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# Strong Instability of Solitary Waves for Inhomogeneous Nonlinear Schrödinger Equations

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**Abstract**

This paper studies the inhomogeneous nonlinear Schrödinger equations, which may model the propagation of laser beams in nonlinear optics. Using the cross-constrained variational method, a sharp condition for global existence is derived. Then, by solving a variational problem, the strong instability of solitary waves of this equation is proved.

**KEYWORDS:**
Inhomogeneous nonlinear Schrödinger equation; Sharp condition; Variational problem; Solitary wave; Instability

## 1 | INTRODUCTION

In this paper, we are concerned with the inhomogeneous nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |x|^{-b}|u|^{p-1}u = 0, \quad t \geq 0, x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $0 < b < \min\{2, N\}$ and $1 + \frac{4-2b}{N} \leq p < 2^* - 1$. Here $2^*$ is defined by $2^* = \frac{2N-2b}{N-2}$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$. (1.1) may model the propagation of laser beams in nonlinear optics (see e.g. [1,9,18] for more details).

We impose an initial value for (1.1)

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Genoud and Stuart [13] established the local well-posedness for the Cauchy problem (1.1)-(1.2) in the energy space $H^1(\mathbb{R}^N)$ (see also [4,10]). In [20], Merle investigated the existence and nonexistence of minimal mass blowup solution for some certain types of inhomogeneity. Furthermore, a sharp sufficient condition for global existence and blowup was studied for $1 + \frac{4-2b}{N} \leq p < 2^* - 1$(see [5,8,12]). But their proofs strongly depend on a sharp Gagliardo-Nirenberg inequality, which is equivalent to compute a complicated constrained variational problem, using the Schwartz symmetrization and variational computation. The stability of standing waves was demonstrated for $N \geq 2$ (see e.g.[3,6,13,15]), while the instability of solitary waves was proved for $N \geq 2$ (see e.g. [7,16,19]).

Note that their results do not include the case $N = 1$. So inspired by above arguments, on the one hand, in the present paper, we expect to look for sharp sufficient conditions for global existence and blowup of the solutions; On the other hand, we also hope to prove the strong instability of solitary waves. The related problem was considered by Ardila and Cardoso [2], however, the sharp sufficient conditions by our given are different from their results and we extend their strong instability results.

The rest of this paper is organized as follows. In Section 2, we state some elementary preliminaries. In Section 3, we prove the existence of solitary waves. In Section 4, we construct cross-constrained variational problems and generated invariant flows. Furthermore, we derive a sharp criterion of global existence and blowup of solutions. In Section 5, the strong instability of standing waves, via a new compactness and generated invariant flows, is showed.

**Notation.** Throughout this paper, we denote the Lebesgue norms $\|v\|_{L^2(\mathbb{R}^N)}$ by $\|v\|_2$. The symbol $c$ denote various positive constants, the exact value of which is not essential to the analysis.

## 2 | PRELIMINARIES

First, we state the local well-posedness of the Cauchy problem (1.1)-(1.2). More precisely,
**Proposition 2.1.** For any $u_0 \in H^1(\mathbb{R}^N)$, there exists $T = T_{max}(u_0) \in (0, \infty]$ and a unique solution $u \in C((0,T); H^1(\mathbb{R}^N))$ of the Cauchy problem (1.1)-(1.2) such that either $T = \infty$ (global existence) or $T < \infty$ and $\lim_{t\to T} \|u(t)\|_{H^1} = \infty$ (finite time blowup). Moreover, the solution $u(t)$ satisfies the conservations of mass and energy

$$\|u(t)\|_2^2 = \|u_0\|_2^2, \quad E(u(t)) = E(u_0) \tag{2.1}$$

for all $t \in (0, T]$, where the energy $E$ is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx. \tag{2.2}$$

Next, we define a functional space

$$\Sigma := \{v \in H^1(\mathbb{R}^N) : xv \in L^2(\mathbb{R}^N)\},$$

which is more smaller than $H^1(\mathbb{R}^N)$. Then the variance

$$J(t) = \int_{\mathbb{R}^N} |x|^2|u(t)|^2dx$$

is well defined. Thus we have the virial theorem.
**Proposition 2.2.** Suppose that $1 + \frac{4-2b}{N} \le p < 2^* - 1$. Let $u_0 \in \Sigma$ and $u(t)$ be a solution of the Cauchy problem (1.1)-(1.2). Then the virial identity is given by

$$J''(t) = 8\|\nabla u\|_2^2 - \frac{8\theta}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx, \tag{2.3}$$

where

$$\theta = \frac{N(p-1)}{2} + b. \tag{2.4}$$

Note that $\theta \ge 2$ for $1 + \frac{4-2b}{N} \le p < 2^* - 1$.

