{\beta_j}}(x)\|_2 \leq \frac{1}{\Gamma(1-\beta_j)\sqrt{1-2\beta_j}}\sqrt{(\frac{1}{3})^k\frac{Mn^{k+1}}{(k+1)!}}. \tag{18}$$

Where

$$M_{i_1,\cdots,i_{k+1}} = \sup_{x\in\Omega}\left|\frac{\partial^{k+1}f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(x)\right|, \quad M = \max_{\substack{i_j=1,\cdots,n \\ j=1,\cdots,k+1}}\{M_{i_1,\cdots,i_{k+1}}\}.$$

Proof. Since for every $m = 0, \cdots, k+1$, $\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(x) \in C(\Omega)$, then according to Theorem 6.2 we have Taylor expansion with the remainder of f at $x_0 = 0$ as follows

$$\begin{aligned}f(x) &= \sum_{m=0}^{k}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{m!}\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}(0)\, x_{i_1}\cdots x_{i_m} \\ &+ \sum_{i_1=1}^{n}\cdots\sum_{i_{k+1}=1}^{n}\int_0^1\frac{1}{k!}(1-t)^k\frac{\partial^{k+1}f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(tx)dt\, x_{i_1}\cdots x_{i_{k+1}}dt.\end{aligned}$$

Therefore, $f(x)$ can be approximated by removing the remainder that denoted by $\hat{f}_{k,\cdots,k}$. The error bound is computed as follows

$$\begin{aligned}|f(x) - f_{k,\cdots,k}(x)| &\leq |f(x) - \hat{f}_{k,\cdots,k}(x)| \\ &= |\sum_{i_1=1}^{n}\cdots\sum_{i_{k+1}=1}^{n}\int_0^1\frac{1}{k!}(1-t)^k\frac{\partial^{k+1}f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(tx)dt\, x_{i_1}\cdots x_{i_{k+1}}| \\ &\leq \sum_{i_1=1}^{n}\cdots\sum_{i_{k+1}=1}^{n}\frac{M_{i_1,\cdots,i_{k+1}}}{(k+1)!}|x_i\cdots x_{k+1}| \\ &= \frac{n^{k+1}M}{(k+1)!}x_1\cdots x_{k+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f(x)-f_{k,\cdots,k}(x)\|_2 &\le \left(\int_0^1\cdots\int_0^1 \frac{M^2(n^{k+1})^2}{((k+1)!)^2}x_1^2\cdots x_{k+1}^2\,dx_1\cdots dx_{k+1}\right)^{\frac{1}{2}}\\ &= \sqrt{(\frac{1}{3})^{k+1}}\frac{Mn^{k+1}}{(k+1)!}.\end{aligned}$$

Let $R_k := \sum_{i_1=1}^{n}\cdots\sum_{i_{k+1}=1}^{n}\int_0^1\frac{1}{k!}(1-t)^k\frac{\partial^{k+1}f}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(tx)x_{i_1}\cdots x_{i_{k+1}}dt$, $\hat{R}_k := \frac{n^{k+1}M}{(k+1)!}x_1\cdots x_{k+1}$, then $R_k \le \hat{R}_k$. Therefore, there exists $b(x)$ such that $\hat{R}_k - b(x) = R_k$. If let $\hat{f}_{k,\cdots,k}(x) - b(x)$ be an approximate of f, then

$$\begin{aligned}\left|\frac{\partial^{\beta_j}f}{\partial x_j^{\beta_j}}(x)-\frac{\partial^{\beta_j}f_{k,\cdots,k}}{\partial x_j^{\beta_j}}(x)\right| &\le \left|\frac{\partial^{\beta_j}f}{\partial x_j^{\beta_j}}(x)-\frac{\partial^{\beta_j}(\hat{f}_{k,\cdots,k}(x)-b(x))}{\partial x_j^{\beta_j}}\right|\\ &\le \left|\frac{n^{k+1}M}{(k+1)!}x_1\cdots x_j^{-\beta_j}\cdots x_{k+1}\frac{1}{\Gamma(1-\beta_j)}\right|\end{aligned}$$

and

$$\left\|\frac{\partial^{\beta_j}f}{\partial x_j^{\beta_j}}(x)-\frac{\partial^{\beta_j}f_{k,\cdots,k}}{\partial x_j^{\beta_j}}(x)\right\|_2 \le \frac{1}{\Gamma(1-\beta_j)\sqrt{1-2\beta_j}}\sqrt{(\frac{1}{3})^k}\frac{Mn^{k+1}}{(k+1)!}$$

□

Consequently, $\|f(x)-f_{k,\cdots,k}(x)\|_2 \to 0$ and $\|\frac{\partial^{\beta_j}f}{\partial x_j^{\beta_j}}(x)-\frac{\partial^{\beta_j}f_{k,\cdots,k}}{\partial x_j^{\beta_j}}(x)\|_2 \to 0$ as $k\to\infty$.

Theorem 6.4. *Let $D^{\alpha,\beta}$ be the R-L fractional derivative operational matrix based on 1-DFMLPs of order $0<\beta<1$. Let $f\in L^2[0,1]$ can be approximated as $f(s)\simeq f_k(s)=\sum_{i=0}^{k}f_iL_i(s,\alpha)=F^T\Phi(s;\alpha)$, where $f_i=(2i\alpha+1)\int_0^1 f(s)L_i(s,\alpha)ds$. Then*

$$\|{}_0D_s^{\beta}f_k-F^TD^{\alpha,\beta}\Phi(s;\alpha)\|_2 \le \frac{k\alpha\lambda^k}{(k+1)!\sqrt{2k+3}}\sum_{i=0}^{k}|f_i|\sum_{r=0}^{i}|a_{ri}|,$$

where $\lambda_i=\sup\left|\frac{d^{k+1}s^{i\alpha-\beta}}{ds^{k+1}}\right|$, and $\lambda^k=\max\{\lambda_i|i=0,\cdots,k\}$. If we increase the number of 1-DFMLPs, then the error tends to zero.

