Stability and Hopf bifurcation analysis of fractional order nonlinear financial system with time delay

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Abstract

In this paper, we study a fractional order time delay for nonlinear financial system. By using Laplace transformation, stability and Hopf bifurcation analysis have been done for the model. Furthermore, numerical simulation has been carried out for better understanding of our results.

Keywords: Financial System; Stability, Hopf bifurcation; Time-delay; Fractional differential equation ;Caputo fractional derivative.

Mathematics Subject Classification (2010) : $93A30; 37N35; 34D20; 34E05; 34K28$

1. Introduction

Recently, the complex dynamics of financial and economical systems has become an exceptionally noticeable issue in both micro and macro economics which is described in the book by Shone[1]. Ma and Chen[[2],[3]] have studied the bifurcation topological structure and the global complicated character of a kind of nonlinear financial system in both of the papers. There are a lot of ways to control chaos which is described in the papers by Ott et. al[4], Pecora and Carroll[5], Tanaka et. al[6], Zhang et. al[7], Bian et. al[8], Zhang et. al[9], Liu and Cheng[10], Wang and Cheng [11], Rafikov and Balthazar[12], Peruzzi et. al[13], Li et. al[14], Wang et. al[15], Wang et. al[16] .

Fractional differential equations and delay differential equations have evolved as an emerging field of research due to its wide spread applications in various fields as the mentioned by Podlubny[17] and Gyori and Ladas[18].

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Xin and Li[19] have done the 0-1 test for chaos in a fractional order financial system with investment incentive. Some economic process have time-delay characteristics as discussed in the papers by Cesare and Sportelli[20], Fanti and Manfredi[21], Neamtu et. al[22], SenGupta[23], Issaka and SenGupta[24] and it has been observed that ordinary differential equations are not suitable to appropriately study the above problems. Jiang et. al[25] have investigated a double delayed feed back control of nonlinear finance systems.

Several researchers are also working in the field of fractional delay differential equations due to its usefulness in various fields. Zhen et. al[26] proposed a delayed fractional order financial system and discussed the complex dynamical behavior of the financial model by numerical simulation.

In this paper we have discussed the stability and Hopf bifurcation of the nonlinear fractional order time delay model for a financial system. We have taken our model from the paper[26], which describes the financial system of a particular countrty i.e.

$$D^{\alpha_1}x(t) = z + (y(t-\xi) - a)x \tag{1}$$

$$D^{\alpha_2}y(t) = 1 - by - x^2(t-\xi) \tag{2}$$

$$D^{\alpha_3}z(t) = -x(t-\xi) - cz \tag{3}$$

where $\xi > 0$ represents the delay term of the system. $x(t)$, $y(t)$ and $z(t)$ are the interest rate, the investment demand and the price index respectively, a, b, c are the nonnegative constants. a is the saving amount, b is the cost per investment, c is the elasticity of demand of the commercial markets.

Let us consider $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. So the model becomes,

$$D^{\alpha}x(t) = z + (y(t-\xi) - a)x \tag{4}$$

$$D^{\alpha}y(t) = 1 - by - x^2(t-\xi) \tag{5}$$

$$D^{\alpha}z(t) = -x(t-\xi) - cz \tag{6}$$

with the initial functions,
$x(t) = \phi_1(t)$, $y(t) = \phi_2(t)$, $z(t) = \phi_3(t)$, $t \in [-\xi, 0]$,

where D^{α} is the Caputo fractional derivative[17].

The paper is organised as follows. Section 1 is the introduction to the paper. Section 2 contains the stability and Hopf bifurcation analysis of the model. The numerical simulation given in section 3, which illustrates our result. Section 4 contains the conclusion part of the paper.

2. Quantitative Analysis

In this section, we have done stability and bifurcation analysis with time delay, where ξ is selected as a bifurcation parameter of nonlinear financial model (4)-(6).

The equilibrium is obtained by setting $D^\alpha x(t) = D^\alpha y(t) = D^\alpha z(t) = 0$, in the model (4)-(6). So the equilibrium point is $\left(\sqrt{1-b(a+\frac{1}{c})}, a+\frac{1}{c}, -\frac{1}{c}\sqrt{1-b(a+\frac{1}{c})}\right)$.