Finally, we recall the following weakly sequential continuity and symmetric rearrangement inequalities.
**Lemma 2.3.** ([10,11,13]) Assume $N \ge 1$, $0 < b < \min\{2, N\}$, $1 + \frac{4-2b}{N} \le p < 2^* - 1$. Let $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ be such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} |x|^{-b}|v_n|^{p+1}dx \to \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx \quad as \quad n \to \infty. \tag{2.5}$$

**Lemma 2.4.**([14,16]) Let $v^*$ be the Schwarz symmetric rearrangement of $v$, then we have

$$\|\nabla v^*\|_2^2 \le \|\nabla v\|_2^2, \quad \|v^*\|_2^2 \le \|v\|_2^2, \tag{2.6}$$

$$\int_{\mathbb{R}^N} |x|^{-b}|v^*|^{p+1}dx > \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx. \tag{2.7}$$

By a standing wave, we mean a solution of (1.1) with the form $u(t,x) = e^{i\omega t}\phi_\omega(x)$, where $\omega \in \mathbb{R}$ is a frequency, and $\phi_\omega \in H^1(\mathbb{R}^N)\backslash\{0\}$ is a solution of the stationary equation

$$-\Delta\phi_\omega + \omega\phi_\omega - |x|^{-b}|\phi_\omega|^{p-1}\phi_\omega = 0, \ \ x \in \mathbb{R}^N. \tag{2.8}$$

In order to prove the instability of solitary waves for (1.1), we need to introduce the following functionals:

$$I(v) = \frac{1}{2}(\|\nabla v\|_2^2 + \omega\|v\|_2^2) - \frac{1}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx, \tag{2.9}$$

$$S(v) = \|\nabla v\|_2^2 + \omega\|v\|_2^2 - \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx, \tag{2.10}$$

$$P(v) = \|\nabla v\|_2^2 - \frac{\theta}{p+1} \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx. \tag{2.11}$$

Here $\theta$ is defined by (2.4). It's clear that these functionals are well defined based on Sobolev embedding theorem.

## 3 | EXISTENCE OF SOLITARY WAVES

We consider a constrained variational problem

$$d_\omega = \inf_{\{v\in H^1(\mathbb{R}^N)\setminus\{0\}, S(v)=0\}} I(v). \tag{3.1}$$

Then we have the following result.

**Lemma 3.1.** Let $1 + \frac{4-2b}{N} \le p < 2^* - 1$ and $\omega > 0$, then $d_\omega > 0$ and

$$d_\omega = \min_{\{v\in H^1(\mathbb{R}^N)\setminus\{0\}, S(v)=0\}} I(v). \tag{3.2}$$

*Proof.* For $\omega > 0$, there exists a $v \in H^1(\mathbb{R}^N)\setminus\{0\}$ such that $S(v) = 0$. Then one has

$$\|\nabla v\|_2^2 + \omega\|v\|_2^2 = \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx.$$

Using the following Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx \le c\|\nabla v\|_2^\theta \|v\|_2^{p+1-\theta}$$

together with the Young's inequality, it follows that

$$\|\nabla v\|_2^2 + \omega\|v\|_2^2 \le c(\|\nabla v\|_2^2 + \omega\|v\|_2^2)^{\frac{p+1}{2}}.$$

Thus $p > 1$ yields that

$$\|\nabla v\|_2^2 + \omega\|v\|_2^2 \ge c > 0. \tag{3.3}$$

On the other hand, we have

$$I(v) = \frac{p-1}{2(p+1)}(\|\nabla v\|_2^2 + \omega\|v\|_2^2). \tag{3.4}$$

Combine (3.3) and (3.4), we see that $I(v) \ge c > 0$. This implies that $d_\omega > 0$.

