Approximate controllability of nonlocal impulsive neutral integro-differential equations with finite delay

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Abstract

In this paper, we apply the resolvent operator theory and an approximating technique to derive the existence and controllability results for nonlocal impulsive neutral integro-differential equations with finite delay in a Hilbert space. To establish the results, we take the impulsive functions as a continuous function only, and we assume that the nonlocal initial condition is Lipschitz continuous function in the first case and continuous functions only in the second case. The main tools applied in our analysis are semigroup theory, the resolvent operator theory, an approximating technique, and fixed point theorems. Finally, we illustrate the main results with the help of two examples.

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1. Introduction

Let E and V be two Hilbert spaces. Consider the following nonlocal and impulsive neutral integro-differential system with finite delay:

$$\begin{cases} \frac{d}{dt}\left[w(t)+h(t,w_t)\right]+Aw(t)= & \int_0^t \eta(t-r)w(r)\,dr+Bv(t) \\ & +g(t,w_t), \quad t\in J=[0,b],\ t\neq t_j, \\ \Delta w|_{t=t_j}= & \mathcal{I}_j(w_{t_j}), \quad j=1,2,\cdots,k, \\ w(t)= & q(t)+\Phi(w)(t), \quad t\in[-\tau,0] \end{cases} \tag{1.1}$$

where $0=t_0<t_1<\ldots<t_k<t_{k+1}=b$; $\tau>0$; $-A$ generates an analytic and compact semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators in a Hilbert space E; $\eta(t)$ is a closed linear operator on $D(A)$ for each $t\geq 0$; the time history function w_t is defined by $w_t(s)=w(t+s)$, $s\in[-\tau,0]$, and belongs to the space $\mathscr{E}=\{w\colon[-\tau,0]\to E \mid w(\cdot)$ is continuous at all point except at a finite number of points s_j at which $w(s_j^+)$ and $w(s_j^-)$ exist and $w(s_j)=w(s_j^-)\}$; $v(\cdot)$ is the control function in a Hilbert space $L^2(J,V)$; the operator $B\colon V\to E$ is linear and continuous; the nonlinear functions $g,\,h\colon[0,b]\times\mathscr{E}\to E$ are continuous; $\mathcal{I}_j\colon\mathscr{E}\to E,\ j=1,2,\ldots,k$ are impulsive functions, $\Delta w(t_j)$ defines the jump of a function w at t_j as $\Delta w(t_j)=w(t_j^+)-w(t_j^-)$; $q\in\mathscr{E}$; and Φ maps continuously

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from the space $PC([-\tau, b], E)$ to $\mathscr{E}$, here $PC([-\tau, b], E) = \{w\colon [-\tau, b] \to E \mid w(t)$ is continuous at $t \neq t_j$, $w(t_j^+)$ and $w(t_j^-)$ both exist, and $w(t_j) = w(t_j^-)\}$.

In recent years, many researchers have paid their attention to the study of neutral integro-differential equations which model many physical phenomena arising in electronics, fluid dynamics, chemical kinetics, etc. In papers [1, 2], the authors have derived the representation of the solutions of integro-differential equations by resolvent operators. The resolvent operator is useful to solve the integro-differential equations in weak as well as strict sense. For the study of abstract integro-differential equations via analytic resolvent operators, we refer readers to the books [3, 4] and the papers [5–11], and references therein.

The concept of controllability was firstly introduced by Kalman in 1960. It is now widely used not only in control theory but in numerous fields such as quantum systems theory, control of electric bulk power systems, reactor control, chemical process control, aerospace engineering, etc. The controllability problem is to find a control function such that the state of the dynamical system can be steered to the required final state. The approximate controllability means that we can steer the system to an arbitrarily small neighborhood of a final state, whereas exact controllability steers to an exact final state. In such a way, the approximate controllability is completely adequate in applications. For readers, we refer to some interesting and important controllability results [5, 12–20] concerning semi-linear or nonlinear differential systems in which semigroup theory and some fixed point theorems are used. In paper [12, 21], the authors studied the approximate controllability of nonlocal and impulsive semilinear system using an approximating technique. Yan and Lu [15] applied an analytic α-resolvent operator to establish the approximate controllability of multi-valued impulsive infinite delay stochastic partial integro-differential equation of fractional order in Hilbert spaces. Jeet [5] applied the resolvent operator theory to establish the approximate controllability of the nonlocal neutral differential equations of finite delay.

Moreover, the theory of impulsive differential equations or inclusions has now become an interesting area of investigation because these equations or inclusions describe the dynamics of the process in which abrupt changes, discontinuous jumps occur at certain moments of time such as shocks and natural disasters. We refer the readers to the books [22, 23] and some papers [6, 24–27] for the basic results of impulsive differential systems.

In this paper, we apply the resolvent operator theory and an approximating technique to derive the existence and controllability results for nonlocal impulsive neutral integro-differential equations (1.1) with finite delay in a Hilbert space. We use the weaker conditions on the impulsive functions and the nonlocal initial condition to derive the results, that is, we take the impulsive functions as a continuous function only, and we assume that the nonlocal initial condition is Lipschitz continuous function in the first case and continuous functions only in the second case. To the best of our information, the approximate controllability of nonlocal and impulsive neutral integro-differential equations (1.1) with a finite delay has not been yet studied using such weaker conditions.

We organize the rest of the paper as follows: some basic definitions, notations, hypotheses, and preliminary results are introduced in section 2. In section 3, we establish the existence and the controllability results for the system (1.1) using the resolvent operator theory and approximating method. In the final section, two examples are given to demonstrate the application of our main results.

2. Preliminaries

In this section, we first introduce some notations, basic definitions, hypotheses, and preliminary results that will be required throughout the paper.

Let $\mathcal{L}(E)$ be a Banach space of bounded linear operators from E into itself with operator norm. Assume that the operator $-A : X \to X$ generates a compact and analytic semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators in a Hilbert space E. We denote the domain $D(A^\beta)$ of A^β by E_β endowed with a norm $\|A^\beta \cdot\|$, here $0 < \beta < 1$.

Definition 2.1. *A one-parameter family $\{\mathcal{R}(t)\}_{t\geq 0}$ in $\mathcal{L}(E)$ is said to be a resolvent operator for the abstract integro-differential Cauchy problem*

$$\begin{cases} \frac{d}{dt}w(t) + Aw(t) = \int_0^t \eta(t-r)w(r)\,dr, \quad t \in J, \\ w(0) = w_0 \in E \end{cases} \tag{2.1}$$

if the following conditions are verified:

(a) The family $\{\mathcal{R}(t)\}_{t\geq 0}$ is strongly continuous, and $\mathcal{R}(0)w = w$ for all $w \in E$

(b) For $w \in D(A)$, $\mathcal{R}(\cdot)w \in C([0,\infty), D(A)) \cap C^1((0,\infty), E)$, and

$$\frac{d}{dt}\mathcal{R}(t)w + A\mathcal{R}(t)w = \int_0^t \eta(t-r)\mathcal{R}(r)w\,dr, \tag{2.2}$$

$$\frac{d}{dt}\mathcal{R}(t)w + \mathcal{R}(t)Aw = \int_0^t \mathcal{R}(t-r)\eta(r)w\,dr, \tag{2.3}$$

for each $t \geq 0$.

For more detail on the theory of resolvent operators, we refer to the papers [11, 28]. We assume the following conditions throughout the paper:

(K1) The operator $-A\colon D(A) \subseteq E \to E$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E and $\rho(-A) \supset \Lambda_\theta = \{\gamma \in \mathbf{C} \setminus \{0\} \colon |\arg(\gamma)| < \theta\}$ and $\|S(\gamma, -A)\| \leq M_0|\gamma|^{-1}$ for some constants $M_0 > 1$, $\theta \in (\pi/2, \pi)$ and for each $\gamma \in \Lambda_\theta$, where $S(\gamma, -A)$ is the resolvent of $-A$.

(K2) The operator $\eta(t)\colon D(\eta(t)) \subseteq E \to E$ is linear and closed with $D(A) \subseteq D(\eta(t))$ for each $t \geq 0$. For any $w \in D(A)$, the function $\eta(\cdot)w$ is strongly measurable on $(0,\infty)$. There is a $\varrho(\cdot) \in L^1_{loc}(\mathbf{R}^+)$ such that $\widehat{\varrho}(\gamma)$ can be obtained for $\mathbf{Re}(\gamma) > 0$ and $\|\eta(t)w\| \leq \varrho(t)\|w\|_1$ for each $t > 0$ and $w \in D(A)$, here $\widehat{\varrho}$ denotes the Laplace transform of ϱ. In addition, the function $\widehat{\eta}\colon \Lambda_{\pi/2} \to \mathcal{L}(D(A), E)$ has an analytical extension (still denoted by $\widehat{\eta}$, here $\widehat{\eta}$ is the Laplace transform of η) to Λ_θ such that $\|\widehat{\eta}(\gamma)w\| \leq \|\widehat{\eta}(\gamma)\|\,\|w\|_1$ for each $w \in D(A)$, and $\|\widehat{\eta}(\gamma)\| \to 0$ as $|\gamma| \to \infty$.

(K3) There is a subspace $X \subseteq D(A)$ that is dense in $D(A)$ and a constant $C_1 > 0$ such that $\widehat{\eta}(\gamma)(X) \subseteq D(A)$, $\|A\widehat{\eta}(\gamma)w\| \leq C_1\|w\|$ for each $w \in X$ and $\gamma \in \Lambda_\theta$.