2.1. Asymptotic stability of the equilibria:

Let us consider the linear transform: $\bar{x} = x - x_0$, $\bar{y} = y - y_0$, $\bar{z} = z - z_0$. After using the above transform in equations (4)-(6), we get the following linearized systems,

$$D^\alpha \bar{x} = \bar{z} + \bar{y}(t-\xi)x_0 + (y_0 - a)\bar{x} \tag{7}$$

$$D^\alpha \bar{y} = -b\bar{y} - 2x_0\bar{x}(t-\xi) \tag{8}$$

$$D^{\alpha_3} \bar{z} = -\bar{x}(t-\xi) - c\bar{z} \tag{9}$$

By taking Laplace transform on both sides of (7)-(9), we get,

$$s^\alpha X_1(s) = s^{\alpha-1}\phi_1(0) + X_3(s) + x_0 e^{-s\xi}X_2(s) + x_0 e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_2(t)dt + (y_0 - a)X_1(s) \tag{10}$$

$$s^\alpha X_2(s) = s^{\alpha-1}\phi_2(0) - bX_2(s) - 2x_0 e^{-s\xi}X_1(s) - 2x_0 e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_1(t)dt \tag{11}$$

$$s^\alpha X_3(s) = s^{\alpha-1}\phi_3(0) - e^{-s\xi}X_1(s) + e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_1(t)dt - cX_3(s) \tag{12}$$

where, $\bar{x} = \phi_1(0)$, $\bar{y} = \phi_2(0)$, $\bar{z} = \phi_3(0)$, $X_1(s) = L[\bar{x}(t)]$, $X_2(s) = L[\bar{y}(t)]$, $X_3(s) = L[\bar{z}(t)]$.

The equation (10)-(12), can be written as,

$$\begin{pmatrix} s^\alpha - y_0 + a & -x_0 e^{-s\xi} & -1 \\ 2x_0 e^{-s\xi} & s^\alpha + b & 0 \\ e^{-s\xi} & 0 & s^\alpha + c \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix} = \begin{pmatrix} s^{\alpha-1}\phi_1(0) + x_0 e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_2(t)dt \\ s^{\alpha-1}\phi_2(0) - 2x_0 e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_1(t)dt \\ s^{\alpha-1}\phi_3(0) - e^{-s\xi}\int_{-\xi}^{0} e^{-s\xi}\phi_1(t)dt \end{pmatrix}$$

The characteristic matrix is

$$\begin{pmatrix} s^\alpha - y_0 + a & -x_0 e^{-s\xi} & -1 \\ 2x_0 e^{-s\xi} & s^\alpha + b & 0 \\ e^{-s\xi} & 0 & s^\alpha + c \end{pmatrix}$$

The characteristic equation is

$$F(s) = P(s) + 2x_0^2 s^\alpha e^{-2s\xi} + s^\alpha e^{-s\xi} + 2x_0^2 c e^{-2s\xi} + b e^{-s\xi} = 0 \tag{13}$$

where $P(s) = s^{3\alpha} + A_1 s^{2\alpha} + A_2 s^{\alpha} + A_3$.

The values of A_1, A_2 and A_3 are

$$\begin{aligned} A_1 &= b + c - y_0 + a, \\ A_2 &= bc - y_0 c - y_0 b + ac + ab, \\ A_3 &= abc - y_0 bc. \end{aligned}$$

Let $s = iv = v\left(cos(\frac{\pi}{2}) + isin(\frac{\pi}{2})\right)$, $v > 0$. By substituting s in (13), we get

$$(B_1 + iC_1) + (B_2 + iC_2)e^{-i2v\xi} + (B_3 + iC_3)e^{-iv\xi} + 2x_0^2 ce^{-i2v\xi} + be^{-iv\xi} = 0$$

where $B_1 = v^{3\alpha} cos\frac{3\alpha\pi}{2} + A_1 v^{2\alpha} cos\alpha\pi + A_2 v^{\alpha} cos\frac{\alpha\pi}{2} + A_3$,

$C_1 = v^{3\alpha} sin\frac{3\alpha\pi}{2} + A_1 v^{2\alpha} sin\alpha\pi + A_2 v^{\alpha} sin\frac{\alpha\pi}{2}$,

$B_2 = 2x_0^2 v^{\alpha} cos\frac{\alpha\pi}{2}$,

$C_2 = 2x_0^2 v^{\alpha} sin\frac{\alpha\pi}{2}$,

$B_3 = 2v^{\alpha} cos\frac{\alpha}{2}$,

$C_3 = 2v^{\alpha} sin\frac{\alpha\pi}{2}$.