Let $\{v_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of (3.1), then we have $S(v_n) = 0$ and $I(v_n) \to d_\omega$ as $n \to \infty$. From (3.4), it follows that

$$\|\nabla v_n\|_2^2 + \omega\|v_n\|_2^2 \le \frac{2(p+1)}{p-1} d_\omega + c. \tag{3.5}$$

We deduce from (3.3) and (3.5) that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is a bounded in $H^1(\mathbb{R}^N)$. Thus there exits a subsequence $\{v_n\}_{n\in\mathbb{N}}$ (here we still denote a subsequence $\{v_{n_k}\}_{n_k\in\mathbb{N}}$ by $\{v_n\}_{n\in\mathbb{N}}$ for simplify) and a function $\phi_\omega(x) \in H^1(\mathbb{R}^N)\setminus\{0\}$ such that $v_n \rightharpoonup \phi_\omega(x)$. From weakly lower semicontinuity, it follows that

$$\|\nabla\phi_\omega(x)\|_2^2 + \omega\|\phi_\omega(x)\|_2^2 \le \liminf_{n\to\infty}(\|\nabla v_n\|_2^2 + \omega\|v_n\|_2^2). \tag{3.6}$$

Lemma 2.3 implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-b}|v_n|^{p+1}dx = \int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega(x)|^{p+1}dx. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$S(\phi_\omega(x)) \le \liminf_{n\to\infty} S(v_n) = 0$$

and

$$\begin{aligned} I(\phi_\omega(x)) = \lim_{n\to\infty} I(v_n) &= \lim_{n\to\infty} \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |x|^{-b}|v_n|^{p+1}dx \\ &= \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega(x)|^{p+1}dx \\ &= d_\omega. \end{aligned}$$

Then we claim that $S(\phi_\omega(x)) = 0$. Indeed, assume $S(\phi_\omega(x)) < 0$, let $\phi_\omega^\lambda = \lambda\phi_\omega$ for all $\lambda > 0$, then

$$S(\phi_\omega^\lambda) = \lambda^2(\|\nabla\phi_\omega\|_2^2 + \omega\|\phi_\omega\|_2^2) - \lambda^{p+1}\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega|^{p+1}dx. \tag{3.8}$$

Since $p > 1$, thus there exists $0 < \lambda_* < 1$ satisfies $S(\phi_\omega^{\lambda_*}) = 0$. We deduce from (3.1) that

$$\begin{aligned} d_\omega \le I(\phi_\omega^{\lambda_*}) &= \frac{p-1}{2(p+1)}\lambda_*\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega|^{p+1}dx \\ &< \frac{p-1}{2(p+1)}\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega|^{p+1}dx \\ &= d_\omega. \end{aligned}$$

This is a contradiction. Thus $\phi_\omega(x)$ is a minimizer of (3.2).

**Lemma 3.2.** Assume $\phi_\omega(x)$ is a minimizer of (3.2), then $\phi_\omega(x)$ is also a solution of (2.8).

*Proof.* Since $\phi_\omega(x)$ is a minimizer of (3.2), then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $\mu S'(\phi_\omega(x)) = I'(\phi_\omega(x))$. It follows that

$$\mu\langle S'(\phi_\omega(x)), \phi_\omega(x)\rangle = \langle I'(\phi_\omega(x)), \phi_\omega(x)\rangle = S(\phi_\omega(x)) = 0$$

From $\phi_\omega(x) \ne 0$, $S(\phi_\omega(x)) = 0$ and (3.8) , one has

$$\begin{aligned} \langle S'(\phi_\omega(x)), \phi_\omega(x)\rangle &= \partial_\lambda S(\lambda\phi_\omega(x))\Big|_{\lambda=1} \\ &= 2(\|\nabla\phi_\omega(x)\|_2^2 + \omega\|\phi_\omega(x)\|_2^2) - (p+1)\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega(x)|^{p+1}dx \\ &= -(p-1)\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega(x)|^{p+1}dx \\ &< 0. \end{aligned}$$

Therefore, we have $\mu = 0$ and $\phi_\omega(x)$ is also a solution of (2.8).

**Lemma 3.3.** Every minimizer of Lemma 3.1 is of the form $u(t,x) = e^{i\vartheta}\phi_\omega(x)$, where $\vartheta \in \mathbb{R}$ and $\phi_\omega(x)$ is a positive, spherically symmetric and radially decreasing real-valued function.