Proof. The R-L fractional derivative of order $0<\beta<1$ from every $L_i(s,\alpha)$ is as follows

$${}_0D_s^{\beta}L_i(s,\alpha)=\sum_{r=0}^{i}a_{ri}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1-\beta)}s^{r\alpha-\beta},$$

then

$$\begin{aligned}\|{}_0D_s^{\beta}f_k-F^TD^{\alpha,\beta}\Phi(s;\alpha)\|_2 &\le \sum_{i=0}^{k}|f_i|\sum_{r=0}^{i}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1-\beta)}|a_{ri}|\left\|s^{r\alpha-\beta}-\sum_{l=0}^{k}(2\alpha l+1)\sum_{t=0}^{l}\frac{a_{tl}}{(r+t)\alpha-\beta+1}L_l(s,\alpha)\right\|_2\\ &\le \sum_{i=0}^{k}|f_i|\sum_{r=0}^{i}\frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+1-\beta)}|a_{ri}|\frac{\lambda_r}{(k+1)!\sqrt{2k+3}}\\ &\le \sum_{i=0}^{k}|f_i|\sum_{r=0}^{i}|a_{ri}|\frac{r\alpha\lambda_r}{(k+1)!\sqrt{2k+3}}\\ &\le \frac{k\alpha\lambda^k}{(k+1)!\sqrt{2k+3}}\sum_{i=0}^{k}|f_i|\sum_{r=0}^{i}|a_{ri}|.\end{aligned}$$

Obviously, by increasing the number of 1-DFMLPs, the error of obtained fractional derivative operational matrix tends to zero. $\square$

Theorem 6.5. *Let $f \in L^2(\Omega)$, $0 < \beta_j < 1$. If f can be approximated by*

$$f_{k,\cdots,k}(x) = F^T\Phi(x;\alpha) = \sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k} f_{i_1\cdots i_n}L_{i_1}(x_1,\alpha_1)\cdots L_{i_j}(x_j,\alpha_j)\cdots L_{i_n}(x_n,\alpha_n),$$

then

$$\left\|\frac{\partial^{\beta_j} f_{k,\cdots,k}}{\partial x_j^{\beta_j}} - F^T D_j^{\alpha_j,\beta_j}\Phi(x;\alpha)\right\|_2 \le \frac{k\alpha_j\lambda_j^k}{(k+1)!\sqrt{2k+3}}\sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\sum_{r=0}^{i_j}|a_{ri_j}|.$$

Where $\lambda_{ij} = \sup\left|\dfrac{d^{k+1}x_j^{i\alpha-\beta}}{ds^{k+1}}\right|$, and $\lambda_j^k = \max\{\lambda_{ij}|i=0,\cdots,k\}$. By increasing the number of n-DFMLPs the error bound of the obtained operational matrix of R-L partial fractional derivative tends to zero.

Proof. By taking L^2-norm of the difference between $\dfrac{\partial^{\beta_j} f_{k,\cdots,k}}{\partial x_j^{\beta_j}}$ and $F^T D_j^{\alpha_j,\beta_j}\Phi(x;\alpha)$, we have

$$\begin{aligned}
&\left\|\frac{\partial^{\beta_j} f_{k,\cdots,k}}{\partial x_j^{\beta_j}} - F^T D_j^{\alpha_j,\beta_j}\Phi(x;\alpha)\right\|_2 \le \sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\|L_{i_1}(x_1,\alpha_1)\|_2\cdots\|L_{i_{j-1}}(x_{j-1},\alpha_{j-1})\|_2\times\\
&\sum_{r=0}^{i_j}\left|a_{ri_j}\frac{\Gamma(r\alpha_j+1)}{\Gamma(r\alpha_j+1-\beta_j)}\right|\|x_j^{r\alpha_j-\beta_j}-\sum_{l=0}^{k}\big((2\alpha_j l+1)\sum_{s=0}^{l}\frac{a_{sl}}{(s+r)\alpha_j-\beta_j+1}\big)L_l(x_l,\alpha_l)\|_2\times\\
&\|L_{i_{j+1}}(x_{j+1},\alpha_{j+1})\|_2\cdots\|L_{i_n}(x_n,\alpha_n)\|_2\le\\
&\sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\frac{1}{2\alpha_1 i_1+1}\cdots\frac{1}{2\alpha_{j-1}i_{j-1}+1}\left(\frac{k\alpha_j\lambda_j^k}{(k+1)!\sqrt{2k+3}}\sum_{r=0}^{i_j}|a_{ri_j}|\right)\frac{1}{2\alpha_{j+1}i_{j+1}+1}\cdots\frac{1}{2\alpha_n i_n+1}\le\\
&\frac{k\alpha_j\lambda_j^k}{(k+1)!\sqrt{2k+3}}\sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\sum_{r=0}^{i_j}|a_{ri_j}|.
\end{aligned}$$

Consequently, if $k\to\infty$, then the error bound tends to zero. $\square$

Theorem 6.6. *Let $f \in L^2(\Omega)$ and $f_{k,\cdots,k} \in \Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$ be the exact solution and the approximation solution of Equation ([12](#)), respectively. Assume the functions $a_i(x_1,\cdots,x_n)$ and fractional partial derivatives $\dfrac{\partial^{\beta_i} f}{\partial x_i^{\beta_i}}(x_1,\cdots,x_n)$, $i=1,\cdots,n$, are bounded with bounds μ_i and η_i, respectively. Then the error bound of proposed method is*

$$\begin{aligned}
\|E_{k,\cdots,k}\|_2 \;\le\; &\sum_{j=1}^{n}\Bigg(\mu_j\Bigg(\frac{1}{\Gamma(1-\beta_j)\sqrt{1-2\beta_j}}\sqrt{(\frac{1}{3})^k}\frac{Mn^{k+1}}{(k+1)!}+\frac{k\alpha_j\lambda_j^k}{(k+1)!\sqrt{2k+3}}\sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\sum_{r=0}^{i_j}|a_{ri_j}|\Bigg)+\\
&\eta_j\frac{\rho_j n^{k+1}}{(k+1)!\sqrt{2k+3}}\Bigg).
\end{aligned}$$

Where for every $j=1,\cdots,n$, $\rho^j = \max\{\rho^j_{j_1,\cdots,j_{k+1}}\}$ and $\rho^j_{j_1,\cdots,j_{k+1}} = \sup_{x\in\Omega}\left|\dfrac{\partial^{k+1}a_i}{\partial x_{j_1}\cdots\partial x_{j_{k+1}}}\right|$.