So,

$$(B_1+iC_1)+(B_2+iC_2)(cos2v\xi-isin2v\xi)+(B_3+iC_3)(cos2v\xi-isin2v\xi)+b(cosv\xi-isinv\xi) = 0$$

By separating real and imaginary part,

$$\begin{aligned} M_1 cos2v\xi + C_2 sin2v\xi &= -M_2 cosv\xi - (B_1 + C_3 sinv\xi) \\ -M_1 sin2v\xi + C_2 cos2v\xi &= M_2 sinv\xi - (C_1 + C_3 cosv\xi) \end{aligned} \tag{14}$$

Squaring and adding the above equation,

$$M_1^2 + C_2^2 - M_2^2 - B_1^2 - C_3^2 - C_1^2 = (2B_1C_3 - 2M_2C_1)sinv\xi + (2M_2B_1 + 2C_1C_3)cosv\xi \tag{15}$$

where, $M_1 = B_2 + 2x_0^2 c$, $M_2 = B_3 + b$. Since $sinv\xi = \pm\sqrt{1 - cos^2 v\xi}$, we will find following two cases.

Case I: Let $sinv\xi = \sqrt{1 - cos^2 v\xi}$. So (15) becomes,

$$L_1 cos^2 v\xi + L_2 cosv\xi + L_3 = 0, \tag{16}$$

where, $L_1 = 2M_2B_1 + 2C_1C_3 + (2B_1C_1 - 2M_2C_1)^2$

$$L_2 = -2(2M_2B_1 + 2C_1C_3)(M_1^2 + C_1^2 - M_2^2 - B_1^2 - C_3^2 - C_1^2)$$

$$L_3 = (M_1^2 + C_1^2 - M_2^2 - B_1^2 - C_3^2 - C_1^2)^2 - (2B_1C_1 - 2M_2C_1)^2.$$

Let $cosv\xi = f_1(v)$ and $sinv\xi = f_2(v)$ into (15), where $f_1(v)$ and $f_2(v)$ are functions of v. Moreover, $f_1^2(v) + f_2^2(v) = 1$. From $cosv\xi = f_1(v)$, we can obtain,

$$\xi_1 = \frac{1}{v}\left[arccosf_1(v) + 2n\pi\right], \quad n = 0, 1, 2, ... \tag{17}$$

Case II: Let $sinv\xi = -\sqrt{1 - cos^2v\xi}$. Similarly, by substituting $cosv\xi = g_1(v)$, $sinv\xi = g_2(v)$, where $\bar{f}_1(v)$, and $g_2(v)$ are functions of v. As we know, ${g_1}^2(v) + {g_2}^2(v) = 1$. From $g_1(v) = cosv\xi$, we obtain ,

$$\xi_2 = \frac{1}{v}\left[arccosg_2(v) + 2n\pi\right], \quad n = 0, 1, 2, ... \tag{18}$$

Let $\xi_0 = \min\{\xi_1, \xi_2\}$ be the bifuraction point.

Thus we have established the following theorem.

Theorem 2.1.1: When $\xi < \xi_0$, the equilibrium point (x^*, y^*, z^*) of the nonlinear financial model (4)-(6) is asymptotically stable.

2.2. Hopf bifurcation analysis

Differentiate (13) with respect to ξ, we obtain ,

$$\frac{ds}{d\xi} = \frac{4x_0^2s^{\alpha+1}e^{-2s\xi} - s^{\alpha+1}e^{-s\xi} - 4x_0^2cse^{-2s\xi} - bse^{-s\xi}}{P_1'(s) + 2x_0^2\alpha s^{\alpha-1}e^{-2s\xi} - 4x_0^2\xi s^{\alpha}e^{-2s\xi} + \alpha s^{\alpha-1}e^{-s\xi} - \xi s^{\alpha}e^{-s\xi} - 4x_0^2c\xi e^{-2s\xi} - b\xi e^{-s\xi}} \tag{19}$$