In the continuation, for $s > 0$ and $\vartheta \in (\pi\backslash 2, \theta)$, $\Lambda_{s,\vartheta} = \{\gamma \in C \setminus \{0\} \colon |\gamma| > s, |\arg(\gamma)| < \vartheta\}$, $\Gamma_{s,\vartheta}$, $\Gamma^i_{s,\vartheta}$, $i = 1, 2, 3$, are the paths $\Gamma^1_{s,\vartheta} = \{te^{i\vartheta} \colon t \geq s\}$, $\Gamma^2_{s,\vartheta} = \{se^{i\xi} \colon -\vartheta \leq \xi \leq \vartheta\}$, $\Gamma^3_{s,\vartheta} = \{te^{-i\vartheta} \colon t \geq s\}$ and $\Gamma_{s,\vartheta} = \cup_{i=1}^3 \Gamma^i_{s,\vartheta}$ oriented in a positive sense. Let

$$\Omega(G) = \{\gamma \in C \colon G(\gamma) = (\gamma I + A - \widehat{\eta}(\gamma))^{-1} \in \mathcal{L}(E)\}.$$

If $\mathcal{R}(\cdot)$ is a resolvent operator for the system (2.1), then the Laplace transform of (2.3) gives

$$\widehat{\mathcal{R}}(\cdot)(\gamma I + A - \widehat{\eta}(\gamma))w = w, \quad \forall\, w \in D(A).$$

Then it follows from [11, Lemma 2.2] and the inverse Laplace transform that $\mathcal{R}(\cdot)$ is the only resolvent operator for the system (2.1). Thus the resolvent operator $\{\mathcal{R}(t)\}_{t\geq 0}$ is defined as

$$\mathcal{R}(t) = \begin{cases} \frac{1}{2i\pi} \int_{\Gamma_{s,\vartheta}} e^{\gamma t} G(\gamma)\, d\gamma, & t > 0, \\ I, & t = 0. \end{cases} \tag{2.4}$$

We recall some results regarding $\mathcal{R}(\cdot)$ which will be required in many of our subsequent results:

Lemma 2.1 (see [11])**.** *The map $\mathcal{R} \colon [0, \infty) \to \mathcal{L}(E)$ is strongly continuous and exponential bounded. Also there exists a positive constant m_α such that $\|A^\alpha \mathcal{R}(t)w\| \leq m_\alpha t^{-\alpha}\|w\|$ for each $w \in E$ and $0 \leq \alpha < 1$.*

Theorem 2.2 ([29, Theorem 2.3.3])**.** *Let $-A$ be the infinitesimal generator of a C_0 semigroup $T(t)$. The semigroup $T(t)$ is compact if and only if $S(\gamma, A)$ is compact for $\gamma \in \rho(A)$ and $T(t)$ is continuous in the uniform operator topology for $t > 0$.*

Lemma 2.3 (see [11])**.** *The operator $\mathcal{R}(t)$ is compact for all $t > 0$ if $S(\gamma_0, A)$ is compact for some $\gamma_0 \in \rho(A)$.*

Lemma 2.4 (see [30])**.** *The operator $\mathcal{R}(t)$ is continuous in the uniform operator topology of $L(E)$ for $t > 0$.*

We now derive a variation of parameters formula to represent the solution of (1.1) in the form of resolvent operator theory.

Theorem 2.5. *Suppose that the functions $g \colon J \times Z \to Z$, $h \colon J \times Z \to Z$ and $B \colon U \to Z$ are continuous, and $q(0) + \Phi(w)(0) \in D(A)$. If $w(\cdot)$ is a classical solution of system (1.1) on the interval $[0, b]$, then*

$$\begin{aligned} w(t) = {} & \mathcal{R}(t)[q(0) + \Phi(w)(0) + h(0, w_0)] - h(t, w_t) + \int_0^t A\mathcal{R}(t-r)h(r, w_r)\, dr \\ & - \int_0^t \int_0^r \eta(r-s)\mathcal{R}(t-r)h(s, w_s)\, ds\, dr + \sum_{0<t_j<t} \mathcal{R}(t-t_j)\mathcal{I}_j(w_{t_j}) \\ & + \int_0^t \mathcal{R}(t-r)\left[g(r, w(r)) + Bv(r)\right] dr, \quad t \in J. \end{aligned} \tag{2.5}$$

Proof. Let $\widehat{w}(\lambda) = \int_0^\infty e^{-\lambda s} w(s)\,ds$, $\widehat{g}(\lambda) = \int_0^\infty e^{-\lambda s} g(s, w(s))\,ds$, $\widehat{h}(\lambda) = \int_0^\infty e^{-\lambda t} h(s, w_s)\,ds$, $\widehat{\eta}(\lambda) = \int_0^\infty e^{-\lambda s} \eta(s)\,ds$ and $\widehat{v}(\lambda) = \int_0^\infty e^{-\lambda s} Bv(s)\,ds$, for $\lambda > 0$. If we take $t \in [0, t_1]$ and integrate both side of (1.1) from 0 to t, we get

$$\begin{aligned} w(t) = q(0) + \Phi(w)(0) + h(0, w_0) - h(t, w_t) + \int_0^t \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] dr \\ + \int_0^t \int_0^r \eta(r-s) w(s)\,ds\,dr, \quad t \in [0, t_1], \end{aligned} \tag{2.6}$$

provided that integral in (2.6) exists. Let $t \in (t_1, t_2]$. If we integrate both side of (1.1) from t_1^+ to t, then we get

$$\begin{aligned} w(t) &= w(t_1^+) + h(t_1^+, w_{t_1^+}) - h(t, w_t) + \int_{t_1}^t \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] \\ &\quad + \int_{t_1}^t \int_0^r \eta(r-s) w(s)\,ds\,dr \\ &= w(t_1) + \mathcal{I}_1(w_{t_1}) + h(t_1, w_{t_1}) - h(t, w_t) + \int_{t_1}^t \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] dr \\ &\quad + \int_{t_1}^t \int_0^r \eta(r-s) w(s)\,ds\,dr. \end{aligned} \tag{2.7}$$

Putting $t = t_1$ in (2.6), we get

$$\begin{aligned} w(t_1) &= q(0) + \Phi(w)(0) + h(0, w_0) - h(t_1, w_{t_1}) \\ &\quad + \int_0^{t_1} \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] dr + \int_0^{t_1} \int_0^r \eta(r-s) w(s)\,ds\,dr. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we get

$$\begin{aligned} w(t) &= q(0) + \Phi(w)(0) + \mathcal{I}_1(w_{t_1}) + h(0, w_0) - h(t, w_t) \\ &\quad + \int_0^t \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] dr + \int_0^t \int_0^r \eta(r-s) w(s)\,ds\,dr, \quad t \in (t_1, t_2]. \end{aligned} \tag{2.9}$$

Similarly, if we take $t \in (t_2, t_3]$ and integrate both side of (1.1) from t_2^+ to t, we get

$$\begin{aligned} w(t) &= q(0) + \Phi(w)(0) + h(0, w_0) - h(t, w_t) + \sum_{j=1}^{2} \mathcal{I}_j(w_{t_j}) \\ &\quad + \int_0^t \left[-Aw(r) + g(r, w(r)) + Bv(r)\right] dr + \int_0^t \int_0^r \eta(r-s) w(s)\,ds\,dr, \quad t \in (t_2, t_3]. \end{aligned} \tag{2.10}$$

Continuing in this process, we get

$$w(t) = q(0) + \Phi(w)(0) + h(0, w_0) - h(t, w_t) + \sum_{0 < t_j < t} \mathcal{I}_j(w_{t_j})$$

$$+\int_0^t [-Aw(r)+g(r,w(r))+Bv(r)]\,dr+\int_0^t\int_0^r \eta(r-s)w(s)\,ds\,dr, \quad t\in J. \tag{2.11}$$

Since

$$\begin{aligned}\widehat{w}(\lambda) &= \int_0^\infty e^{-\lambda t}w(t)\,dt\\ &= \int_0^{t_1} e^{-\lambda t}w(t)\,dt+\int_{t_1}^{t_2} e^{-\lambda t}w(t)\,dt+\cdots+\int_{t_k}^\infty e^{-\lambda t}w(t)\,dt,\end{aligned}$$

we obtain from (2.6), (2.9), (2.10) and (2.11) that

$$\begin{aligned}\widehat{w}(\lambda) &= \sum_{j=1}^k \int_{t_j}^\infty e^{-\lambda t}\mathcal{I}_j(w_{t_j})\,dt+\int_0^\infty e^{-\lambda t}\Big[q(0)+\Phi(w)(0)+h(0,w_0)-h(t,w_t)\\ &\quad +\int_0^t \{-Aw(r)+g(r,w(r))+Bv(r)\}\,dr+\int_0^t\int_0^r \eta(r-s)w(s)\,ds\,dr\Big]\,dt\\ &= \frac{1}{\lambda}\,[q(0)+\Phi(w)(0)+h(0,w_0)]-\widehat{h}(\lambda)+\sum_{j=1}^k \frac{e^{-\lambda t_j}}{\lambda}\mathcal{I}_j(w_{t_j})\\ &\quad +\frac{1}{\lambda}\,[-A\widehat{w}(\lambda)+\widehat{g}(\lambda)+\widehat{v}(\lambda)]+\frac{1}{\lambda}\widehat{\eta}(\lambda)\widehat{w}(\lambda)\\ &= \widehat{\mathcal{R}}(\lambda)\,[q(0)+\Phi(w)(0)+h(0,w_0)]-\lambda\widehat{\mathcal{R}}(\lambda)\widehat{h}(\lambda)\\ &\quad +\widehat{\mathcal{R}}(\lambda)\sum_{j=1}^k e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})+\widehat{\mathcal{R}}(\lambda)\,[\widehat{g}(\lambda)+\widehat{v}(\lambda)]\\ &= \int_0^\infty e^{-\lambda t}\mathcal{R}(t)\,[q(0)+\Phi(w)(0)+h(0,w_0)]\,dt-\lambda\int_0^\infty e^{-\lambda t}\left[\int_0^t \mathcal{R}(t-r)h(r,w_r)\,dr\right]dt\\ &\quad +\widehat{\mathcal{R}}(\lambda)\sum_{j=1}^k e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})+\int_0^\infty e^{-\lambda t}\left[\int_0^t \mathcal{R}(t-r)\,\{g(r,w(r))+Bv(r)\}\,dr\right]dt. \end{aligned}\tag{2.12}$$

In view of inverse Laplace transformation, we get

$$\begin{aligned}L^{-1}\left\{\lambda\int_0^\infty e^{-\lambda t}\ \left[\int_0^t \mathcal{R}(t-r)h(r,w_r)\,dr\right]dt\right\} &= \frac{d}{dt}\int_0^t \mathcal{R}(t-r)h(r,w_r)\,dr\\ &= h(t,w_t)-\int_0^t A\mathcal{R}(t-r)h(r,w_r)\,dr+\int_0^t R(t-r)\int_0^r \eta(r-s)h(s,w_s)\,ds\,dr\end{aligned}$$

and

$$\begin{aligned}L^{-1}\left\{\frac{e^{-\lambda t_j}}{\lambda}\mathcal{I}_j(w_{t_j})\right\} &= u_j(t-t_j)\quad \text{(say)}\\ &= \begin{cases}0, & t<t_j\\ \mathcal{I}_j(w_{t_j}), & t\ge t_j.\end{cases}\end{aligned}$$