Proof. For every $j=1,\cdots,n$, due to be bounded $a_i(x_1,\cdots,x_n)$, $(a_i(x_1,\cdots,x_n))_{k,\cdots,k}$ is also bounded, $|(a_i(x_1,\cdots,x_n))_{k,\cdots,k}| \le \mu_i$. Then

$$
\begin{aligned}
&\|E_{k,\cdots,k}\|_2 = \|(a_1)_{k,\cdots,k}\, F^T D_1^{\alpha_1,\beta_1}\Phi(x;\alpha) + \cdots + (a_n)_{k,\cdots,k}\, F^T D_n^{\alpha_n,\beta_n}\Phi(x;\alpha) - g\|_2 = \\
&\|(a_1)_{k,\cdots,k}\, F^T D_1^{\alpha_1,\beta_1}\Phi(x;\alpha) - a_1\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}} + \cdots + (a_n)_{k,\cdots,k}\, F^T D_n^{\alpha_n,\beta_n}\Phi(x;\alpha) - a_n\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}\|_2 \le \\
&\|(a_1)_{k,\cdots,k}\, F^T D_1^{\alpha_1,\beta_1}\Phi(x;\alpha) - a_1\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}}\|_2 + \cdots + \|(a_n)_{k,\cdots,k}\, F^T D_n^{\alpha_n,\beta_n}\Phi(x;\alpha) - a_n\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}\|_2 \le \\
&\|(a_1)_{k,\cdots,k}\|_2\|\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}} - F^T D_1^{\alpha_1,\beta_1}\Phi(x;\alpha)\|_2 + \|\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}}\|_2\|a_1 - (a_1)_{k,\cdots,k}\|_2 + \cdots + \\
&\|(a_n)_{k,\cdots,k}\|_2\|\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}} - F^T D_n^{\alpha_n,\beta_n}\Phi(x;\alpha)\|_2 + \|\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}\|_2\|a_n - (a_n)_{k,\cdots,k}\|_2 \le \\
&\|(a_1)_{k,\cdots,k}\|_2\left(\|\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}} - \frac{\partial^{\beta_1} f_{k,\cdots,k}}{\partial x_1^{\beta_1}}\|_2 + \|\frac{\partial^{\beta_1} f_{k,\cdots,k}}{\partial x_1^{\beta_1}} - F^T D_1^{\alpha_1,\beta_1}\Phi(x;\alpha)\|_2\right) + \|\frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}}\|_2\|a_1 - (a_1)_{k,\cdots,k}\|_2 + \cdots + \\
&\|(a_n)_{k,\cdots,k}\|_2\left(\|\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}} - \frac{\partial^{\beta_n} f_{k,\cdots,k}}{\partial x_n^{\beta_n}}\|_2 + \|\frac{\partial^{\beta_1} f_{k,\cdots,k}}{\partial x_n^{\beta_n}} - F^T D_n^{\alpha_n,\beta_n}\Phi(x;\alpha)\|_2\right) + \|\frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}\|_2\|a_n - (a_n)_{k,\cdots,k}\|_2 \cdot
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|E_{k,\cdots,k}\|_2 \quad \le \quad & \sum_{j=1}^{n}\left(\mu_j\left(\frac{1}{\Gamma(1-\beta_j)\sqrt{1-2\beta_j}}\sqrt{(\frac{1}{3})^k}\frac{Mn^{k+1}}{(k+1)!} + \frac{k\alpha_j\lambda_j^k}{(k+1)!\sqrt{2k+3}}\sum_{i_1=0}^{k}\cdots\sum_{i_j=0}^{k}\cdots\sum_{i_n=0}^{k}|f_{i_1\cdots i_n}|\sum_{r=0}^{i_j}|a_{ri_j}|\right) + \right. \\
& \left.\eta_j\frac{\rho_j n^{k+1}}{(k+1)!\sqrt{2k+3}}\right).
\end{aligned}
$$

Where M and λ_j^k are defined in Theorems 6.3, 6.5, respectively. □

Remark 6.1. *According to Theorem 6.6, if $k\to\infty$, then $E_{k,\cdots,k}\to 0$ and the proposed method is convergence.*

7 Illustrative test examples

In this section, given examples are provided to demonstrate the efficiency and accuracy of our proposed method. In all our examples, n, $(k+1)^n$ denotes the number of variables and bases, respectively. If $f\in L^2(\Omega)$ and $f_{k,\cdots,k}\in \Phi_{k,\cdots,k}^{\alpha_1,\cdots,\alpha_n}(\Omega)$ be the exact solution and the approximate solution, respectively, the absolute errors between them are

$$|f(x_1,\cdots,x_n) - f_{k,\cdots,k}(x_1,\cdots,x_n)|, \quad (x_1,\cdots,x_n)\in\Omega\cdot$$

The maximum absolute errors are calculated by

$$\max_{\substack{i=1,\cdots,n\\ i_j=1,\cdots,(k+1)^n}} \{|f(x_1^{j_1},\cdots,x_n^{j_n}) - f_{k,\cdots,k}(x_1^{j_1},\cdots,x_n^{j_n})|\},$$

where $(x_1^{j_1},\cdots,x_n^{j_n}) = (\frac{2j_1-1}{(k+1)^n},\frac{2j_2-1}{(k+1)^n},\cdots,\frac{2j_n-1}{(k+1)^n})$.
Plots of maximum absolute errors are displayed by

$$m_a(t) := \max_{\substack{i=1,\cdots,n\\ i_j=1,\cdots,(k+1)^n}} \{|f(x_1^{j_1},t,x_3^{j_3},\cdots,x_n^{j_n}) - f_{k,\cdots,k}(x_1^{j_1},t,x_3^{j_3},\cdots,x_n^{j_n})|\}\cdot$$

Example 7.1. *Consider the following two-dimensional FCE*

$$\Gamma(\frac{4}{3})x^{\frac{1}{3}}\cos^2(x)\frac{\partial^{\frac{1}{3}}u}{\partial x^{\frac{1}{3}}}(x,y)+\Gamma(\frac{4}{3})y^{\frac{1}{3}}\sin^2(x)\frac{\partial^{\frac{1}{3}}u}{\partial y^{\frac{1}{3}}}(x,y)=\Gamma(\frac{5}{3})x^{\frac{2}{3}}y^{\frac{2}{3}},$$

with exact solution $u(x,y)=x^{\frac{2}{3}}y^{\frac{2}{3}}$. *The values of the exact solution and the approximate solution for* $k=5,7,9$, $\alpha_1=\alpha_2=\dfrac{1}{3}$ *are reported in Table 1. Also, we report the absolute error in Table 2. The accuracy and efficiency of our method are reported in Figures 1 and 2.*

Table 1: The numerical result of Example 7.1

x=y	Exact solutions	Approximate solutions		
		$k=5$	$k=7$	$k=9$
0	0	-8.6499e-6	0.0004093593	-0.0156772165
0.1	0.04641588833	0.0464312147	0.0464183583	0.0463053392
0.2	0.1169607095	0.1169464422	0.1169606529	0.1170104943
0.3	0.2008298851	0.2007922429	0.2008264283	0.2008591848
0.4	0.2947225199	0.2946863795	0.2947207296	0.2946557013
0.5	0.3968502629	0.3968372254	0.3968516564	0.3967873063
0.6	0.5060595992	0.5060788853	0.5060619996	0.5061126173
0.7	0.6215328012	0.6215769416	0.6215334091	0.6216589573
0.8	0.7426542134	0.7426969020	0.7426523442	0.7426491573
0.9	0.8689404462	0.8689354930	0.8689386362	0.8687528073