From (17), we consider the numerator term ,

$$\begin{aligned}
& 4x_0^2s^{\alpha+1}e^{-2s\xi} - s^{\alpha+1}e^{-s\xi} - 4x_0^2cse^{-2s\xi} - bse^{-s\xi} \\
= \; & 4x_0^2v^{\alpha+1}\left(cos\frac{(\alpha+1)\pi}{2}cos2v\xi - icos\frac{(\alpha+1)\pi}{2}sin2v\xi + isin\frac{(\alpha+1)\pi}{2}cos2v\xi + sin\frac{(\alpha+1)\pi}{2}sin2v\xi\right) \\
- \; & v^{\alpha+1}\left(cos\frac{(\alpha+1)\pi}{2}cosv\xi - icos\frac{(\alpha+1)\pi}{2}sinv\xi + isin\frac{(\alpha+1)\pi}{2}cosv\xi + sin\frac{(\alpha+1)\pi}{2}sinv\xi\right) \\
- \; & i4x_0^2cvcos2v\xi - 4x_0^2cvsin2v\xi - ibvcosv\xi - bvsinv\xi \\
= \; & M_1 + iM_2
\end{aligned}$$

where $M_1 = 4x_0^2v^{\alpha+1}cos\left(\frac{(\alpha+1)\pi}{2} - 2v\xi\right) - v^{\alpha+1}cos\left(\frac{(\alpha+1)\pi}{2} - v\xi\right) - 4x_0^2(vsin2v\xi + bvsinv\xi)$

$M_2 = 4x_0^2v^{\alpha+1}sin\left(\frac{(\alpha+1)\pi}{2} - 2v\xi\right) - v^{\alpha+1}sin\left(\frac{(\alpha+1)\pi}{2} - v\xi\right) - 4x_0^2(vcos2v\xi + bvcosv\xi).$

Let us consider the denominator term from (19),

$$
\begin{aligned}
& P_1'(s) + 2x_0^2\alpha s^{\alpha-1}e^{-2s\xi} - 4x_0^2\xi s^{\alpha}e^{-2s\xi} + \alpha s^{\alpha-1}e^{-s\xi} - \xi s^{\alpha}e^{-s\xi} - 4x_0^2c\xi e^{-2s\xi} - b\xi e^{-s\xi} \\
= \ & 3\alpha v^{3\alpha-1}\left(cos\frac{(3\alpha-1)\pi}{2} + isin\frac{(3\alpha-1)\pi}{2}\right) + 2\alpha A_1 v^{2\alpha-1}\left(cos\frac{(2\alpha-1)\pi}{2} + isin\frac{(2\alpha-1)\pi}{2}\right) \\
+ \ & A_2\alpha v^{\alpha-1}\left(cos\frac{(\alpha-1)\pi}{2} + isin\frac{(\alpha-1)\pi}{2}\right) \\
+ \ & 2x_0^2\alpha v^{\alpha-1}\left(cos\frac{(\alpha-1)\pi}{2}cos2v\xi - icos\frac{(\alpha-1)\pi}{2}sin2v\xi + isin\frac{(\alpha-1)\pi}{2}cos2v\xi + sin\frac{(\alpha-1)\pi}{2}sin2v\xi\right) \\
- \ & 4x_0^2\xi v^{\alpha}\left(cos\frac{\alpha\pi}{2}cos2v\xi - icos\frac{\alpha\pi}{2}sin2v\xi + isin\frac{\alpha\pi}{2}cos2v\xi + sin\frac{\alpha\pi}{2}sin2v\xi\right) \\
+ \ & \alpha v^{\alpha-1}\left(cos\frac{(\alpha-1)\pi}{2}cosv\xi - icos\frac{(\alpha-1)\pi}{2}sinv\xi + isin\frac{(\alpha-1)\pi}{2}cosv\xi + sin\frac{(\alpha-1)\pi}{2}sinv\xi\right) \\
- \ & \xi v^{\alpha}\left(cos\frac{\alpha\pi}{2}cosv\xi - icos\frac{\alpha\pi}{2}sinv\xi + isin\frac{\alpha\pi}{2}cosv\xi + sin\frac{\alpha\pi}{2}sinv\xi\right) \\
- \ & 4x_0^2c\xi(cos2v\xi - isin2v\xi) - b\xi(cosv\xi - isinv\xi) \\
= \ & N_1 + iN_2
\end{aligned}
$$