*Proof.* Since $\|\nabla(|u|)\|_2^2 \le \|\nabla u\|_2^2$, it follows that

$$I(|u|) \le I(u), \quad S(|u|) \le S(u) = 0.$$

Thus we deduce that $|u|$ is also a minimizer of (3.1) and

$$\|\nabla(|u|)\|_2^2 = \|\nabla u\|_2^2. \tag{3.9}$$

From Lemma 3.2, we deduce from a standard elliptic regularity that $u(x) \in C^1(\mathbb{R}^N, \mathbb{C})$. Using the strong maximum principle, we get $|u| > 0$ and $u(x) \in C^1(\mathbb{R}^N, \mathbb{C}\backslash\{0\})$.

Put $\varphi(x) := \frac{u(x)}{|u(x)|}$, then we have $|\varphi(x)| = 1$ and

$$\begin{aligned} \nabla u &= |u|\nabla\varphi + \varphi\nabla(|u|) \\ &= \varphi[|u|\bar{\varphi}\nabla\varphi + |\nabla u|]. \end{aligned}$$

Therefore, we get $|\nabla u|^2 = |\nabla\varphi|^2|u|^2 + |\nabla|u||^2$. Since (3.9), we get $\|\varphi\nabla u\|_2 = 0$, which implies that $|\nabla\varphi| = 0$. Hence $\varphi$ is a constant with $|\varphi| = 1$. We see that there exists a $\vartheta \in \mathbb{R}$ such that $u(x) = e^{i\vartheta}\phi_\omega(x)$, where $\phi_\omega(x) = |u(x)| > 0$.

Let $\phi_\omega^{n*}$ be the Schwartz symmetrization of $\phi_\omega^n(x)$. From Lemma 2.4, we have

$$S(\phi_\omega^{n*}) \le S(\phi_\omega^n(x)) = 0, \quad I(\phi_\omega^{n*}) \le I(\phi_\omega^n(x)).$$

Thus there exists $0 < \lambda_1 \le 1$ such that $\psi_\omega^n(x) = \lambda_1\phi_\omega^{n*}(x)$ and satisfies $S(\psi_\omega^n(x)) = 0$ and $I(\psi_\omega^n(x)) = d_\omega$. We deduce that $\{\psi_\omega^n(x)\}_{n\in\mathbb{N}}$ is also a minimizing sequence of (3.2). It is clear that $\{\psi_\omega^n(x)\}_{n\in\mathbb{N}}$ is bounded, then there exist a positive, spherically symmetric and radially decreasing real function $\phi_\omega(x) \in H^1(\mathbb{R}^N)$ such that $\psi_\omega^n(x) \rightharpoonup \phi_\omega(x)$. By similar proofs of Lemma 3.1, we also prove that $\phi_\omega(x)$ is also a minimizer of (3.2).

## 4 | SHARP CONDITION OF GLOBAL EXISTENCE

Here we define another variational problem

$$d_M = \inf_{v \in M} I(v), \tag{4.1}$$

where

$$M = \{v \in H^1(\mathbb{R}^N), P(v) = 0, S(v) < 0\}.$$

Then we have the following results.

**Lemma 4.1.** Let $1 + \frac{4-2b}{N} \le p < 2^* - 1$ and $\omega > 0$. Then

(i) The set $M$ is not empty;

(ii) $d_M > 0$ for all $v \in M$.

*Proof.* (i) For (2.8), multiplying $\bar{v}$ and integrating with respect to $x$ on $\mathbb{R}^N$, one has $S(v) = 0$. Moreover, multiplying $x \cdot \nabla \bar{v}$ and integrating with respect to $x$ on $\mathbb{R}^N$, it follows that

$$\frac{2-N}{2}\|\nabla v\|_2^2 - \frac{N}{2}\omega\|v\|_2^2 + \frac{N-b}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx = 0. \tag{4.2}$$

Since $S(v) = 0$, by (4.2), we see that $P(v) = 0$. Set

$$v_\lambda(x) = \lambda^{\frac{2-b}{p-1}} v(\lambda x) \quad\quad for \quad \lambda > 0.$$