Therefore

$$L^{-1}\left\{\widehat{\mathcal{R}}(\lambda)e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})\right\} = \frac{d}{dt}\int_0^t \mathcal{R}(t-r)u_j(r-t_j)\,dr,$$

here $j = 1, 2, \cdots, k$. If $t \in [0, t_1]$, then

$$L^{-1}\left\{\widehat{\mathcal{R}}(\lambda)e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})\right\} = \frac{d}{dt}\int_0^t \mathcal{R}(t-r)u_j(r-t_j)\,dr = 0$$

for each $j = 1, 2, \cdots, k$. If $t \in (t_n, t_{n+1}]$ for some $n \in \{1, 2, \cdots, k\}$, then

$$\begin{aligned}
L^{-1}\left\{\sum_{j=1}^{k}\widehat{\mathcal{R}}(\lambda)e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})\right\} &= \sum_{j=1}^{k}\frac{d}{dt}\int_0^t \mathcal{R}(t-r)u_j(r-t_j)\,dr \\
&= \sum_{j=1}^{n}\frac{d}{dt}\int_{t_j}^t \mathcal{R}(t-r)\mathcal{I}_j(w_{t_j})\,dr \\
&= \sum_{j=1}^{n}\left[\mathcal{I}_j(w_{t_j}) + \int_{t_j}^t R'(t-r)\mathcal{I}_j(w_{t_j})\,dr\right] \\
&= \sum_{j=1}^{n}\mathcal{R}(t-t_j)\mathcal{I}_j(w_{t_j}).
\end{aligned}$$

That is, if $t \in [0, b]$, then

$$L^{-1}\left\{\sum_{j=1}^{k}\widehat{\mathcal{R}}(\lambda)e^{-\lambda t_j}\mathcal{I}_j(w_{t_j})\right\} = \sum_{0<t_j<t}\mathcal{R}(t-t_j)\mathcal{I}_j(w_{t_j}).$$

Thus if we apply inverse Laplace transform to both sides of (2.12), we obtain the same equation as given in (2.5). Hence the theorem is proved. □

Definition 2.2. *A continuous function $w\colon [-\tau, b] \to E$ is said to be a mild solution of* (1.1) *if $w(t) = q(t) + \Phi(w)(t)$, $t \in [-\tau, 0]$, and $w(t)$ satisfies the integral equation* (2.5) *for each $t \in [0, b]$.*

Lemma 2.6 ([13])**.** *If a set $K \subset PC(J, E)$ satisfies that the set K is uniformly bounded in $PC(J, E)$, K is a family of equicontinuity function in (t_j, t_{j+1}) for each $j = 0, 1, \cdots, k$, and sets $K(t) = \{w(t)\colon w \in K, t \in J \setminus t_1, t_2, \cdots, t_k\}$, $K(t_j^+) = \{w(t_j^+)\colon w \in K\}$ and $K(t_j^-) = \{w(t_j^-)\colon w \in K\}$ are relatively compact in E, then K is relatively compact in $PC(J, E)$.*

Definition 2.3 ([14])**.** *The system* (1.1) *is said to be approximately controllable on J if for each final state $z_b \in E$ and for any $\epsilon > 0$ there exists a control $v(\cdot) \in L^2(J, V)$ such that the mild solution $w(\cdot, v)$ of* (1.1) *satisfies that $\|w(b, v) - z_b\| < \epsilon$.*

We now define the controllability operator $\Gamma_0^b : E \to E$ and the resolvent operator $S(\epsilon, \Gamma_0^b)\colon E \to E$ as

$$\Gamma_0^b = \int_0^b \mathcal{R}(b-r)BB^*\mathcal{R}^*(b-r)dr,$$

$$S(\epsilon, \Gamma_0^b) = (\epsilon I + \Gamma_0^b)^{-1}, \quad \epsilon > 0,$$

where $\mathcal{R}^*$ and B^* are the adjoint of the operators $\mathcal{R}$ and B respectively.

Remark 2.7 (see [8, 14])**.** *Theorem 4.1.7 of [31] yields that $\epsilon S(\epsilon, \Gamma_0^b) \to 0$ strongly as $\epsilon \to 0^+$ if and only if $\langle x, \Gamma_0^b x\rangle = \int_0^b \|B^*\mathcal{R}^*(b-r)x\|^2\, dr > 0$ for each non-zero $x \in E$, that is, it is equivalent to saying that $B^*\mathcal{R}^*(b-r)x = 0$ implies $x = 0$.*

Finally, we recall Kuratowski's measure of noncompactness and its properties which will be used in the next section.

Definition 2.4 (see [32, 33])**.** *Let $\mathscr{B}(E)$ be a collection of bounded subsets of E. A function $\mu\colon \mathscr{B}(E) \to \mathbb{R}^+$, defined by*

$$\mu(\mathcal{D}) = \inf\{\epsilon > 0 : \mathcal{D} \subset \bigcup_{j=1}^{m} \mathcal{D}_j,\ diam(\mathcal{D}_j) < \epsilon\, (j = 1, 2, \cdots, m \in \mathbb{N})\}, \quad \mathcal{D} \in \mathscr{B}(E),$$

is called the Kuratowski's measure of noncompactness.

Lemma 2.8 (see [32, 33])**.** *If K_1, K_2 and K are bounded subsets of a Banach space E, then the following statements are true:*

(i) K *is relatively compact set in* E *if and only if* $\mu(K) = 0$.

(ii) $\mu(K_1) \le \mu(K_2)$ *if* $K_1 \subset K_2$.

(iii) $\mu(K_1 + K_2) \le \mu(K_1) + \mu(K_2)$.

(iv) $\mu(cK) \le |c|\mu(K)$ *for any* $c \in \mathbb{R}$.

3. Main Results

Since the semigroup $T(t)$ is compact, Theorem 2.2 and Lemma 2.3 show that the resolvent operator $\mathcal{R}(t)$ is compact for each $t > 0$. Because of Lemma 2.1, we can assume that $\sup_{t\in J} \|\mathcal{R}(t)\| \le M$ for some $M > 1$ and let $M_1 = \sup_{m\in\mathbb{N}} \left\|\mathcal{R}(\frac{1}{m})\right\|$. We define the set $E_l = \{w \in PC([-\tau, b], E)\colon \|w\|_{PC} \le l\}$, here $l > 0$ is any number. Let $\mathbb{N}$ be a set of natural number in this section. We make the following hypotheses to prove our subsequent main results:

(A0) $\epsilon S(\epsilon, \Gamma_0^b) \to 0$ as $\epsilon \to 0^+$ in strong operator topology.

(A1) The map $\eta(\cdot)z\colon J \to E$ is continuous for each $z \in D(A^{1-\beta})$, and there exists a function $a(\cdot) \in L^1(J, \mathbb{R}^+$ such that

$$\|\eta(s)R(t)\|_{\mathcal{L}(D(A^{\frac{\beta}{2}}), E)} \le Ma(s)t^{\beta-1}.$$

(A2) There is a $\beta \in (0, 1)$ such that the function $h\colon J \times \mathscr{E} \to E_\beta$ satisfies that for each $\varphi \in \mathscr{E}$, the function $h(\cdot, \varphi)$ is strongly measurable in E_β over the interval J. There is also a $l_h > 0$ such that

$$\|h(t, \varphi_1) - h(s, \varphi_2)\|_\beta \le l_h\{|t - s| + \|\varphi_1 - \varphi_2\|_{\mathscr{E}}\}, \quad \forall\, t, s \in J \quad \text{and} \quad \varphi_1, \varphi_2 \in \mathscr{E}.$$

Let $L_h = h(0, O)$, here O is a zero element in $\mathscr{E}$.

(A3) The function $I_j\colon \mathscr{E} \to E, \quad j = 1, 2 \ldots, k$, are continuous operators. There are positive constants c_j such that

$$\|I_j(\varphi)\| \leq c_j(\|\varphi\|_{\mathscr{E}+1}), j = 1, 2, \ldots, k, \ \varphi \in \mathscr{E}.$$

(A4) The function $g\colon [0, b] \times \mathscr{E} \to E$ satisfies the following

(i) The map $g(t, \cdot)$ is continuous from $\mathscr{E}$ to E for all $t \in J$ and the map $g(\cdot, \varphi)$ is strongly measurable for each $\varphi \in \mathscr{E}$

(ii) There is $\varpi_l(\cdot) \in L^2([0, b], \mathbb{R}^+)$ for each $l > 0$ such that

$$\sup\{\|g(t, \varphi)\|\colon \|\varphi\|_{\mathscr{E}} \leq l\} \leq \varpi_l(t), \quad \text{for a.e. } t \in J,$$

and

$$\lim_{l\to\infty} \inf \frac{1}{l}\|\varpi_l\|_{L^2} = p < +\infty.$$

(A5) For each $w \in PC([-\tau, b], E)$ with $\|w\|_{PC} < \infty$, the control $v(\cdot) = v(\cdot, w) \in V$ is bounded, i.e. there exist a constant $\lambda > 0$ such that $\|v(t) = \|v(t, w)\| \leq \lambda\|w\|_{PC}$ for each $t \in J$. The map $v(t, \cdot)\colon PC([-\tau, b], E) \to V$ is continuous for each $t \in J$.

Lemma 3.1. *If the conditions (A1) and (A4) are satisfied and $v(\cdot) \in V$ is bounded, then the operator $F\colon E_l \to \mathcal{C}([-\tau, b], E)$, defined by*

$$(Fw)(t) = \begin{cases} \int_0^t \mathcal{R}(t-s)\,[g(r, w_r) + Bv(r)]\,dr & t \in J, \\ 0, \quad t \in [-\tau, 0], \end{cases} \tag{3.1}$$

is completely continuous.

Proof. We First show that F is continuous on E_l. Let $\{w^{(m)}\} \subset E_l$ be any sequence such that $w^{(m)} \to w \in E_l$ as $m \to \infty$. We get for $t \in J$ that

$$\|g\left(t, w_t^{(m)}\right)) - g\left(t, w_t\right))\| \leq 2\varpi_l(t) \quad \text{and} \quad \|Bv\left(t, w^{(m)}(t)\right) - B\left(t, w\right))\| < \infty.$$

Then, from the Lebesgue Dominated Convergence theorem, we get

$$\begin{aligned} \|(Fw^{(m)})(t) - (Fw)(t)\| \leq M \int_0^b &\|g\left(r, w_r^{(m)}\right) - g(r, w_r)\|\,dr \\ &+ M\int_0^b \|Bv\left(r, w^{(m)}(r)\right) - Bv\left(r, w(r)\right)\|\,dr \\ &\to 0 \text{ as } m \to \infty \text{ independent of } t \in J. \end{aligned}$$

Thus F is continuous on E_l.