Table 2: Absolute errors of Example 7.1

x=y	k=5	k=7	k=9
0	8.6499e-6	4.093593e-4	1.56772165e-2
0.1	1.532637e-5	2.46997e-6	1.1054913e-4
0.2	1.42673e-5	5.66e-8	4.97848e-5
0.3	3.76422e-5	3.4568e-6	2.92997e-5
0.4	3.61404e-5	1.7903e-6	6.68186e-5
0.5	1.30375e-5	1.3935e-6	6.29566e-5
0.6	1.92861e-5	2.4004e-6	5.30181e-5
0.7	4.41404e-5	6.079e-7	1.261561e-4
0.8	4.26886e-5	1.8692e-6	5.0561e-6
0.9	4.9532e-6	1.8100e-6	1.876389e-4

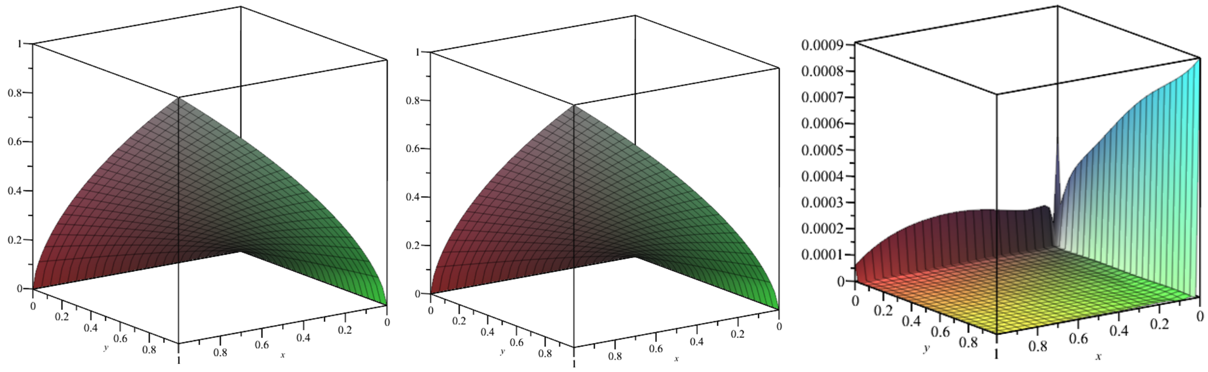

Figure 1: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.1. Note that $k = 7$.

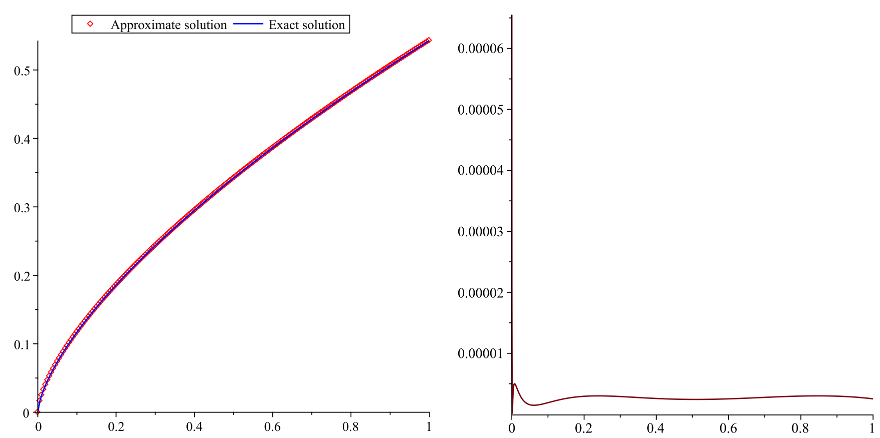

Figure 2: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.1. Note $k = 7$, $x = 0.4$.

Example 7.2. *We consider the following two-dimensional FCE*

$$\frac{\partial^{\frac{1}{4}} u}{\partial x^{\frac{1}{4}}}(x,y) + \frac{\partial^{\frac{1}{4}} u}{\partial y^{\frac{1}{3}}}(x,y) = \frac{1}{\Gamma(\frac{3}{4})}(x^2 y^{-\frac{1}{4}} - x^{-\frac{1}{4}} y^2) + \frac{2}{\Gamma(\frac{11}{4})}(x^{\frac{7}{4}} - y^{\frac{7}{4}}),$$

with exact solution $u(x,y) = x^2 - y^2$. We apply the proposed method in this article for this example. The values of the exact solution and the approximate solution are reported in Table 3. Also the absolute errors are reported in Tables 4 for $k = 7,8,9$, $\alpha_1 = \alpha_2 = \dfrac{1}{4}$. The accuracy and efficiency of our method are reported in Figures 3 and 4.

Table 3: The numerical result of Example 7.2

x=y	Exact solution	Approximate solutions		
		$k=7$	$k=8$	$k=9$
0	0	-1.9999e-10	8.500000258e-11	-5.058621551e-10
0.1	0	-8.8e-10	8.e-11	-5.10735623e-10
0.2	0	-1.09e-9	8.e-11	-5.1080572e-10
0.3	0	4.5e-10	9.e-11	-5.0001182e-10
0.4	0	1.2e-10	1.e-10	-4.789065e-10
0.5	0	-3.4e-10	1.e-10	-4.738289e-10
0.6	0	-2.9e-10	1.e-10	-4.582606e-10
0.7	0	-1.5e-10	1.e-10	-5.447979e-10
0.8	0	-5.3e-10	1.e-10	-5.015494e-10
0.9	0	1.94e-9	1.e-10	-4.869799e-10

Table 4: Absolute errors of Example 7.2

$x=y$	$k=7$	$k=8$	$k=9$
0	1.9999e-10	8.500000258e-11	5.058621551e-10
0.1	8.8e-10	8.e-11	5.10735623e-10
0.2	1.09e-9	9.e-11	5.1080572e-10
0.3	4.5e-10	9.e-11	5.0001182e-10
0.4	1.2e-10	1.e-10	4.789065e-10
0.5	3.4e-10	1.e-10	4.738289e-10
0.6	2.9e-10	1.e-10	4.582606e-10
0.7	1.5e-10	1.e-10	5.447979e-10
0.8	5.3e-10	1.e-10	5.015494e-10
0.9	1.94e-9	1.e-10	4.869799e-10