where

$$
\begin{aligned}
N_1 = \ & 3\alpha v^{3\alpha-1}cos\frac{(3\alpha-1)\pi}{2} + 2\alpha A_1 v^{2\alpha-1}cos\frac{(2\alpha-1)\pi}{2} + A_2\alpha v^{\alpha-1}cos\frac{(\alpha-1)\pi}{2} \\
+ \ & 2x_0^2\alpha v^{\alpha-1}cos\left(\frac{(\alpha-1)\pi}{2} - 2v\xi\right) - 4x_0^2\xi v^{\alpha}cos\left(\frac{\alpha\pi}{2} - 2v\xi\right) + \alpha v^{\alpha-1}cos\left(\frac{(\alpha-1)\pi}{2} - v\xi\right) \\
- \ & \xi v^{\alpha}cos\left(\frac{\alpha\pi}{2} - v\xi\right) - 4x_0^2c\xi cos2v\xi - b\xi cosv\xi
\end{aligned}
$$

$$
\begin{aligned}
N_2 = \ & 3\alpha v^{3\alpha-1}sin\frac{(3\alpha-1)\pi}{2} + 2\alpha A_1 v^{2\alpha-1}sin\frac{(2\alpha-1)\pi}{2} + A_2\alpha v^{\alpha-1}sin\frac{(\alpha-1)\pi}{2} \\
+ \ & 2x_0^2\alpha v^{\alpha-1}sin\left(\frac{(\alpha-1)\pi}{2} - 2v\xi\right) - 4x_0^2\xi v^{\alpha}sin\left(\frac{\alpha\pi}{2} - 2v\xi\right) + \alpha v^{\alpha-1}sin\left(\frac{(\alpha-1)\pi}{2} - v\xi\right) \\
- \ & \xi v^{\alpha}sin\left(\frac{\alpha\pi}{2} - v\xi\right) + 4x_0^2c\xi sin2v\xi + b\xi sinv\xi.
\end{aligned}
$$

Thus, (19) can be written as,

$$
\begin{aligned}
\frac{ds}{d\xi}|_{\xi=\xi_0, v=v_0} &= \frac{M_1 + iM_2}{N_1 + iN2} \\
\Rightarrow Re\left(\frac{ds}{d\xi}\right)_{\xi=\xi_0, v=v_0} &= \frac{M_1N_1 + M_2N_2}{M_1^2 + N_2^2}
\end{aligned}
$$

Thus the following result is established.

Theorem 2.2.1 When $\xi = \xi_0$, the nonlinear finance model (4)-(6) has a Hopf bifurcation.

3. Numerical Simulations

In this section we have done the numerical simulation, which illustrates our result. Plots are done by using MATLAB software. Figure 1 shows the phase diagram of fractional order system ((4)-(6)) is asymptotically stable and have a Hopf bifurcation when $\xi = 0.8$. Figure 2 shows the phase diagram of fractional order system ((4)-(6)) is asymptotically stable and have a Hopf bifurcation when $\xi = 1.1$.