Next we show the equicontinuity of $F(E_l)$ on J. For any $s_1, s_2 \in J$ with $s_1 < s_2$ and $w \in E_l$, we have

$$\|Fw(s_2) - Fw(s_1)\| \leq \left\|\int_0^{s_1} [\mathcal{R}(s_2 - r) - \mathcal{R}(s_1 - r)]\,[g(r, w_r) + Bv(r)]\,dr\right\|$$

$$+\left\|\int_{s_1}^{s_2}\mathcal{R}(s_2-r)\left[g(r,w_r)+Bv(r)\right]dr\right\|$$
$$=I_1+I_2.$$

In view of assumption (H1), it can be easily seen that $I_2\to 0$ as $s_2\to s_1$ independent of $w\in E_l$. Let $\epsilon\in(0,s_1)$ be any number. Then, from Lemma 2.4, the operator $\mathcal{R}(t)$ is uniformly continuous for $t>0$, and therefore

$$\begin{aligned}I_1\leq\ &\left[\int_0^{s_1-\epsilon}\varpi_l(r)\,dr+\|B\|\,\|v(\cdot)\|_V\ \{s_1-\epsilon\}\right]\sup_{r\in[0,s_1-\epsilon]}\|\mathcal{R}(s_2-r)-\mathcal{R}(s_1-r)\|\\ &+2M\int_{s_1-\epsilon}^{s_1}\varpi_l(r)\,dr+2M\epsilon\|B\|\,\|v(\cdot)\|_V\\ &\to 0\text{ as }s_1\to s_2\text{ and }\epsilon\to 0\text{ independent of }w\in E_l.\end{aligned}$$

Hence $F(E_l)$ is equicontinuous on J. Further, we shall show that the set $\{(Fw)(t):\ w\in E_l\}$, $t\in J$, is relatively compact in E. Let $t\in(0,b]$ be any fixed number. For any $0<\gamma<\frac{1}{2}$, we have

$$\begin{aligned}\|A^\gamma Fw(t)\|\leq&\int_0^t\|A^\gamma\mathcal{R}(t-r)\|\left[\|g(r,w_r)\|+\|Bv(r)\|\right]dr\\ \leq&\ m_\gamma\frac{b^{1-2\gamma}}{1-2\gamma}\|\varpi_l\|_{L^2}+\frac{b^{1-\gamma}}{1-\gamma}\|B\|\,\|v(\cdot)\|_V\\ <&\ \infty.\end{aligned}$$

Thus $\{A^\gamma Fw(t)\colon w\in E_l\}$ is bounded in E. Since $\{(Fw)(t)\}=\{0\}$ for each $t\in[-\tau,0]$, compactness of the operator $(-A)^{-\gamma}$ implies that $\{Fw(t)\colon w\in E_l\}$ is relatively compact in E for each $t\in[-\tau,b]$. Hence, by Arzela-Ascoli theorem, we get that $F\colon E_l\to\mathcal{C}([-\tau,b],E)$ is a continuous and compact map. □

Case I: Φ satisfies Lipschitz condition.

We make the following assumption for Φ to prove the approximate controllability of (1.1):

(A6) The function $\Phi\colon PC([-\tau,b],E)\to\mathscr{E}$ satisfies Lipschitz condition, i.e., there is a constant $L_\Phi>0$ such that
$$\|\Phi(w^{(1)})-\Phi(w^{(2)})\|_{\mathscr{E}}\leq L_\Phi\|w_0^{(1)}-w_0^{(2)}\|_{\mathscr{E}},\quad\text{and }\|\Phi(w)\|_{\mathscr{E}}\leq L_\Phi(1+\|w_0\|_{\mathscr{E}})$$
for any $w,w^{(1)},w^{(2)}\in PC([-\tau,b],E)$.

For the sake of convenience, we write

$$N_1=ML_\Phi+l_h\left\{C_\beta(1+M)+\left(m_{1-\beta}+M\int_0^b a(r)\,dr\right)\frac{b^\beta}{\beta}\right\}.$$

Consider the following approximate impulsive system of (1.1):

$$\left\{\begin{array}{rl}\frac{d}{dt}\left[w(t)+h(t,w_t)\right]+Aw(t)= & \int_0^t\eta(t-r)w(r)\,dr+Bv(t)\\ & +g(t,w_t),\quad t\in J=[0,b],\ t\neq t_j,\\ \Delta w|_{t=t_j}= & \mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_j(w_{t_j}),\quad j=1,2,\cdots,k,\\ w(t)= & q(t)+\Phi(w)(t),\quad t\in[-\tau,0],\end{array}\right.\tag{3.2}$$

where $m\in\mathbb{N}$.

Lemma 3.2. *Suppose that (A1)-(A6) hold. Then the approximate system* (3.2) *has a mild solution in some E_l for each $m \in \mathbb{N}$ provided that*

$$N_1 + M \left\{ M_1 \sum_{j=1}^{k} c_j + p\sqrt{b} + \|B\| \lambda b \right\} < 1. \tag{3.3}$$

Proof. Take the control $v(\cdot)$ as $v(t) = v(t, w(b))$, $w \in E_l$. We define an operator $\mathcal{T}_m \colon E_l \to PC([-\tau, b], E)$ by

$$(\mathcal{T}_m w)(t) = (F_{\Phi+h} w)(t) + (F_m w)(t) + (Fw)(t), \; t \in [-\tau, b], \tag{3.4}$$

where the map F is given by (3.1), and the map $F_{\Phi+h}$ and F_m are defined respectively as below

$$(F_{\Phi+h} w)(t) = \begin{cases} \mathcal{R}(t)[q(0) + \Phi(w)(0) + h(0, w_0)] - h(t, w_t) + \int_0^t A\mathcal{R}(t-r)h(r, w_r)\, dr \\ \quad - \int_0^t \int_0^r \eta(r-s)\mathcal{R}(t-r)h(s, w_s)\, ds\, dr & t \in J, \\ q(t) + \Phi(w)(t), & t \in [-\tau, 0] \end{cases} \tag{3.5}$$

and

$$(F_m w)(t) = \begin{cases} \sum_{0<t_j<t} \mathcal{R}(t-t_j)\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_j(w_{t_j}), & t \in J, \\ 0 & t \in [-\tau, 0]. \end{cases} \tag{3.6}$$

It is clear that any fixed point of the map $\mathcal{T}_m$ is a mild solution of (3.2). Firstly we claim that the $\mathcal{T}_m(E_l) \subset E_l$ for some $l > 0$. If this is not true, then there would exist $w^{(l)} \in E_l$ for each $l > 0$ such that $\|\mathcal{T}_m w^{(l)}(t)\| > l$ for some $t \in J$. Therefore

$$\begin{aligned} l &< \|(\mathcal{T}_m w^{(l)})(t)\| \\ &\leq M\left[\|q\| + L_\Phi\left(\|(w^{(l)})\|_{PC} + 1\right) + C_\beta\left(l_h\|(w^{(l)})\|_{PC} + L_h\right)\right] \\ &\quad + C_\beta l_h\left(t + \|(w^{(l)})\|_{PC}\right) + C_\beta L_h + \frac{m_{1-\beta} t^\beta}{\beta}\left[l_h\left(t + \|(w^{(l)})\|_{PC}\right) + L_h\right] \\ &\quad + M\int_0^t (t-r)^{\beta-1}\int_0^r a(r-s)\left[l_h\left(s + \|(w^{(l)})\|_{PC}\right) + L_h\right] ds\, dr \\ &\quad + MM_1\sum_{j=1}^{k} c_j\left(\|(w^{(l)})\|_{PC} + 1\right) + M\sqrt{b}\,\|\varpi_l\|_{L^2} + M\lambda b\|B\|\left\|(w^{(l)})\right\|_{PC}. \end{aligned} \tag{3.7}$$

Dividing both sides of (3.7) by l and taking $l \to \infty$, we get

$$1 < N_1 + M\left\{M_1\sum_{j=1}^{k} c_j + p\sqrt{b} + \|B\|\lambda b\right\}.$$

This contradicts (3.3). So we can say that there exists a constant $l > 0$ such that $\mathcal{T}_m(E_l) \subseteq E_l$.

Let $w^{(1)}, w^{(2)} \in E_l$ any elements. Then we obtain from the hypothesis that

$$\|(F_{\Phi+h} w^{(1)}) - (F_{\Phi+h} w^{(2)})\|_{PC} \leq N_1\|w^{(1)} - w^{(2)}\|_{PC}. \tag{3.8}$$

Thus from the hypothesis of the theorem we see that the map $F_{\Phi+h}$ is a contraction on E_l.

Finally we shall show that F_m is completely continuous on E_l. It is easy to see from the hypothesis (A3) that F_m is continuous on E_l. Take $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_k = (t_k, b]$. We can rewrite F_m as the following form.

$$(F_m w)(t) = \begin{cases} 0, & t \in [-\tau, t_1], \\ \mathcal{R}(t - t_1)\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_1(w_{t_1}), & t \in J_1, \\ \cdots, & \\ \sum_{j=1}^{k} \mathcal{R}(t - t_j)\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_j(w_{t_j}), & t \in J_k. \end{cases}$$

The continuity of $\mathcal{I}_1$ and compactness of $\mathcal{R}(t)$ for $t > 0$ yield that the set

$$\left\{\mathcal{R}(t - t_1)\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_1(w_{t_1}) \colon w \in E_l\right\}$$

for each $t \in J_1$ and the set

$$\left\{F_m w(t_1^+) \colon w \in E_l\right\} = \left\{\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_1(w_{t_1}) \colon w \in E_l\right\}$$

are precompact in E. Take any $s_1, s_2 \in J_1$ with $s_1 < s_2$ and $w \in E_l$. Then we get from Lemma 2.4 that

$$\begin{aligned} &\left\|\mathcal{R}(s_2 - t_1)\mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}) - \mathcal{R}(s_1 - t_1)\mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1})\right\| \\ &\quad \le M_1 c_1(l + 1)\|\mathcal{R}(s_2 - t_1) - \mathcal{R}(s_1 - t_1)\| \\ &\quad\quad \to 0 \text{ as } s_1 \to s_2 \text{ independent of } w \in E_l. \end{aligned}$$

Therefore the set $\{F_m w \colon w \in E_l\}$ is equicontinuous on J_1. Similarly we can obtain the same for each the interval $J_j, j = 2, 3, \cdots, k$. Hence, we get from Lemma 2.6, that $\{F_m w \colon w \in E_l\}$ is relatively compact in $PC([J, E)$.

We have now shown that $F_{\Phi+h}$ is a contraction and $F_m + F$ is completely continuous. Hence we obtain from Krasnoselskii's fixed point theorem that $\mathcal{T}_m \colon E_l \to E_l$ has a fixed point in E_l. That is, the approximate system (3.2) has a mild solution. □

Theorem 3.3. *If all conditions of Lemma 3.2 hold, then the delay system* (1.1) *has a mild solution on* $[-\tau, b]$.

Proof. Let

$$\mathcal{H} = \left\{w^{(m)} \in E_l \colon w^{(m)} = \mathcal{T}_m w^{(m)}\right\}_{m=1}^{\infty} \subset E_l.$$

Then

$$\mathcal{H}(t) = \left\{w^{(m)}(t) \colon w^{(m)} \in \mathcal{H}\right\} \quad \text{and} \quad \mathcal{H}(t_i^+) = \left\{w^{(m)}(t_i^+) \colon w^{(m)} \in \mathcal{H}\right\},$$

where $t \in [-\tau, b]$ and $w^{(m)}(t_i^+)$ denotes the right limit of $w^{(m)}$ at $t_i, i = 1, 2, \ldots, k$.