Put $\alpha = \frac{N+2-(N-2)p-2b}{p-1}$, $\beta = \frac{N+4-Np-2b}{p-1}$. From $1 + \frac{4-2b}{N} \le p < 2^* - 1$, we have

$$\alpha > 0; \quad \beta \le 0; \quad \alpha = \beta + 2.$$

In addition,

$$S(v_\lambda) = \lambda^\alpha(\|\nabla v\|_2^2 - \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx) + \lambda^\beta\omega\|v\|_2^2.$$

$$P(v_\lambda) = \lambda^\alpha(\|\nabla v\|_2^2 - \frac{\theta}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx).$$

Thus $S(v) < 0$ implies that there exists a unique $\lambda_* > 1$ such that $S(v_{\lambda_*}) < 0$. On the other hand, from $P(v) = 0$, we have $P(v_{\lambda_*}) = 0$ for all $\lambda_* > 1$. Therefore $v_{\lambda_*} \in M$.

(ii) From $S(v) < 0$, we have $v \neq 0$. Since $P(v) = 0$, then it follows from (2.9) and (2.11) that

$$I(v) = \frac{\theta - 2}{2\theta}\|\nabla v\|_2^2 + \frac{1}{2}\omega\|v\|_2^2. \tag{4.3}$$

From $\theta \ge 2$, $v \neq 0$, $\omega > 0$ and (4.3), we see that $I(v) \ge 0$ for all $v \in M$. Thus $d_M \ge 0$. It is clear that there must divide into two cases.

(a) $\theta > 2$ for the case $1 + \frac{4-2b}{N} < p < 2^* - 1$,

(b) $\theta = 2$ for the case $p = 1 + \frac{4-2b}{N}$.

First, we deal with the case (a). By $S(v) < 0$, we have

$$\|\nabla v\|_2^2 + \omega\|v\|_2^2 < \int_{\mathbb{R}^N} |x|^{-b}|v|^{p+1}dx < c(\|\nabla v\|_2^2 + \omega\|v\|_2^2)^{\frac{p+1}{2}}. \tag{4.4}$$

Then (4.4) implies that

$$\|\nabla v\|_2^2 + \omega\|v\|_2^2 \ge c > 0. \tag{4.5}$$

Therefore it follows from (4.3) and (4.5) that $I(v) \ge c > 0$. By (4.1), we get $d_M > 0$ for the case (a).

Next, we treats the case (b). From (4.3), we have

$$I(v) = \frac{1}{2}\omega\|v\|_2^2. \tag{4.6}$$

If $d_M = 0$, then from (4.6), we see that there exists a sequence $\{v_n\} \subset M$ such that $Q(v_n) = 0$, $S(v_n) < 0$ and $\lim_{n\to\infty} I(v_n) = 0$. By (4.6), then

$$\lim_{n\to\infty} \|v_n\|_2^2 = 0.$$

Since $S(v_n) < 0$, it follows from Gagliardo-Nirenberg inequality that

$$\|\nabla v_n\|_2^2 + \omega\|v_n\|_2^2 < c\|\nabla v_n\|_2^2(\omega\|v_n\|_2^2)^{\frac{p-1}{2}}. \tag{4.7}$$

For $n$ sufficiently large, we have

$$\|\nabla v_n\|_2^2 + \omega\|v_n\|_2^2 \geq c\|\nabla v_n\|_2^2(\omega\|v_n\|_2^2)^{\frac{p-1}{2}}. \tag{4.8}$$

Thus (4.7) and (4.8) are contradictory. Then we get $d_M > 0$ for the case (b). This completes this proof.

Take

$$d = \min\{d_\omega, d_M\}, \tag{4.9}$$

then the following theorem stands.

**Theorem 4.2.** Suppose $1 + \frac{4-2b}{N} \leq p < 2^* - 1$ and $\omega > 0$, then $d > 0$.

*Proof.* Lemma 3.1 and Lemma 4.1 imply that $d > 0$.

Now we define the following sets

$$K_1 = \{v \in H^1(\mathbb{R}^N), I(v) < d, P(v) < 0, S(v) < 0\},$$
$$K_2 = \{v \in H^1(\mathbb{R}^N), I(v) < d, P(v) > 0, S(v) < 0\},$$
$$K_3 = \{v \in H^1(\mathbb{R}^N), I(v) < d, S(v) < 0\},$$
$$K_4 = \{v \in H^1(\mathbb{R}^N), I(v) < d, S(v) > 0\}.$$

Then the following results hold.