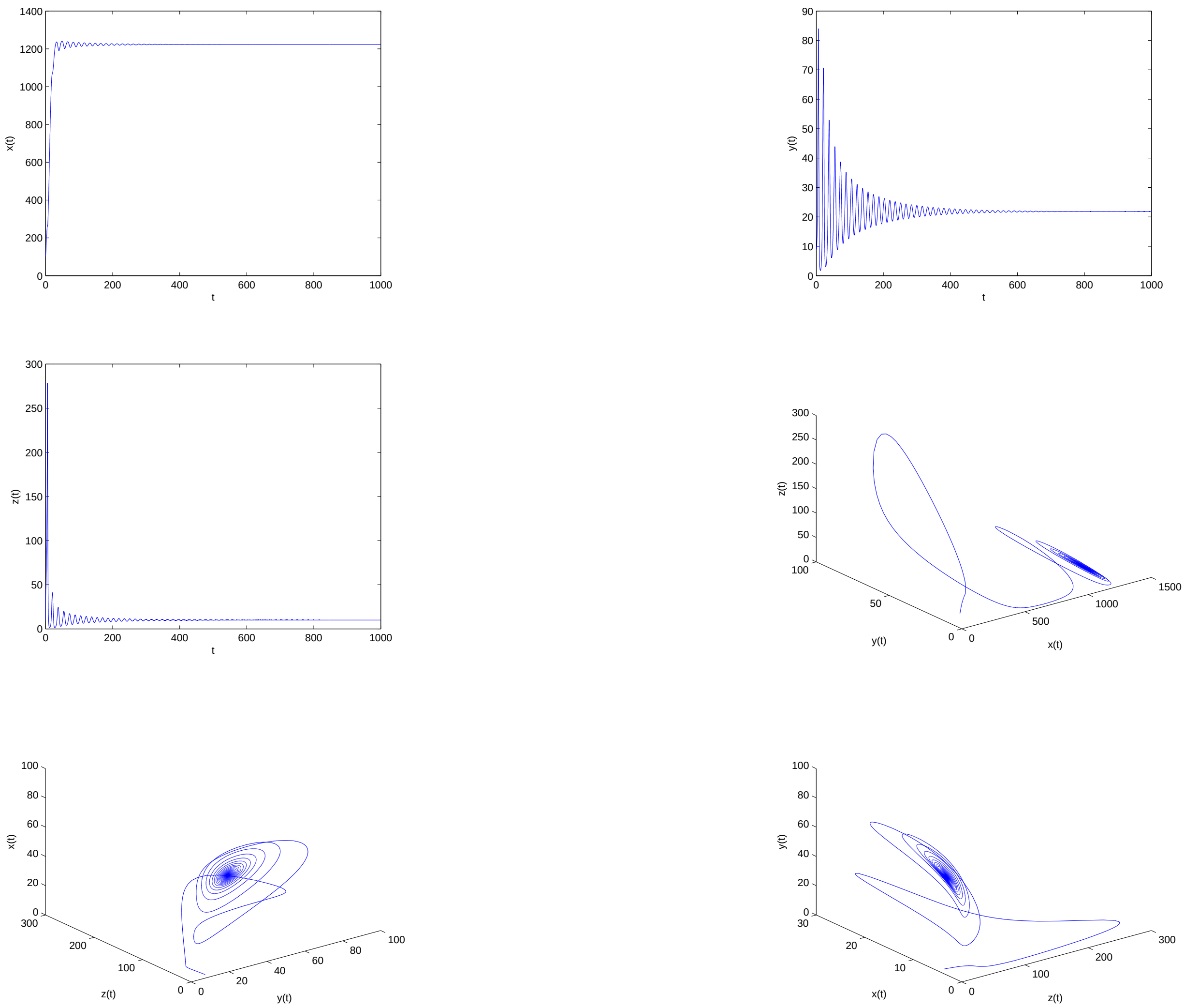

Figure 1: The phase diagrams of the system (4)-(6) undergoes Hopf bifurcation when $\xi = 0.8$