Step 1. The sets $\mathcal{H}(t_1)$ and $\mathcal{H}(t_1^+)$ are precompact in E.

Take $w^{(m)} \in \mathcal{H}$ with $\mathcal{T}_m w^{(m)} = w^{(m)}$. Therefore

$$w^{(m)}(t) = (F_{\Phi+h} w^{(m)})(t) + (F w^{(m)})(t), \; t \in [-\tau, t_1] \quad \text{and}$$

$$w^{(m)}(t_1^+) = (F_{\Phi+h}w^{(m)})(t_1) + (Fw^{(m)})(t_1) + \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}).$$

From the hypotheses we can easily show that

$$\left\| F_{\Phi+h}w^{(1)} - F_{\Phi+h}w^{(2)} \right\|_{[-\tau,t_1]} \leq N_1 \left\| w^{(1)} - w^{(2)} \right\|_{[-\tau,t_1]}, \tag{3.9}$$

where $\|\cdot\|_{[-\tau,t_1]} = \sup_{r\in[-\tau,t_1]} \|\cdot(r)\|$.

Since the map F is completely continuous, we obtain from the measure of noncompactness and (3.9) that

$$\begin{aligned} \mu\left(\left\{ w^{(m)}\Big|_{[-\tau,t_1]} \right\}\right) &\leq \mu\left(\left\{ F_{\Phi+h}\ w^{(m)}\Big|_{[-\tau,t_1]} \right\}\right) + \mu\left(\left\{ F\ w^{(m)}\Big|_{[-\tau,t_1]} \right\}\right) \\ &\leq N_1\mu\left(\left\{ w^{(m)}\Big|_{[-\tau,t_1]} \right\}\right) + 0. \end{aligned}$$

This implies that $\mu\left(\left\{ w^{(m)}\big|_{[-\tau,t_1]} \right\}\right) = 0$. Since $\|w_{t1}^{(2)} - w_{t1}^{(1)}\| \leq \|w^{(2)} - w^{(1)}\|_{[-\tau,t_1]}$, we get

$$0 \leq \mu\left(\left\{ w_{t_1}^{(m)} \right\}\right) \leq \mu\left(\left\{ w^{(m)}\Big|_{[-\tau,t_1]} \right\}\right) = 0.$$

That is, the set $\left\{ w_{t_1}^{(m)} \right\}_{m=1}^{\infty}$ is precompact. Therefore we can let $w_{t_1}^{(m)} \to \varphi$ in $\mathscr{E}$ as $m \to \infty$. Then

$$\begin{aligned} \left\| \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) - \mathcal{I}_1(\varphi) \right\| &\leq \left\| \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) - \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(\varphi) \right\| \\ &\quad + \left\| \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(\varphi) - \mathcal{I}_1(\varphi) \right\| \\ &\quad \to 0 \text{ as } m \to \infty. \end{aligned}$$

Thus the set $\{w^{(m)}(t_1^+)\}_{m=1}^{\infty}$ is precompact. That is, $\mathcal{H}(t_1)$ and $\mathcal{H}(t_1^+)$ are precompact in E.

Step 2. The family $\left\{ F_m w^{(m)}|_{[-\tau,t_2]}\ :\ w^{(m)} \in \mathcal{H} \text{ and } \mathcal{T}_m w^{(m)} = w^{(m)} \right\}_{m=1}^{\infty}$ in $PC([-\tau,t_2],E)$ is precompact.

Since the set $\left\{ w_{t_1}^{(m)} \right\}_{m=1}^{\infty}$ is precompact, we let $w_{t_1}^{(m)} \to \varphi$ in $\mathscr{E}$ as $m \to \infty$. Therefore, for each $\epsilon > 0$, there exists an integer $N > 0$ such that

$$\left\| \mathcal{I}_1\left(w_{t_1}^{(m)} \right) - \mathcal{I}_1(\varphi) \right\| < \frac{\epsilon}{3M},\ \forall\, m \geq N.$$

From the strong continuity of $R(t)$, we can find a number $\delta > 0$ such that if $0 < t < \delta$, we have

$$\left\| \mathcal{R}(t)\mathcal{I}_1\left(w_{t_1}^{(m)} \right) - \mathcal{I}_1\left(w_{t_1}^{(m)} \right) \right\| < \frac{\epsilon}{3},\ \forall\, m = 1, 2, \cdots, N-1,$$

and

$$\|\mathcal{R}(t)\mathcal{I}_1(\varphi) - \mathcal{I}_1(\varphi)\| < \frac{\epsilon}{3}.$$

Let $s_1, s_2 \in J_1$ with $s_1 < s_2$ and $w \in E_l$. If $s_1 \neq t_1^+$, then we obtain from Lemma 2.4,

$$\left\| \mathcal{R}(s_2 - t_1)\mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) - \mathcal{R}(s_1 - t_1)\mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) \right\|$$

$$\leq M_1 c_1(l+1)\Big\|\mathcal{R}(s_2-t_1)-\mathcal{R}(s_1-t_1)\Big\|$$
$$\to 0 \text{ as } s_1 \to s_2 \text{ independent of } m.$$

If $s_1 = t_1^+$, then we have two choices of m, i.e. either $m \in \{1,2,\cdots,N-1\}$ or $m \geq N$. We now choose $s_2 - t_1 < \delta$. Thus if $m \in \{1,2,\cdots,N-1\}$, we have

$$\left\|\mathcal{R}(s_2-t_1)\mathcal{I}_1\left(w_{t_1}^{(m)}\right)-\mathcal{I}_1\left(w_{t_1}^{(m)}\right)\right\| < \frac{\epsilon}{3},$$

and if $m \geq N$, we have

$$\begin{aligned}
&\left\|\mathcal{R}(s_2-t_1)\mathcal{I}_1(w_{t_1}^{(m)})-\mathcal{I}_1(w_{t_1}^{(m)})\right\| \\
&\qquad \leq \left\|\mathcal{R}(s_2-t_1)\mathcal{I}_1(w_{t_1}^{(m)})-\mathcal{R}(s_2-t_1)\mathcal{I}_1(\varphi)\right\| \\
&\qquad\quad + \left\|\mathcal{R}(s_2-t_1)\mathcal{I}_1(\varphi)-\mathcal{I}_1(\varphi)\right\| + \left\|\mathcal{I}_1(\varphi)-\mathcal{I}_1(w_{t_1}^{(m)})\right\| \\
&\qquad < \epsilon.
\end{aligned}$$

Therefore the set

$$\left\{F_m w^{(m)} \colon w^{(m)} \in \mathcal{H}\right\}_{m=1}^{\infty}\Big|_{J_1} = \left\{\mathcal{R}(\cdot - t_1)\mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_1(w_{t_1}^{(m)}) \colon w^{(m)} \in \mathcal{H} \text{ and } \cdot \in J_1\right\}_{m=1}^{\infty}$$

is equicontinuous on J_1. Let $w^{(m)} \in \mathcal{H}$ with $\mathcal{T}_m w^{(m)} = w^{(m)}$ and $t \in J_1$. Then

$$\begin{aligned}
(F_m w^{(m)})(t) &= \mathcal{R}(t-t_1)\mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) \quad \text{and} \\
(F_m w^{(m)})(t_1^+) &= \mathcal{R}\Big(\frac{1}{m}\Big)\mathcal{I}_1(w_{t_1}^{(m)}) = w^{(m)}(t_1^+) - w^{(m)}(t_1).
\end{aligned}$$

Since the sets $\mathcal{H}(t_1)$ and $\mathcal{H}(t_1^+)$ are precompact, and the compactness of the operator $R(t)$, the set $\big\{(F_m w^{(m)})(t) \colon w^{(m)} \in \mathcal{H}\big\}_{m=1}^{\infty}$ are precompact in E for each $t \in J_1$.

Hence, from Lemma 2.6, the set $\big\{F_m w^{(m)} \colon w^{(m)} \in \mathcal{H} \text{ and } \mathcal{T}_m w^{(m)} = w^{(m)}\big\}_{m=1}^{\infty}\Big|_{[-\tau,t_2]}$ is precompact in $PC([-\tau,t_2],E)$.

Step 3. The family $\big\{F_m w^{(m)} \colon w^{(m)} \in \mathcal{H} \text{ and } \mathcal{T}_m w^{(m)} = w^{(m)}\big\}_{m=1}^{\infty}$ is precompact in the space $PC([-\tau,b],E)$.

If we repeat the Step 1. and Step 2. for all the intervals J_2 to J_k, then we get from Lemma 2.6 that $\big\{F_m w^{(m)} \colon w^{(m)} \in \mathcal{H} \text{ and } \mathcal{T}_m w^{(m)} = w^{(m)}\big\}_{m=1}^{\infty}$ is a precompact set in $PC([-\tau,b],E)$.

Step 4. The impulsive delay system (1.1) has a mild solution on $[-\tau,b]$.

Let $w^{(m)} \in \mathcal{H}$ with $\mathcal{T}_m w^{(m)} = w^{(m)}$ for $m = 1,2,\cdots$. That is,

$$w^{(m)}(t) = F_{\Phi+h} w^{(m)}(t) + F_m w^{(m)}(t) + F w^{(m)}(t),\ t \in [-\tau,b]. \tag{3.10}$$

Since the sets $\big\{F_m w^{(m)} \colon w^{(m)} \in \mathcal{H}\big\}_{m=1}^{\infty}$ and $\big\{F w^{(m)} \colon w^{(m)} \in \mathcal{H}\big\}_{m=1}^{\infty}$ are relatively compact, and $F_{\Phi+h}$ is a contraction on E_l, we get from Kuratowski's measure of noncompactness that

$$\mu(\{w^{(m)}\}) = \mu(\{\mathcal{T}_m(w^{(m)})\}) \leq \mu(\{F_{\Phi+h}(w^{(m)})\}) + \mu(\{F_m(w^{(m)})\}) + \mu(\{F(w^{(m)})\})$$

$$\le N_1\mu(\{w^{(m)}\}).$$

Then $\mu(\{w^{(m)}\}) = 0$ as $0 < N_1 < 1$. This means that $\{w^{(m)}\} \subset PC(J,E)$ is precompact. We can now assume without loss of generality that $w^{(m)} \to w$ in $PC(J,E)$ as $m \to \infty$. Taking the limit $m \to \infty$ in both sides of (3.10), we get

$$w(t) = F_{\Phi+h}w(t) + \sum_{0<t_j<t} \mathcal{R}(t-t_j)\mathcal{I}_j(w_{(t_j)}) + Fw(t),\ t \in [-\tau, b].$$

Hence the impulsive delay system (1.1) has a mild solution $w \in PC([-\tau, b], E)$. This completes the proof. □

In the following theorem, we prove the approximate controllability of the system (1.1).