**Theorem 4.3.** Assume $1 + \frac{4-2b}{N} \leq p < 2^* - 1$ and $\omega > 0$, then $K_j, j = 1, 2, 3, 4$ is invariant under the flow generated by the Cauchy problem (1.1)-(1.2). More precisely, if $u_0 \in K_j$, then the corresponding solution $u(t, x)$ of the Cauchy problem (1.1)-(1.2) also satisfies $u(t, x) \in K_j$ for any $t \in [0, T)$.

*Proof.* We prove only for $K_1$, the rest of other cases being similar. Let $u_0 \in K_1$, then there exists a unique $u(t) \in C([0, T); H^1(\mathbb{R}^N))$ with $T \leq \infty$ such that $u(t)$ is a solution of the Cauchy problem (1.1)-(1.2). By (1.3), we have

$$I(u(t)) = I(u_0) \tag{4.10}$$

for any $t \in [0, T)$. Therefore $I(u_0) < d$ yields that $I(u(t)) < d$ for any $t \in [0, T)$.

Suppose that there exists $t_1 \in (0, T)$ such that $S(u(t_1)) \geq 0$. Then, by the continuity of the function $t \mapsto S(u(t))$, one has $t_0 \in (0, t_1]$ such that $S(u(t_0)) = 0$. From (4.10), we have $u(t_0) \neq 0$. Then it follows from (3.1) and (4.9) that $I(u(t_0)) \geq d_\omega \geq d$, which contradicts with $I(u(t_0)) < d$ for $t \in [0, T)$. Therefore $S(u(t)) < 0$ for all $t \in [0, T)$.

Suppose that there exists $t_2 \in (0, T)$ such that $P(u(t_2)) \geq 0$. Then, by the continuity of the function $t \mapsto P(u(t))$, one has $t_3 \in (0, t_1]$ such that $P(u(t_3)) = 0$. From $S(u(t_3)) < 0$, we have $u(t_3) \in M$. Then it follows from (4.1) and (4.9) that $I(u(t_3)) \geq d_M \geq d$, which contradicts with $I(u(t_3)) < d$ for $t \in [0, T)$. Therefore $P(u(t)) < 0$ for all $t \in [0, T)$.

**Remark 4.4.** Let $1 + \frac{4-2b}{N} \leq p < 2^* - 1$ and $\omega > 0$. Then

$$\left\{v \in H^1(\mathbb{R}^N)\backslash\{0\}, I(v) < d\right\} = K_1 \cup K_2 \cup K_4.$$

Now we sate the sharp criteria of global existence and blowup.

**Theorem 4.5.** Let $1 + \frac{4-2b}{N} \leq p < 2^* - 1$ and $\omega > 0$. Assume $u_0 \in K_2 \cup K_4$, then the solution $u(t)$ of the Cauchy problem (1.1)-(1.2) exists globally in $t \in (0, \infty)$.

*Proof.* If $u_0 \in K_2 \cup K_4$, then it follows from Theorem 4.3 that the Cauchy problem (1.1)-(1.2) has a unique solution $u(t, x) \in C([0, T); H^1(\mathbb{R}^N))$ and satisfies $u(t, x) \in K_2 \cup K_4$ for $t \in [0, T)$. Fixed $t \in [0, T)$, we denote $u(t, x) = u$.

We first treat the case $u \in K_2$. From $P(u) > 0$ and $I(u) < d$, one has

$$\frac{\theta - 2}{2\theta}\|\nabla u\|_2^2 + \frac{1}{2}\omega\|u\|_2^2 < d. \tag{4.11}$$

In the following we have to divide the proof into two cases: $\theta = 2$ and $\theta > 2$.