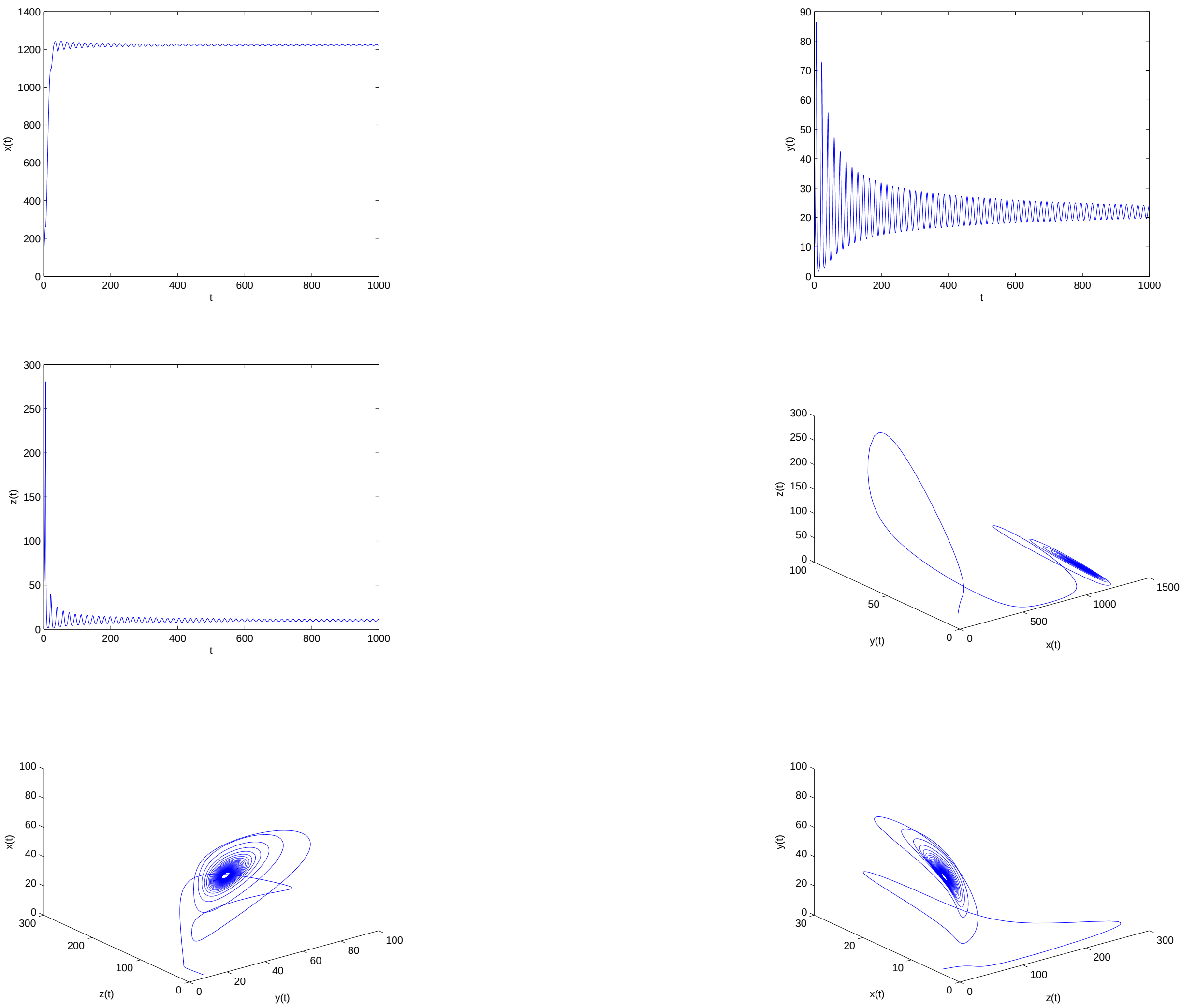

Figure 2: The phase diagrams of the system (4)-(6) undergoes Hopf bifurcation when $\xi = 1.1$

4. Conclusion

In this paper,we have examined a fractional order time delay nonlinear financial model. The proposed model focuses on the dynamic interaction between interest rate, investment demand and price index. Some conditions on stability and Hopf bifurcation have been derived for the model by using Laplace transformation. Further numerical simulation has been carried out for the purpose of better understanding of our results.

References

[1] R. Shone, *Economic dynamics, Cambridge*, Cambridge University Press, 2002.

[2] J. H. Ma , Y. S. Chen, *Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (i)*, AppliedMathematics andMechanics, vol. 22, no. 11, pp. 1240–1251, 2001.

[3] J. H. Ma , Y. S. Chen, *Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (ii)* Applied Mathematics and Mechanics, vol. 22, pp. 1375–1382, 2001.

[4] E. Ott, C. Grebogi, J. A. Yorke, *Controlling chaos*, Physical Review Letters, vol. 64, no. 11, pp. 1196–1199, 1990.

[5] L. M. Pecora , T. L. Carroll, *Synchronization in chaotic systems*, Physical Review Letters, vol. 64, no. 8, pp. 821–824, 1990.

[6] K. Tanaka, T. Ikeda, H. O. Wang, *A unified approach to controlling chaos via an lmi-based fuzzy control systemdesign IEEE Transactions on Circuits and Systems I*, Fundamental eory and Applications, vol. 45, no. 10, pp. 1021–1040, 1998.

[7] T. Zhang, X. Meng, Y. Song, T. Zhang, *A stage-structured predator-prey SI model with disease in the prey and impulsive effects*, Mathematical Modelling and Analysis, vol. 18, no. 4, pp. 505–528, 2013.