Theorem 3.4. *Assume that hypotheses (A0)-(A3), (A4)(i) and (A6) are satisfied, and the functions $h\colon J \times \mathscr{E} \to E_\beta$, $\mathcal{I}_j\colon E \to E, (j = 1, 2 \ldots, k)$, $g\colon J \times E \to E$, and $\Phi\colon PC(J,E) \to \mathscr{E}$ are uniformly bounded. If $\eta(\cdot)A^\beta = A^\beta\eta(\cdot)$ and $N_1 < 1$, then the system (1.1) is approximately controllable on $[-\tau, b]$.*

Proof. Let $z_b \in E$ be any arbitrary final state. For any $\varepsilon > 0$, we fix a control $v(t)$ as

$$v(t) := v_\varepsilon(t,w) = B^*\mathcal{R}^*(b-t)S(\varepsilon, \Gamma_0^b)\Upsilon(w), \tag{3.11}$$

where

$$\begin{aligned}\Upsilon(w) = z_b - \mathcal{R}(b)\{q(0) + \Phi(w)(0) + h(0,w_0)\} + h(b,w_b) - \int_0^b A\mathcal{R}(b-r)h(r,w_r)\,dr \\ + \int_0^b\int_0^r \eta(r-s)\mathcal{R}(b-r)h(r,w_r)\,ds\,dr - \sum_{j=1}^{j=k}\mathcal{R}(b-t_j)\mathcal{I}_j(w_{t_j}) - \int_0^b \mathcal{R}(b-r)g(r,w_r)\,dr.\end{aligned}$$

Let $\varepsilon > 0$ be any arbitrary number. Define a map $P_\varepsilon\colon PC([-\tau,b],E)$ to $PC([-\tau,b],E)$ as

$$(P_\varepsilon w)(t) = \begin{cases} \mathcal{R}(t)[q(0) + \Phi(w)(0) + h(0,w_0)] - h(t,w_t) + \int_0^t A\mathcal{R}(t-r)h(r,w_r)\,dr \\ -\int_0^t\int_0^r \eta(r-s)\mathcal{R}(t-r)h(r,w_r)\,ds\,dr + \sum_{0<t_j<t}\mathcal{R}(t-t_j)\mathcal{I}_j(w_{t_j}) \\ \qquad + \int_0^t \mathcal{R}(t-s)\,[g(r,w_r) + Bv_\varepsilon(r,w)]\,dr, \qquad t \in J, \\ q(t) + \Phi(w)(t), \qquad t \in [-\tau, 0], \end{cases} \tag{3.12}$$

where the control $v_\varepsilon(t,w)$ is given by (3.11).

It is easy to verify from Lemma 3.2 that the control (3.11) satisfies the condition (A5). We now conclude from Theorem 3.3 that for each $\varepsilon > 0$ the map P_ε has a fixed point in some E_l (this fixed point is a mild solution of (1.1)). Let $w^{(\varepsilon)}$ be a fixed point of P_ε. Then we get that

$$w^{(\varepsilon)}(b) = z_b - \varepsilon S(\varepsilon, \Gamma_0^b)\Upsilon(w^{(\varepsilon)}). \tag{3.13}$$

Let $0 < \gamma < \frac{1}{2}$. By Lemma 3.1, the set $\left\{A^\gamma \int_0^b \mathcal{R}(b-r)g(r,w^{(\varepsilon)}(r))dr\right\}$ is bounded in E. It follows from the compactness of embedding $E_\gamma \hookrightarrow E$ that there exists a subsequence of $\left\{\int_0^b \mathcal{R}(b-r)g(r,w^{(\varepsilon)}(r))dr\right\}$, denoted by itself, such that it converges to say $\overline{g} \in E$. From the hypotheses, the functions $I_j\colon E \to E, (j = 1, 2\ldots, k)$ and $\Phi\colon PC(J, E) \to \mathscr{E}$ are uniformly bounded. By compactness of the operator $\mathcal{R}(t)(t > 0)$, there are subsequences of $\left\{\sum_{j=1}^{j=k} \mathcal{R}(b - t_j)\mathcal{I}_j(w^{(\varepsilon)}(t_j))\right\}$ and $\left\{\mathcal{R}(b)[\Phi(w^{(\varepsilon)})(0) + h(0, w_0)]\right\}$, denoted by themselves respectively, that converge to $\widetilde{\mathcal{I}}$ and $\psi_{\Phi+h}$ respectively.

From the hypothesis we can easily show that $\left\{A^{\beta} h(t, w_t)\right\}$, $\left\{A^{\beta} \int_0^t A\mathcal{R}(t - r)h(r, w_r)\, dr\right\}$ and $\left\{A^{\frac{\beta}{2}} \int_0^b \int_0^r \eta(r - s)\mathcal{R}(b - r)h(r, w_r^{(\varepsilon)})\, ds\, dr\right\}$ are uniformly bounded in E. Then, from the compactness of embedding $E_{\beta} \hookrightarrow E$ and the compactness of embedding $E_{\frac{\beta}{2}} \hookrightarrow E$, we say that there are subsequences of $\{h(b, w_b)\}$, $\left\{\int_0^t A\mathcal{R}(t - r)h(r, w_r)\, dr\right\}$ and $\left\{\int_0^b \int_0^r \eta(r - s)\mathcal{R}(b - r)h(r, w_r^{(\varepsilon)})\, ds\, dr\right\}$, denoted by themselves, such that they converges to say $\overline{h}_1$, $\overline{h}_2$, $\overline{h}_3 \in E$ respectively.

Let $\Psi = z_b - \mathcal{R}(b)q(0) - \psi_{\Phi+h} + \overline{h}_1 - \widetilde{\mathcal{I}} - \overline{h}_2 + \overline{h}_3 - \overline{g}$. Then

$$
\begin{aligned}
\|\Upsilon(w^{(\varepsilon)}) - \Psi\| \le{}& \left\|\mathcal{R}(b)[\Phi(w^{(\varepsilon)})(0) + h(0, w_0)] - \psi_{\Phi+h}\right\| + \|h(b, w_b) - h_1\| \\
&+ \left\|\sum_{j=1}^{j=k} \mathcal{R}(b - t_j)\mathcal{I}_j(w^{(\varepsilon)}(t_j)) - \widetilde{\mathcal{I}}\right\| + \left\|\int_0^t A\mathcal{R}(t - r)h(r, w_r)\, dr - \overline{h}_2\right\| \\
&+ \left\|\int_0^b \int_0^r \eta(r - s)\mathcal{R}(b - r)h(r, w_r^{(\varepsilon)})\, ds\, dr - \overline{h}_3\right\| \\
&+ \left\|\int_0^b \mathcal{R}(b - r)g(r, w^{(\varepsilon)}(r))dr - \overline{g}\right\| \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{3.14}
$$

Using (3.13), (3.14) and (A0), we obtain

$$
\begin{aligned}
\|w^{(\varepsilon)}(b) - z_b\| &\le \|\varepsilon S(\varepsilon, \Gamma_0^b)(\Psi)\| + \|\varepsilon S(\varepsilon, \Gamma_0^b)\|\, \|\Upsilon(w^{(\varepsilon)}) - \Psi\| \\
&\le \|\varepsilon S(\varepsilon, \Gamma_0^b)(\Psi)\| + \|\Upsilon(w^{(\varepsilon)}) - \Psi\| \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Hence the system (1.1) is approximate controllable on $[-\tau, b]$. □

Case II: Φ is continuous only in $PC([-\tau, b], E)$.

In this case, we shall extend the results from a Lipschitz continuous function to a continuous function Φ. We now consider the following assumptions on Φ:

(A7) The nonlocal function $\Phi\colon PC([-\tau, b], E) \to \mathscr{E}$ is continuous and has the following properties:

(i) There is a constant $k_{\Phi} > 0$ such that

$$\|\Phi(w)\|_{\mathscr{E}} \le k_{\Phi}\|w\|_{PC} \quad \forall w \in PC([-\tau, b]).$$

(ii) For any $l > 0$, there exists a number $\alpha \in (0, t_1)$ (α depends on l) such that if $w^{(1)}, w^{(2)} \in E_l$ with $w^{(1)}(t) = w^{(2)}(t), t \in [\alpha, b]$, then $\Phi(w^{(1)}) = \Phi(w^{(2)})$.

For the sake of convenience, we write

$$N_2 = l_h \left\{C_{\beta}(1 + M) + \left(m_{1-\beta} + M \int_0^b a(r)\, dr\right) \frac{b^{\beta}}{\beta}\right\}.$$

For any $m \in \mathbb{N}$, a set of natural number, we consider the following approximate impulsive system of (1.1):

$$\begin{cases} \frac{d}{dt}w(t) = & Aw(t) + \int_0^t \eta(t-r)w(r)\,dr + Bv(t) \\ & \quad +g(t, w_t), \quad t \in J = [0,b],\ t \neq t_j, \\ \Delta w|_{t=t_j} = & \mathcal{R}\left(\frac{1}{m}\right)\mathcal{I}_j(w_{t_j}), \quad j = 1, 2, \cdots, k, \\ w(t) = & \mathcal{R}\left(\frac{1}{m}\right)[q(t) + \Phi(w)(t)], \quad t \in [-\tau, 0]. \end{cases} \tag{3.15}$$

Lemma 3.5. *Assume that the hypotheses (A1)-(A5) and (A7) are satisfied. Then the approximate system* (3.15) *has a mild solution in some E_l for each $m \in \mathbb{N}$ provided that*

$$N_2 + M\left\{M_1\left(k_\Phi + \sum_{j=1}^{k} c_j\right) + p\sqrt{b} + \|B\|\,\lambda b\right\} < 1. \tag{3.16}$$

Proof. We choose the control $v(\cdot)$ as $v(t) = v(t, w(b)),\ w \in E_l$ such that it satisfies (A5). Define an operator $\mathcal{T}_m^*\colon E_l \to PC([-\tau, b], E)$ as

$$(\mathcal{T}_m^* w)(t) = (F_{(\Phi,m)}w)(t) + (F_h w)(t) + (F_m w)(t) + (Fw)(t), \tag{3.17}$$

where the map F and F_m are defined by (3.1) and (3.6) respectively, and the map $F_{(\Phi,m)}$ and F_h are defined respectively as below:

$$(F_{(\Phi,m)}w)(t) = \begin{cases} \mathcal{R}(t)\mathcal{R}\left(\frac{1}{m}\right)[q(0) + \Phi(w)(0)], & t \in J = [0,b], \\ \mathcal{R}\left(\frac{1}{m}\right)[q(t) + \Phi(w)(t)], & t \in [-\tau, 0], \end{cases} \tag{3.18}$$

and

$$(F_h w)(t) = \begin{cases} \mathcal{R}(t)h(0, w_0) - h(t, w_t) + \int_0^t A\mathcal{R}(t-r)h(r, w_r)\,dr \\ \quad - \int_0^t \int_0^r \eta(r-s)\mathcal{R}(t-r)h(s, w_s)\,ds\,dr & t \in J, \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.19}$$