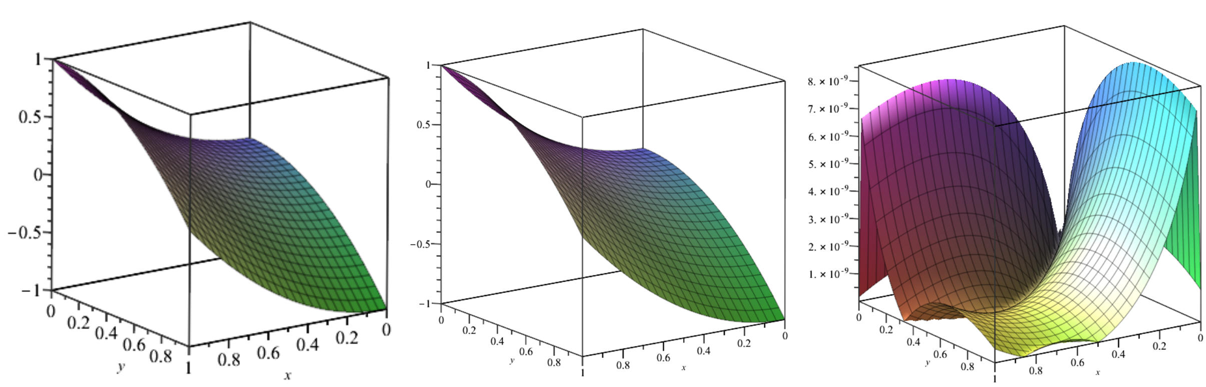

Figure 3: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.2. Note that $k=9$.

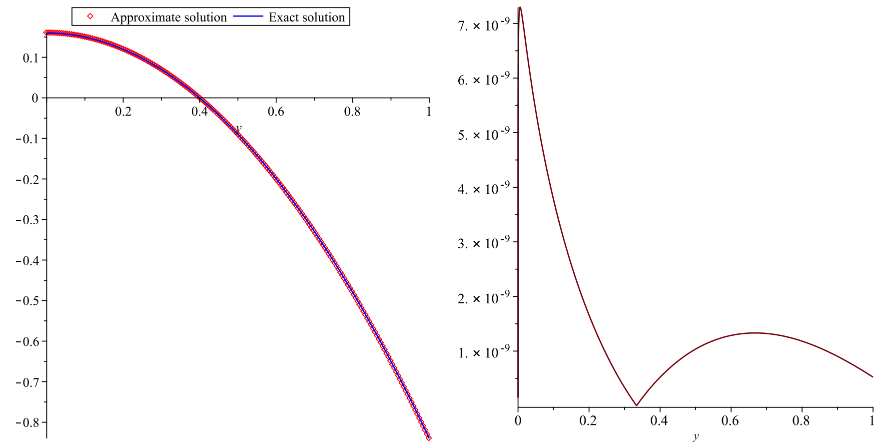

Figure 4: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.2. Note $k = 9$, $x = 0.4$.

Example 7.3. *In this example, the following two-dimensional FCE is considered with exact solution $u(x,y) = x^3y - xy$,*

$$x^{\frac{1}{2}}y\frac{\partial^{\frac{1}{2}}u}{\partial x^{\frac{1}{2}}}(x,y) + xy^{\frac{1}{2}}\frac{\partial^{\frac{1}{2}}u}{\partial y^{\frac{1}{2}}}(x,y) = \frac{xy}{\Gamma(\frac{3}{2})}(x^3 - x - y) + \frac{6}{\Gamma(\frac{7}{2})}x^3y^2.$$

We report the values of the exact solution and the approximate solution in Table 5. Also, the absolute errors are reported in Table 6, for $k = 4,5,6$, $\alpha_1 = \alpha_2 = \dfrac{1}{2}$. The accuracy and efficiency of our method are reported in Figures 5 and 6.

Table 5: The numerical result of Example 7.3

		Approximate solutions		
$x = y$	Exact solution	$k = 4$	$k = 5$	$k = 6$
0	0	-0.0004734844234	0.0000508542	0.0000264210
0.1	-0.0099	-0.009957151397	-0.0099357655	-0.0099000552
0.2	-0.0384	-0.03759784243	-0.0384084745	-0.0383999987
0.3	-0.0819	-0.08139495625	-0.0818235180	-0.0818999617
0.4	-0.1344	-0.1350775202	-0.1343411794	-0.1343999774
0.5	-0.1875	-0.1890817207	-0.1875555650	-0.1874999927
0.6	-0.2304	-0.2316625456	-0.2305476072	-0.2303999956
0.7	-0.2499	-0.2494840616	-0.2499982854	-0.2498999955
0.8	-0.2304	-0.2279805666	-0.2303048199	-0.2303999998
0.9	-0.1539	-0.1515990163	-0.1536882787	-0.1539000045

Table 6: Absolute errors of Example 7.3

$x = y$	$k = 4$	$k = 5$	$k = 6$
0	4.734844234e-4	5.08542e-5	2.64210e-5
0.1	5.7151397e-5	3.57655e-5	5.52e-8
0.2	8.0215757e-4	8.4745e-6	1.3e-9
0.3	5.0504375e-4	7.64820e-5	3.83e-8
0.4	6.775202e-4	5.88206e-5	2.26e-8
0.5	1.5817207e-3	5.55650e-5	7.3e-9
0.6	1.2625456e-3	1.476072e-4	4.4e-9
0.7	4.159384e-4	9.82854e-5	4.5e-9
0.8	2.4194334e-3	9.51801e-5	2.e-10
0.9	2.3009837e-3	2.117213e-4	4.5e-9

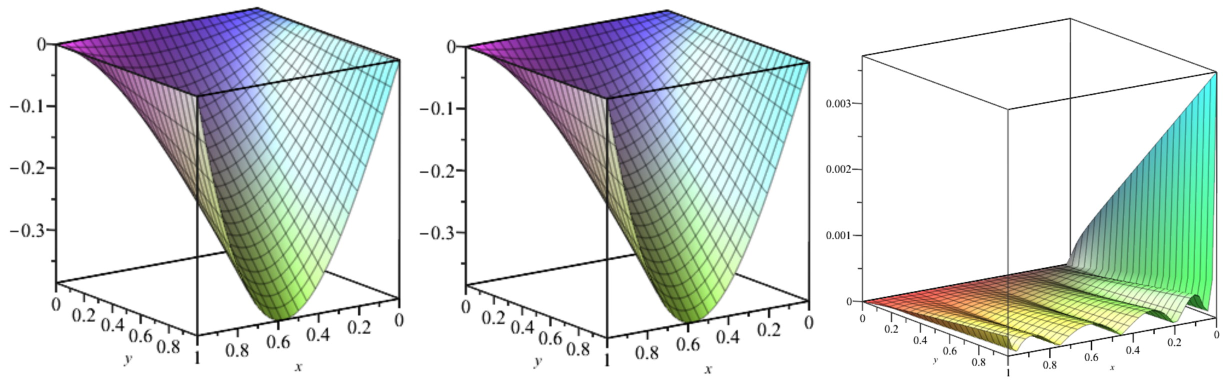

Figure 5: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.3. Note that $k = 5$.