For the case $\theta = 2$, from (4.11), it follows that

$$\frac{1}{2}\omega\|u\|_2^2 < d. \tag{4.12}$$

For $\mu > 0$, we set

$$u^\mu = \mu^{\frac{N-b}{p+1}} u(\mu x).$$

By (2.11), we get

$$P(u^\mu) = \mu^{\frac{2(N-b)-(N-2)(p+1)}{p+1}}\|\nabla u\|_2^2 - \frac{2}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx.$$

Since $P(u) > 0$, then there exists $0 < \mu_* < 1$ such that $P(u^{\mu_*}) = 0$. It follows from (2.9) and (2.11) that

$$I(u^{\mu_*}) = \frac{1}{2}\mu_*^{\frac{2(N-b)-N(p+1)}{p+1}}\omega\|u\|_2^2. \tag{4.13}$$

Combine (4.12) with (4.13), one has

$$I(u^{\mu_*}) < \mu_*^{\frac{2(N-b)-N(p+1)}{p+1}} d. \tag{4.14}$$

Now we see $S(u^{\mu_*})$, which has two possibilities. One is $S(u^{\mu_*}) < 0$, the other is $S(u^{\mu_*}) \geq 0$. For the case $S(u^{\mu_*}) < 0$, note that $P(u^{\mu_*}) = 0$, then (4.1) and (4.9) yield that

$$I(u^{\mu_*}) \geq d_M \geq d > I(u).$$

It follows that

$$I(u) - I(u^{\mu_*}) < 0.$$

That is

$$\begin{aligned} I(u) - I(u^{\mu_*}) =& \frac{1}{2}(1 - \mu_*^{\frac{2(N-b)-(N-2)(p+1)}{p+1}})\|\nabla u\|_{L^2}^2 \\ &+ \frac{1}{2}(1 - \mu_*^{\frac{2(N-b)-N(p+1)}{p+1}})\omega\|u\|_{L^2}^2 \\ &< 0. \end{aligned}$$

It follows that

$$\|\nabla u\|_2^2 + \omega\|u\|_2^2 < c. \tag{4.15}$$

For the case $S(u^{\mu_*}) \geq 0$, it follows from (4.14) that

$$I(u^{\mu_*}) - \frac{1}{p+1}S(u^{\mu_*}) < \mu_*^{\frac{2(N-b)-N(p+1)}{p+1}} d.$$

More precisely,

$$\begin{aligned} I(u^{\mu_*}) - \frac{1}{p+1}S(u^{\mu_*}) =& \frac{p-1}{2(p+1)}[\mu_*^{\frac{2(N-b)-(N-2)(p+1)}{p+1}}\|\nabla u\|_2^2 \\ &+ \mu_*^{\frac{2(N-b)-N(p+1)}{p+1}}\omega\|u\|_2^2] \\ &< \mu_*^{\frac{2(N-b)-N(p+1)}{p+1}} d. \end{aligned}$$

It follows that

$$\|\nabla u\|_2^2 + \omega\|\psi\|_2^2 < c. \tag{4.16}$$

By (4.15) and (4.16), we always get $\|u\|_{H^1(\mathbb{R}^N)}$ is bounded for any $t \in [0, T)$. Thus we get that $u(t, x)$ exists globally in $t \in [0, \infty)$.

For the case $\theta > 2$, from (4.11), we always get

$$\|\nabla u\|_2^2 + \omega\|\psi\|_2^2 < c.$$

Therefore we know that $u(t, x)$ exists globally in $t \in [0, \infty)$.

Thus for $u_0 \in K_2$, we prove that the solution $u(t, x)$ of the Cauchy problem (1.1)-(1.2) exists globally in $t \in [0, \infty)$.

Next, we treat the case $u \in K_4$. From $I(u) < d$ and $S(u) > 0$, one has

$$\|\nabla u\|_2^2 + \omega\|u\|_2^2 < \frac{2(p+1)}{p-1}d.$$

Since $1 + \frac{4-2b}{N} \leq p < 2^* - 1$, then $\|u\|_{H^1(\mathbb{R}^N)}$ is bounded for any $t \in [0, \infty)$. Thus we get that $u(t, x)$ exists globally in $t \in [0, \infty)$.

**Theorem 4.6.** Let $1 + \frac{4-2b}{N} \leq p < 2^* - 1$ and $\omega > 0$. Assume $u_0 \in K_1 \cap \Sigma$, then the solution $u(t)$ of the Cauchy problem (1.1)-(1.2) blows up in a finite time.