We now claim that $\mathcal{T}_m^*(E_l) \subseteq E_l$ for some $l > 0$. Suppose that this is false. Then there would exist $w^{(l)} \in E_l$ for each $l > 0$ such that $\|\mathcal{T}_m^* w^{(l)}(t)\| > l$ for some $t \in [0, b]$. Therefore

$$\begin{aligned} l &< \|(\mathcal{T}_m^* w^{(l)})(t)\| \\ &\leq M\,M_1\left[\|q\| + k_\Phi\|w^{(l)}\|_{PC}\right] + MC_\beta\left(l_h\|(w^{(l)})\|_{PC} + L_h\right) \\ &\quad + C_\beta l_h\left(t + \|(w^{(l)})\|_{PC}\right) + C_\beta L_h + \frac{m_{1-\beta}t^\beta}{\beta}\left[l_h\left(t + \|(w^{(l)})\|_{PC}\right) + L_h\right] \\ &\quad + M\int_0^t (t-r)^{\beta-1}\int_0^r a(r-s)\left[l_h\left(s + \|(w^{(l)})\|_{PC}\right) + L_h\right]ds\,dr \\ &\quad + MM_1\sum_{j=1}^{k} c_j\left(\|(w^{(l)})\|_{PC} + 1\right) + M\sqrt{b}\,\|\varpi_l\|_{L^2} + M\lambda b\|B\|\,\left\|(w^{(l)})\right\|_{PC}. \end{aligned} \tag{3.20}$$

Dividing both sides of (3) by l and then taking $l \to \infty$, we get

$$1 < N_2 + M\left\{M_1\left(k_\Phi + \sum_{j=1}^{k} c_j\right) + p\sqrt{b} + \|B\|\,\lambda b\right\}.$$

This is a contradiction to (3.16). Thus we can say that $\mathcal{T}_m^*$ maps from E_l to E_l for some $l > 0$.

We can easily derive from the assumptions of the theorem, Lemma 3.1 and Lemma 3.2 that F, F_m are completely continuous on E_l and the operator F_h is a contraction with Lipschitz constant N_2. Since the resolvent operator $\mathcal{R}(t)$ is compact for each $t > 0$ and Φ is continuous, the set $F_{(\Phi,m)}$ is completely continuous on E_l for each $m \in \mathbb{N}$.

Hence the operator $F_{(\Phi,m)} + F_m + F$ is a completely continuous and and the operator F_h is a contraction on the closed bounded convex set E_l. Finally, we conclude from Krasnoselskii's fixed point theorem that $\mathcal{T}_m^*$ has a fixed point on E_l. That is, the approximate system (3.15) has a mild solution on E_l. □

Theorem 3.6. *If the family $\{\Phi(E_l)\}$ of functions in $PC([-\tau, 0], E)$ is equicontinous on $[-\tau, 0]$, and all hypotheses of Lemma 3.5 hold, then the delay impulsive system* (1.1) *has a mild solution on $[-\tau, b]$.*

Proof. Let

$$\begin{aligned}
&\mathcal{H}^* = \{w^{(m,*)} \in E_l \colon w^{(m,*)} = \mathcal{T}_m^* w^{(m,*)}\}_{m=1}^{\infty} \subset PC([-\tau, b], E),\\
&\mathcal{H}^*(t) = \{w^{(m,*)}(t) \colon w^{(m,*)} \in \mathcal{H}^*\}, \quad t \in J, \quad \text{and}\\
&\mathcal{H}^*(t_i^+) = \{w^{(m,*)}(t_i^+) \colon w^{(m,*)} \in \mathcal{H}^*\},
\end{aligned}$$

where $w^{(m,*)}(t_i^+)$ $(i = 1, 2, \ldots, k)$ denotes the right limit of $w^{(m,*)}$ at t_i^+.

We are going to show that the set $\{F_{(\Phi,m)} w^{(m,*)} \colon w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is relatively compact in $PC([-\tau, b], E)$.

Take any $s_1, s_2 \in (0, b]$ with $s_1 < s_2$ and $w^{(m,*)} \in \mathcal{H}^*$. By Lemma 2.4, we obtain

$$\begin{aligned}
&\Big\|\mathcal{R}(s_2)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0) - \mathcal{R}(s_1)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0)\Big\|\\
&\qquad\qquad \le k_\Phi M_1 l \|\mathcal{R}(s_2) - \mathcal{R}(s_1)\|\\
&\qquad\qquad\qquad \to 0 \text{ as } s_1 \to s_2 \text{ independent of } w^{(m,*)} \in \mathcal{H}^*.
\end{aligned}$$

Therefore the set $\{\mathcal{R}(\cdot)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)}) \colon w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is equicontinuous on $(0, b]$.

From compactness of resolvent operator $\mathcal{R}(t)(t > 0)$, the set $\{F_{(\Phi,m)} w^{(m,*)}(t) \colon w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is also relatively compact for all $t > 0$. Therefore the set $\mathcal{H}^*(t) \subset E$ is precompact for each $t > 0$. Using the Arzela-Ascoli theorem, $\mathcal{H}^*\big|_{[\alpha,b]}$ is a relatively compact set in $PC([\alpha, b], E)$ for $0 < \alpha \le b$.

For each $w^{(m,*)} \in \mathcal{H}^*$, we define

$$\widetilde{w}^{(m)}(t) = \begin{cases} w^{(m,*)}(t), & t \in [\alpha, b],\\ w^{(m,*)}(\alpha), & t \in [-\tau, \alpha], \end{cases}$$

where α is given in (A7). It is clear that $\{w^{(m,*)}|_{[\alpha,b]}\}_{m=1}^{\infty}$ is a subset of $\mathcal{H}^*\big|_{[\alpha,b]}$, a relatively compact set of $PC([\alpha, b], E)$. So we assume that $w^{(m,*)}|_{[\alpha,b]} \to \overline{w}$ in $PC([\alpha, b], E)$ and set

$$\widetilde{w}(t) = \begin{cases} \overline{w}(t), & t \in [\alpha, b],\\ \overline{w}(\alpha), & t \in [-\tau, \alpha]. \end{cases}$$

Then $\widetilde{w}^{(m)} \to \widetilde{w}$ in $PC([-\tau, b], E)$ as $m \to \infty$. By Hypothesis (A7), $\Phi(\widetilde{w}^{(m)}) = \Phi(w^{(m,*)})$ for each $m \geq 1$ and $\Phi(\widetilde{w}^{(m)}) \to \Phi(\widetilde{w})$ as $m \to \infty$. Therefore for any $t \in [-\tau, 0]$ and $w^{(m,*)} \in \mathcal{H}^*$, it follows that

$$\begin{aligned}\|w^{(m,*)}(t) - [\,q(t) + \Phi(\widetilde{w})(t)\,]\| &= \Big\|\mathcal{R}\Big(\frac{1}{m}\Big)[\,q(t) + \Phi(w^{(m,*)})(t)\,] - [\,q(t) + \Phi(\widetilde{w})(t)\,]\Big\| \\ &\leq \Big\|\mathcal{R}\Big(\frac{1}{m}\Big)q(t) - q(t)\Big\| + M_1\|\Phi(\widetilde{w}^{(m)})(t) - \Phi(\widetilde{w})(t)\| \\ &\quad + \Big\|\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(t) - \Phi(\widetilde{w})(t)\Big\| \\ &\to 0 \text{ as } m \to \infty.\end{aligned}$$

Therefore the set $\{F_{(\Phi,m)}w^{(m,*)}(t)\colon\ w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is precompact in E for each $t \in [-\tau, b]$. It is yet to show that $\{\mathcal{R}(\cdot)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})\colon w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is equicontinuous at $t = 0$ for the right hand-side limit.

Since $\Phi(w^{(m,*)}) = \Phi(\widetilde{w}^{(m)}) \to \Phi(\widetilde{w})$ as $m \to \infty$, there exists an integer $N > 0$ for each $\epsilon > 0$ such that

$$\Big\|\Phi(\widetilde{w}^{(m)}) - \Phi(\widetilde{w})\Big\| < \frac{\epsilon}{3M},\ \forall\, m \geq N. \tag{3.21}$$

From the strong continuity of $R(t)$, we can find a number $\delta > 0$ such that if $0 < t < \delta$, we have

$$\Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0)\Big\| < \frac{\epsilon}{3},\ \forall\, m \in \{1, 2, \cdots, N-1\}, \tag{3.22}$$

and

$$\Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(0) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(0)\Big\| < \frac{\epsilon}{3}\ \text{ for any } m. \tag{3.23}$$

Take $0 < t < \delta$. If $m \in \{1, 2, \cdots, N-1\}$, then (3.22) holds, and if $m \geq N$, then we obtain from (3.21) and (3.23) that

$$\begin{aligned}&\Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0)\Big\| \\ &\qquad \leq \Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0) - \mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(0)\Big\| \\ &\qquad\quad + \Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w}) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(0)\Big\| \\ &\qquad\quad + \Big\|\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(\widetilde{w})(0) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0)\Big\| \\ &\qquad < \epsilon.\end{aligned}$$

That is,

$$\Big\|\mathcal{R}(t)\mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0) - \mathcal{R}\Big(\frac{1}{m}\Big)\Phi(w^{(m,*)})(0)\Big\| < \epsilon \text{ independent of } m.$$

By the hypothesis of the theorem, $\{F_{(\Phi,m)}w^{(m,*)}\colon w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is now equicontinuous on $[-\tau, b]$. Hence, from Arzela-Ascoli theorem, the set $\{F_{(\Phi,m)}w^{(m,*)}\colon\ w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ is precompact in $PC([-\tau, b], E)$.

It is now concluded from Lemma 3.1 and Theorem 3.3 that the set $\{Fw^{(m,*)}: w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ and $\{F_m w^{(m,*)}: \ w^{(m,*)} \in \mathcal{H}^*\}_{m=1}^{\infty}$ are relatively compact in $C([-\tau, b], E)$ and $PC([-\tau, b], E)$ respectively. From the hypotheses, we can easily show that

$$\left\| F_h w^{(1)} - F_h w^{(2)} \right\| \leq N_2 \left\| w^{(1)} - w^{(2)} \right\|,$$

where $w_1, w_2 \in PC([-\tau, b], E)$.