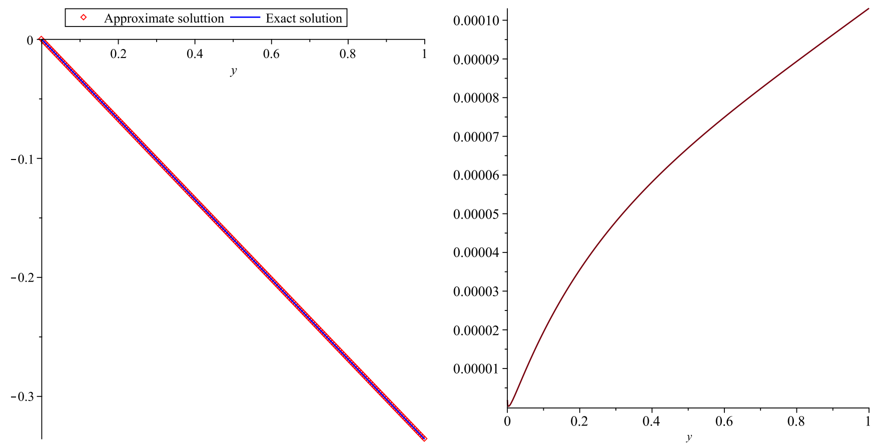

Figure 6: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.3. Note $k = 5$, $x = 0.4$.

Example 7.4. *We consider the following two-dimensional FCE*

$$x^{\frac{7}{10}}y^{\frac{1}{2}}\frac{\partial^{\frac{1}{5}}u}{\partial x^{\frac{1}{5}}}(x,y)+x^{\frac{1}{2}}y^{\frac{7}{10}}\frac{\partial^{\frac{1}{5}}u}{\partial y^{\frac{1}{5}}}(x,y)=\frac{2\Gamma(\frac{3}{2})}{\Gamma(\frac{13}{10})}xy,$$

that its exact solution is $u(x,y)=\sqrt{xy}$. We use of the proposed method, for obtaining the approximate solution. We report the values of the exact solution and the approximate solution in Table 7, for $k=4,5,6$, $\alpha_1=\alpha_2=\frac{1}{5}$. Also, the absolute errors are reported in Table 8.The accuracy and efficiency of our method are reported in Figures 7 and 8.

Table 7: The numerical result of Example 7.4

$x=y$	Exact solution	Approximate solutions $k=4$	$k=5$	$k=6$
0	0	0.004674419445	0.0262889566	0.4420366028
0.1	0.1	0.1000048627	0.0999956947	0.1000144611
0.2	0.2000000000	0.1999874128	0.2000020790	0.2000052542
0.3	0.3000000000	0.2999845807	0.3000047221	0.2999981671
0.4	0.4000000000	0.3999929445	0.4000040266	0.3999966301
0.5	0.5000000000	0.5000056047	0.5000012037	0.4999977514
0.6	0.6000000000	0.6000164656	0.5999977804	0.6000000861
0.7	0.7000000000	0.7000205909	0.6999953504	0.7000023959
0.8	0.8000000000	0.8000140327	0.7999955184	0.8000031058
0.9	0.9000000000	0.8999936201	0.8999999404	0.9000008838

Table 8: Absolute errors of Example 7.4

$x=y$	$k=4$	$k=5$	$k=6$
0	4.674419445e-3	2.62889566e-2	4.420366028e-1
0.1	4.8627e-6	4.3053e-6	1.446112e-5
0.2	1.25872e-5	2.0790e-6	5.2542e-6
0.3	1.54193e-5	4.7221e-6	1.8329e-6
0.4	7.0555e-6	4.0266e-6	3.3699e-6
0.5	5.6047e-6	1.2037e-6	2.2486e-6
0.6	1.64656e-5	2.2196e-6	8.61e-8
0.7	2.05909e-5	4.6496e-6	2.3959e-6
0.8	1.40327e-5	4.4816e-6	3.1058e-6
0.9	6.3799e-6	5.96e-8	8.838e-7

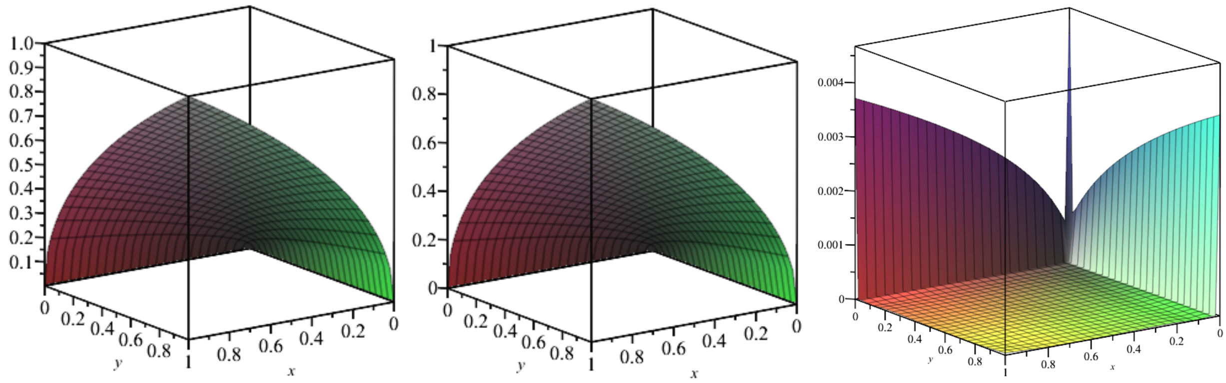

Figure 7: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.4. Note that $k = 4$.