[8] F. Bian,W. Zhao, Y. Song, R. Yue, *Dynamical analysis of a class of prey-predator model with beddington-deangelis functional response, stochastic perturbation, and impulsive toxicant input*, Complexity, vol. 2017, Article ID 3742197, 18 pages, 2017.

[9] T. Zhang,W. Ma, X. Meng, T. Zhang, *Periodic solution of a prey-predator model with nonlinear state feedback control*, Applied Mathematics and Computation, vol. 266, pp. 95–107, 2015.

[10] H. Liu , H. Cheng, *Dynamic analysis of a prey–predator model with state-dependent control strategy and square root response function*, Advances in Difference Equations, vol. 2018, no. 63, 2018.

[11] J.Wang, H. Cheng, H. Liu, Y.Wang, *Periodic solution and control optimization of a prey-predator model with two types of harvesting*, Advances in Difference Equations, vol. 2018, no. 1, 2018.

[12] M. Rafikov , J. M. Balthazar, *On control and synchronization in chaotic and hyperchaotic systems via linear feedback control*, Communications in Nonlinear Science and Numerical Simulation, vol. 13, no. 7, pp. 1246–1255, 2008.

[13] N. J. Peruzzi, F. R. Chavarette, J. M. Balthazar, A. M. Tusset, A. L. Perticarrari, R. M. Brasil, *The dynamic behavior of a parametrically excited time-periodic mems taking into account parametric errors*, Journal of Vibration and Control, vol. 22, no. 20, pp. 4101–4110, 2016.

[14] Y. Li, H. Cheng, J. Wang, Y. Wang, *Dynamic analysis of unilateral diffusion Gompertz model with impulsive control strategy*, Advances in Difference Equations, vol. 2018, no. 32, 2018.

[15] J. Wang, H. Cheng, X. Meng, B. G. Pradeep, *Geometrical analysis and control optimization of a predator-prey model with multi state-dependent impulse*, Advances in Difference Equations, vol. 2017, no. 252, 2017.

[16] J. Wang, H. Cheng, Y. Li, X. Zhang, *The geometrical analysis of a predator-prey model with multi-state dependent impulses*, Journal of Applied Analysis and Computation, vol. 8, no. 2, pp. 427–442, 2018.

[17] I. Podlubny, *Fractional differential equations*, San Diego: Academic Press; 1999.

[18] I. Gyori and G. Ladas, *Oscillation theory of delay differential equation with applications*, Clarendon Press, Oxford (1991).

[19] B. Xin and Y. Li, *0-1 Test for Chaos in a Fractional Order Financial System with Investment Incentive*, Abstract and Applied Analysis, vol. 2013, Article ID 876298, 10 pages, 2013.

[20] L. D. Cesare , M. Sportelli, *A dynamic IS-LM model with delayed taxation revenues*, Chaos, Solitons and Fractals, vol. 25, no. 1, pp. 233–244, 2005.

[21] L. Fanti and P. Manfredi, *Chaotic business cycles and fiscal policy an IS-LM model with distributed tax collection lags*, Chaos, Solitons and Fractals, vol. 32, no. 2, pp. 736–744, 2007.

[22] M. Neamtu, D. Opris, and C. Chilarescu, *Hopf bifurcation in a dynamic IS-LM model with time delay*, Chaos, Solitons and Fractals, vol. 34, no. 2, pp. 519–530, 2007.

[23] I. SenGupta, *Generalized BN-S stochastic volatility model for option pricing*, International journal of eoretical and Applied Finance, Vol.19, no. 02, Article ID 1650014, 2016.

[24] A. Issaka , I. SenGupta, *Analysis of variance based instruments for Ornstein-Uhlenbeck type models: swap and price index*, Annals of Finance, vol. 13, no. 4, pp. 401–434, 2017.

[25] Z. Jiang , Y. Guo , T Zhang, *Double Delayed Feedback Control of a Nonlinear Finance System*,Discrete Dynamics in Nature and Society Vol. 2019, Article ID 7254121, 17 pages .

[26] W. Zhena, H. Xia, S. Guodong, *Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay*, Computers and Mathematics with Applications, Vol. 62, 1531–1539, 2011.