*Proof.* If $u_0 \in K_1 \cap \Sigma$, then it follows from Theorem 4.3 that the Cauchy problem (1.1)-(1.2) has a unique solution $u(t, x) \in C([0, T); H^1(\mathbb{R}^N))$ and satisfies $u(t) \in K_1$. Fixed $t \in [0, T)$, we denote $u(t, x) = u$.

We set

$$u^\lambda(x) = \lambda^{\frac{N}{2}}u(\lambda x) \quad for \quad \lambda > 0.$$

From (2.9) and (2.11), one has that

$$I(u^\lambda) = \frac{\lambda^2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\omega\|u\|_2^2 - \frac{\lambda^\theta}{p+1}\int_{\mathbb{R}^N}|x|^{-b}|u|^{p+1}dx,$$

$$S(u^\lambda) = \lambda^2\|\nabla u\|_2^2 + \omega\|u\|_2^2 - \lambda^\theta \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx.$$

Since $P(u) < 0$, then

$$S(u^\lambda) < (\frac{\theta\lambda^2}{p+1} - \lambda^\theta) \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx + \omega\|u\|_2^2.$$

By $\theta \geq 2$ and $S(u) < 0$, then there exists $\lambda_0 \in (0,\infty)$ such that $S(u^{\lambda_0}) = 0$.

We consider the function

$$f(\lambda) = I(u^\lambda) - \frac{1}{2}\lambda^2 P(u) = \frac{1}{2}\omega\|u\|_2^2 + \frac{\theta\lambda^2 - 2\lambda^\theta}{2(p+1)} \int_{\mathbb{R}^N} |x|^{-b}|u|^{p+1}dx.$$

It is clear that $f(\lambda)$ attains its maximum at $\lambda = 1$.

From (4.9) and $P(u) < 0$, it follows that

$$d \leq I(u^{\lambda_0}) \leq I(u^{\lambda_0}) - \frac{1}{2}\lambda_0^2 P(u) \leq I(\psi) - \frac{1}{2}P(u). \tag{4.17}$$

Moreover, by $I(u) < d$, (4.17) and the Virial identity (2.3), we have

$$J''(t) = 8P(u) \leq 16[I(u) - d] < 0$$

for all $t \in [0,T)$. Thus, it must be the case $T < \infty$, which implies

$$\lim_{t\to T} ||u||_{H^1(\mathbb{R}^N)} = \infty.$$

**Remark 4.7.** From Remark 4.4, thus Theorem 4.5 and Theorem 4.6 provide a sharp threshold for global existence and blowup of the Cauchy problem (1.1)-(1.2).

## 5 | STRONG INSTABILITY

**Theorem 5.1.** Let $1 + \frac{4-2b}{N} \leq p < 2^* - 1$. Assume $\phi_\omega(x)$ be a ground state of (2.8). Then for any $\omega > 0$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is strongly unstable.

*Proof.* First, by the elliptic regularity theory, we see that $\phi_\omega(x) \in H^1(\mathbb{R}^N)$.

Next, from Lemma 3.2, it follows that $S(\phi_\omega) = 0$ and $P(\phi_\omega) = 0$. For $\lambda > 1$, then $P(\lambda\phi_\omega) < 0$ and $S(\lambda\phi_\omega) < 0$. In addition, the function

$$I(\lambda\phi_\omega) = \frac{\lambda^2}{2}(\|\nabla\phi_\omega\|_2^2 + \omega\|\phi_\omega\|_2^2) - \frac{\lambda^{p+1}}{p+1}\int_{\mathbb{R}^N} |x|^{-b}|\phi_\omega|^{p+1}dx$$

obtain its maximum at $\lambda = 1$. Thus, one has $I(\lambda\phi_\omega) < I(\phi_\omega) = d$ for all $\lambda > 1$. Therefore, we see that $\lambda\phi_\omega \in K_1$ for all $\lambda > 1$.

Now we take $\lambda > 1$ and $\lambda$ is sufficiently close to 1 such that

$$\|\lambda\phi_\omega - \phi_\omega\|_{H^1(\mathbb{R}^N)} = (\lambda - 1)\|\phi_\omega\|_{H^1(\mathbb{R}^N)} < \varepsilon.$$

Then take $u_0 = \lambda\phi_\omega$. From Theorem 4.6, the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) with $u(0,x) = u_0(x)$ blows up in a finite time.

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