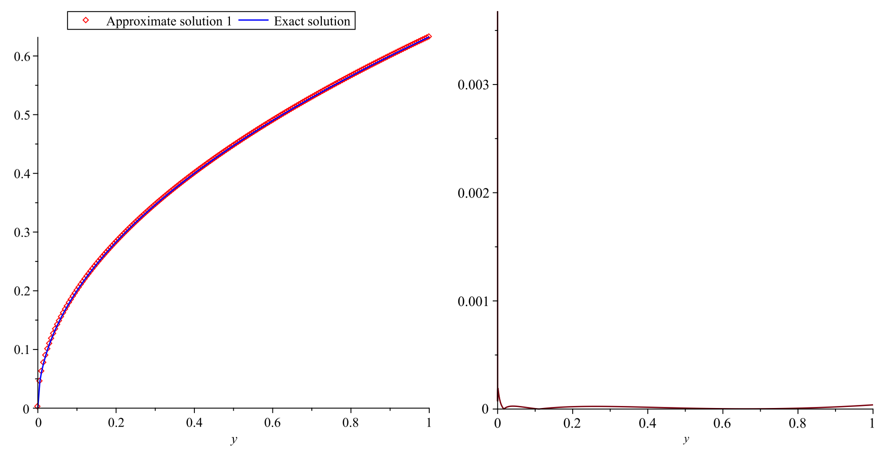

Figure 8: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.4. Note $k = 4$, $x = 0.4$.

Example 7.5. *Consider the following two-dimensional FCE*

$$xy\frac{\partial^{\frac{1}{2}}u}{\partial x^{\frac{1}{2}}}(x,y)+x^{\frac{1}{2}}y^{\frac{1}{2}}\frac{\partial^{\frac{1}{2}}u}{\partial y^{\frac{1}{2}}}(x,y)=x^{\frac{5}{2}}\Big(\frac{y}{\Gamma(\frac{3}{2})}+\frac{2y^2}{\Gamma(\frac{5}{2})}\Big)+x^{\frac{3}{2}}\Big(\frac{y^3}{\Gamma(\frac{3}{2})}+\frac{2y^2}{\Gamma(\frac{5}{2})}\Big),$$

this equation has the exact solution $u(x,y) = x^2y + xy^2$. For numerically solving the assumed equation, we use the proposed method. We report the values of the exact solution and the approximate solution in Table 9, for $k = 4,5,6$, $\alpha_1 = \alpha_2 = \frac{1}{2}$. Also, the absolute errors are reported in Table 10. The accuracy and efficiency of our method are reported in Figures 9 and 10.

Table 9: The numerical result of Example 7.5

$x = y$	Exact solution	Approximate solutions		
		$k = 4$	$k = 5$	$k = 6$
0	0	-6.017889461e-8	-7.53e-8	-7.88e-8
0.1	0.002	0.001999980773	0.0019999836	0.0019999872
0.2	0.016	0.01599999201	0.0159999942	0.0159999929
0.3	0.054	0.05399999630	0.0539999952	0.0539999921
0.4	0.128	0.1279999976	0.1279999948	0.1279999943
0.5	0.250	0.2499999978	0.2499999952	0.2499999970
0.6	0.432	0.4319999977	0.4319999961	0.4319999987
0.7	0.686	0.6859999978	0.6859999971	0.6859999990
0.8	1.024	1.023999998	1.024000000	1.024000000
0.9	1.458	1.457999996	1.457999996	1.457999997

Table 10: Absolute errors with $k = 4, 5, 6$ of Example 7.5

$x = y$	$k = 4$	$k = 5$	$k = 6$
0	6.017889461e-8	7.53e-8	7.88e-8
0.1	1.9227e-8	1.64e-8	1.28e-8
0.2	7.99e-9	5.8e-9	7.1e-9
0.3	3.70e-9	4.8e-9	7.9e-9
0.4	2.4e-9	5.2e-9	5.7e-9
0.5	2.2e-9	4.8e-9	3.0e-9
0.6	2.3e-9	3.9e-9	1.3e-9
0.7	2.2e-9	2.9e-9	1.0e-9
0.8	2.e-9	0	0
0.9	4.e-9	4.e-9	3.e-9

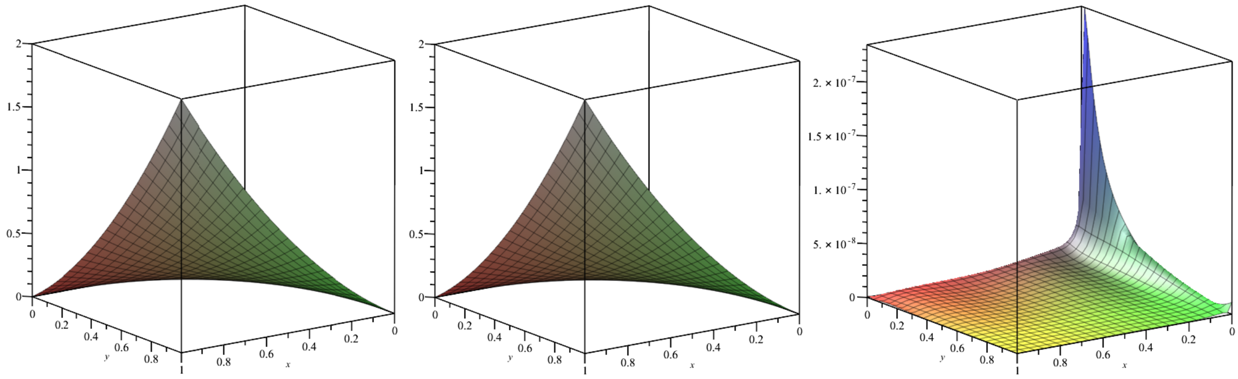

Figure 9: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.5. Note that $k = 6$.

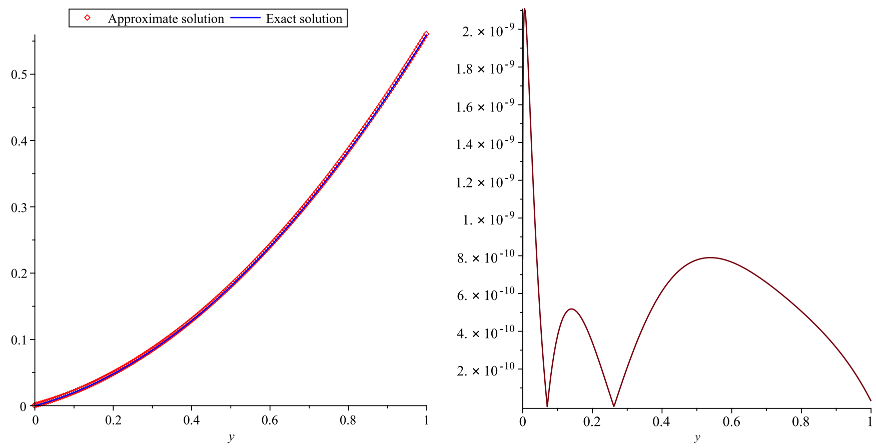

Figure 10: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.5. Note $k=6$, $x=0.4$.