In view of Kuratowski's measure of noncompactness, we have

$$\begin{aligned}
\mu(\{w^{(m,*)}\}) &= \mu(\{\mathcal{T}_m(w^{(m,*)})\}) \\
&\leq \mu(\{F_{(\Phi,m)}(w^{(m,*)})\}) + \mu(\{F_h(w^{(m,*)})\}) + \mu(\{F_m(w^{(m,*)})\}) + \mu(\{F(w^{(m,*)})\}) \\
&\leq N_2 \mu(\{w^{(m,*)}\}).
\end{aligned}$$

Then $\mu(\{w^{(m,*)}\}) = 0$ as $0 < N_2 < 1$. This means that the set $\mathcal{H}^*$ is precompact in $PC([-\tau, b], E)$. We can now assume without loss of generality that $w^{(m,*)} \to w^*$ in $PC(J, E)$ as $m \to \infty$. Taking the limit $m \to \infty$ in both sides of (3.17), we get

$$w^*(t) = \begin{cases} \mathcal{R}(t)[q(0) + \Phi(w^*)(0)] + F_h w^*(t) + \sum_{0<t_j<t} \mathcal{R}(t - t_j)\mathcal{I}_j(w^*_{t_j}) + Fw^*(t), & t \in J = [0, b], \\ q(t) + \Phi(w^*)(t), & t \in [-\tau, 0]. \end{cases}$$

Hence, by Definition 2.2, $w^* \in PC([-\tau, b], E)$ is the mild solution of impulsive neutral integro-differential system (1.1). This completes the proof. $\square$

Theorem 3.7. *Suppose that (A0)-(A3), (A4)(i) and (A7) are satisfied and the functions* $h: J \times \mathscr{E} \to E_\beta$, $\mathcal{I}_j: E \to E, (j = 1, 2 \ldots, k)$, $g: J \times E \to E$, *and* $\Phi: PC(J, E) \to \mathscr{E}$ *are uniformly bounded. If* $\eta(\cdot)A^\beta = A^\beta \eta(\cdot)$ *and* $N_2 < 1$, *then the system* (1.1) *is approximately controllable on* $[-\tau, b]$.

We omit the proof of this theorem because it can be proven similar to Theorem 3.4.

4. Example

Example 4.1. *Consider the following nonlocal neutral integro-differential equations that arise in the theory of heat flow in materials with fading memory:*

$$\begin{cases} \left[\frac{\partial}{\partial t} w(t, x) + \int_0^\pi \zeta(x, y) \cos(w(t - \tau, y))\, dy\right] = \frac{\partial^2 w(t,x)}{\partial x^2} + \int_0^t e^{-\alpha(t-r)} \frac{\partial^2 w(r,x)}{\partial x^2}\, dr \\ \qquad +g(t, w(t - \tau, x)) + v(t, x), \quad t \in [0, b], \ x \in [0, \pi], t \neq t_j, \\ \Delta w(t, x)|_{t=t_j} = \int_0^\pi \frac{p_j(y,x)}{1+(w(t_j-\tau,y))^2}\, dy, \ j = 1, 2, \cdots, k, \\ w(t, 0) = w(t, \pi) = 0, \quad t \in [0, b], \\ w(t, x) = q(t, x) + \sum_{i=1}^3 \varphi_i(t) \sin(w(s_i, x)), \quad t \in [-\tau, 0], \end{cases} \tag{4.1}$$

where $0 < t_1 < t_2 < \cdots < t_k < b$, $0 < s_1 < s_2 < s_3 < b$, $p_j \in C([0, \pi] \times [0, \pi], \mathbb{R}), j = 1, 2, \cdots, k$, φ_i $(i = 1, 2, 3)$ *are real continuous functions, and* g *is a given continuous function.*

Let $E = L^2([0, \pi], \mathbb{R})$. An operator A is defined on E as $Aw = w''$ with $D(A) = \{w \in E :$ w, w' are absolutely continuous $, w'' \in E$ and $w(0) = w(\pi) = 0\}$. In fact, A generates an analytic and compact semigroup $\{T(t), t \geq 0\}$, that is self-adjoint in Hilbert space E. Moreover, the operator A is given by

$$Au = -\sum_{m=1}^{\infty} m^2 \langle u, e_m \rangle e_m, \quad u \in D(A),$$

and semigroup $\{T(t)\}$ is given by

$$T(t)u = \sum_{m=1}^{\infty} \exp(-m^2 t) \langle u, e_m \rangle e_m, \quad u \in E,$$

where $e_m(x) = \sqrt{\frac{2}{\pi}} \sin(m\varphi), m \in \mathbb{N}$. Obviously the set $\{e_m \colon m \in \mathbb{N}\}$ is an orthonormal basis for E.

Furthermore, the operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} u = \sum_{m=1}^{\infty} m \langle u, e_m \rangle e_m, \quad u \in D((-A)^{\frac{1}{2}}),$$

where $D((-A)^{\frac{1}{2}}) = \{u \in E \colon \sum_{m=1}^{\infty} m \langle u, e_m \rangle e_m \in E\}$. Let $B = I$ and $V = D((-A)^{\frac{1}{2}})$ with norm $\| \cdot \|_{\frac{1}{2}} = \|(-A)^{\frac{1}{2}} \cdot \|$.

Define $w(t)(x) = w(t,x)$, $g(t, w_t)(x) = g(t, w(t-\tau, x))$, $\mathcal{I}_j(w_{t_j})(x) = \int_0^{\pi} \frac{p_j(y,x)}{1+(w(t_j-\tau,y))^2}\, dy$, $h(t, w_t)(x) = \int_0^{\pi} \zeta(x,y) \cos(w(t-\tau, y))\, dy$ and $\Phi(w)(t)(x) = \sum_{i=1}^{3} \Phi_i(t) \sin(w(s_i, x))$, here $x \in [0, \pi]$. We also define $\eta(t) \colon D(A) \subset E \to E$ by $\eta(t)w = e^{-\alpha t} Aw$ for $w \in D(A)$.

Using the above notations and conditions, we can represent the system (4.1) in the abstract form (1.1). It's very simple to check that the conditions (K1)-(K3) hold as $\widehat{\eta}(\gamma) = \frac{1}{\gamma+\alpha} A$ and $X = C_0^{\infty}([0,\pi])$, where $C_0^{\infty}([0,\pi])$ denotes the space of infinitely differentiable real valued functions vanishing at 0 and π. Then the linear system of (4.1) has a resolvent operator $\mathcal{R}(\cdot) \colon [0, \infty) \to \mathcal{L}(E)$ defined as

$$\mathcal{R}(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\varrho,\vartheta}} e^{\gamma t} (\gamma I + A - \widehat{\eta}(\gamma))^{-1}\, d\gamma, & t > 0, \\ I, & t = 0. \end{cases} \tag{4.2}$$

It is clear that $\mathcal{I}_j, j = 1, 2, \cdots, k$, are uniformly bounded functions and satisfy the assumption (A3). Let $L_i = \sup_{t \in [-\tau, 0]} \varphi_i(t), i = 1, 2, 3$, and then let $L_\Phi = \max\{L_1, L_2, L_3\}$. Clearly Φ satisfies the Lipschitz condition with Lipschitz constant L_Φ, i.e, (A6) is satisfied. Now we suppose that g is a uniformly bounded function and satisfies the assumption (A4)(i).

Since the semigroup $T(t)$ is compact, we conclude from [29, Theorem 3.3 of Chapter 2] and Lemma 2.3 that the resolvent operator $\mathcal{R}(t)$ is compact for each $t > 0$. For any $w \in E$,

$$\begin{aligned} \langle B^* \mathcal{R}^*(t) w, v \rangle &= \langle \mathcal{R}^*(t) w, v \rangle \\ &= \langle w, \mathcal{R}(t) v \rangle, \quad \forall\, v \in V \text{ and } \forall\, t \in J. \end{aligned}$$

If we let $B^* \mathcal{R}^*(t) w = 0$, then

$$\langle w, \mathcal{R}(t) v \rangle = 0, \quad \forall\, v \in V \text{ and } \forall\, t \in J.$$

At $t = 0$, we get

$$\langle w, v \rangle = 0, \quad \forall\, v \in V.$$

Since V is dense in E, we obtain that $w = 0$. We can now say from Remark 2.7 that the condition (A0) is satisfied. Hence, in view of Theorems 3.3 and 3.4, the nonlocal impulsive neutral integro-differential system (4.1) is approximately controllable on $[-\tau, b]$.

Example 4.2. *Let $E = L^2([0,\pi],\mathbb{R})$. Consider the following neutral integro-differential equations:*

$$\begin{cases} \left[\frac{\partial}{\partial t} w(t,x) + \int_0^\pi \zeta(x,y)\cos(w(t-\tau,y))\,dy\right] = \frac{\partial^2 w(t,x)}{\partial x^2} + \int_0^t e^{-\alpha(t-r)}\frac{\partial^2 w(r,x)}{\partial x^2}\,dr \\ \qquad +g(t, w(t-\tau,x)) + v(t,x), \quad t \in [0,b],\ x \in [0,\pi], t \neq t_j, \\ \Delta w(t,x)|_{t=t_j} = \int_0^\pi \frac{p_j(y,x)}{1+(w(t_j-\tau,y))^2}\,dy,\ j = 1,2,\cdots,k, \\ w(t,0) = w(t,\pi) = 0, \quad t \in [0,b], \\ w(t,x) = q(t,x) + \int_\alpha^b \varphi(r,t)\sin((w(r,x))^2)\,dr, \quad t \in [-\tau, 0], \end{cases} \tag{4.3}$$

where $0 < t_1 < t_2 < \cdots < t_k < b$, $p_j \in C([0,\pi]\times[0,\pi],\mathbb{R}), j = 1,2,\cdots,k$, g is a given continuous function, and $\varphi(r,t) \in C([0,b]\times[-\tau,0]$.

Define the operator A as defined in Example 4.1. Therefore the operator A generates the compact analytic semigroup $T(t)$, $(t \geq 0)$. Let $B = I$ and $V = D((-A)^{\frac{1}{2}})$ with norm $\|\cdot\|_{\frac{1}{2}} = \|(-A)^{\frac{1}{2}}\cdot\|$.

Now we take $\Phi(\cdot)$ as $\Phi(w)(t)(x) = \int_\alpha^b \varphi(r,t)\sin((w(r,x))^2)\,dr$. Clearly Φ is uniformly bounded and satisfies hypothesis (A7). Thus it follows from Example 4.1 that the system (4.3) can be represented in the abstract form (1.1) and functions $\mathcal{I}_j, j = 1,2,\cdots,k$, are uniformly bounded and satisfy the assumption (A3). It is also clear that the linear system of (4.3) has a resolvent operator $\mathcal{R}(\cdot)\colon \mathcal{L}(E)$ given by (4.2). If g is a uniformly bounded function and satisfies the assumption (A4)(i), then, from Example 4.1, all the conditions of Theorem 3.7 are satisfied. Hence the nonlocal impulsive neutral integro-differential system (4.3) is approximately controllable on $[-\tau, b]$.

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