Example 7.6. *We consider the following three-dimensional FCE*

$$\frac{\partial^{\frac{2}{5}}u}{\partial x^{\frac{2}{5}}}(x,y,z)+\frac{\partial^{\frac{2}{5}}u}{\partial y^{\frac{2}{5}}}(x,y,z)+\frac{\partial^{\frac{2}{5}}}{\partial z^{\frac{2}{5}}}(x,y,z)=x^{\frac{3}{5}}z^{\frac{4}{5}}+x^{\frac{2}{5}}y^{\frac{1}{5}}z^{\frac{4}{5}}+x^{\frac{2}{5}}y^{\frac{3}{5}}z^{\frac{2}{5}},$$

with the exact solution $u(x,y)=x^{\frac{2}{5}}y^{\frac{3}{5}}z^{\frac{4}{5}}$. By applying the proposed method, the numerical solution is obtained. We report the values of the exact solution and the approximate solution in Table 11, for $k=5,6,7$, $\alpha_1=\alpha_2=\alpha_3=\dfrac{2}{5}$. Also, the absolute errors are reported in Table 12. The accuracy and efficiency of our method are reported in Figures 11 and 12.

Table 11: The numerical result of Example 7.6

		Approximate solutions		
$x=y$	Exact solution	$k=5$	$k=6$	$k=7$
0	0	-1.17e-8	-4.320239786e-9	-1.449693311e-9
0.1	0.01584893192	0.01584671301	0.01585091164	0.01584974646
0.2	0.05518918646	0.05519821018	0.05518984875	0.05518711764
0.3	0.1145033673	0.1145102563	0.1144977447	0.1145033177
0.4	0.1921799095	0.1921715774	0.1921754713	0.1921835515
0.5	0.2871745888	0.2871538910	0.2871788518	0.2871768983
0.6	0.3987238835	0.3987074991	0.3987348894	0.3987204890
0.7	0.5262310526	0.5262375774	0.5262373841	0.5262249908
0.8	0.6692093138	0.6692419869	0.6692000621	0.6692097898
0.9	0.8272495069	0.8272759868	0.8272331760	0.8272584968

Table 12: Absolute errors of Example 7.6

$x=y$	$k=5$	$k=6$	$k=7$
0	1.17e-8	4.320239786e-9	1.449693311e-9
0.1	2.21891e-6	1.97972e-6	8.1454e-7
0.2	9.02372e-6	6.6229e-7	2.06882e-6
0.3	6.8890e-6	5.6226e-6	4.96e-8
0.4	8.3321e-6	4.4382e-6	3.6420e-6
0.5	2.06978e-5	4.2630e-6	2.3095e-6
0.6	1.63844e-5	1.10059e-5	3.3945e-6
0.7	6.5248e-6	6.3315e-6	6.0618e-6
0.8	3.26731e-5	9.2517e-6	4.760e-7
0.9	2.64799e-5	1.63309e-5	8.9899e-6

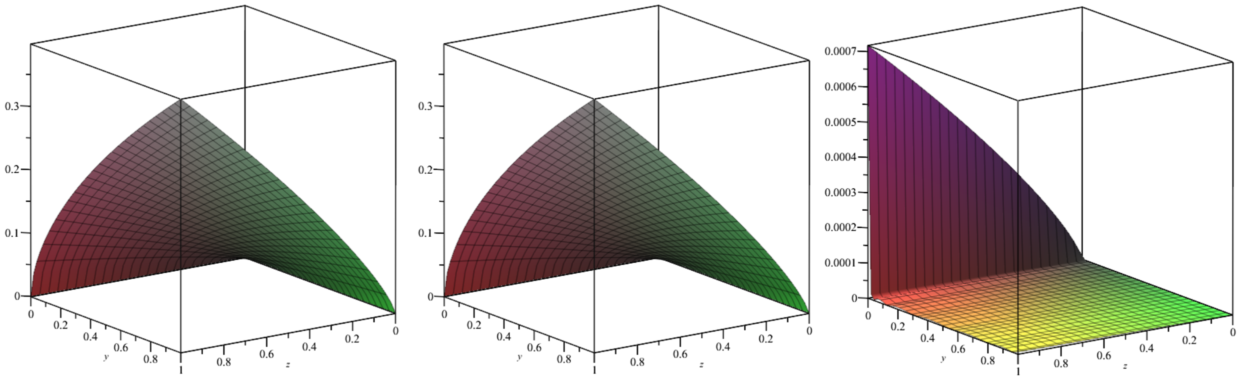

Figure 11: From left to right: the exact solution, the approximate solution, the absolute error of Example 7.6. Note that $k=7$ and $x=0.1$.

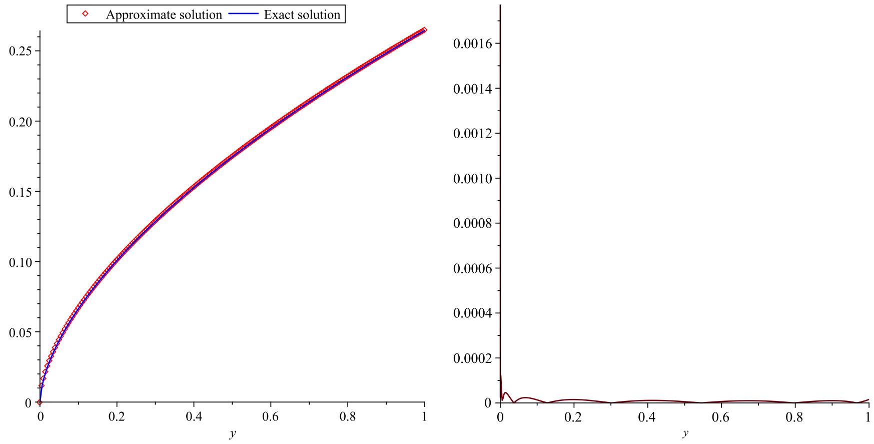

Figure 12: The comparison of the exact solution and the approximate solution (left) and the maximum absolute error (right) of Example 7.6. Note $k=7$, $x=0.4$, $z=0.3$.

8 Conclusion

For the first time, we had numerically solved fractional-order cohomological equations with variable coefficients. Our method had based on the R-L partial fractional derivative operational matrix. Since our equations are n-dimensional, then here n-DFMLPs were introduced. The product operational matrix was presented. Error bound and convergence analysis was investigated. Numerical results in the given examples showed that our method is useful.

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