Symmetries and solutions to dissipative hyperbolic geometric flow

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Abstract: Based on the Lie-symmetric method, we study the solutions of dissipative hyperbolic geometric flows on Riemann surfaces; In the process of simplification, the mixed equations are produced. And the hyperbolic equations are obtained under limited conditions. Considering the Cauchy problem of the hyperbolic equation, the existence and uniqueness conditions of the global solutions are obtained. Finally, the phenomenon of blow up is discussed.

Keywords: dissipative hyperbolic geometric flow, Lie symmetry, group-invariant solutions, blow up

1. Introduction

Hyperbolic geometric flows are of particular importance to the understanding of manifold structure, space-time geometry, modern physics, general relativity and gravity theory. The hyperbolic geometric flow was first proposed by Kong and Liu [1],

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}\left(g, \frac{\partial g}{\partial t}\right) = 0, \tag{1.1}$$

in which g_{ij} is the surface metric, $\mathcal{F}$ is the smooth function of $g, \frac{\partial g}{\partial t}$, R_{ij} is the $Ricci$ curvature tensor.

Liu studied the model [2]:

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} - \beta \frac{\partial g_{ij}}{\partial t}, \tag{1.2}$$

in which β is a positive constant. The classical global solution to the Cauchy problem of dissipative hyperbolic geometric flows is obtained, and it is discussed that the solution brows up. Meanwhile Liu [3] study the mixed initial boundary value problem of hyperbolic geometric flow and proved the global existence of classical solutions.

The lifetimes of classical solutions of hyperbolic geometric flows with two spatial variables with slow decay initial data were studied by Kong, Liu et al [4]. Dai et al. [5] studied hyperbolic

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geometric flow about the ephemeral existence and uniqueness theorem, and the nonlinear stability of hyperbolic geometric flow which is larger than 4D of Euclidean space was demonstrated. And they derived wave equations satisfied by the curvatures. They also obtained the relation that it is hyper geometric flow with the Einstein equation and the Ricci flow. Kong and Wang [6] studied the Einstein's hyperbolic geometric flow, which provides a natural tool to deform the shape of a manifold and to understand the wave character of metrics, the wave phenomenon of the curvature for evolutionary manifolds. The global existence of classical solutions for dissipative hyperbolic geometry flow in accord with the time was studied by Kong et al. [7]. Then Kong et al. [8] discussed the lower bound of life-span which is classical solutions for hyperbolic geometry flow equations with "small" initial data in several space dimensions. And they [9] obtained classical solutions of a dissipative hyperbolic geometry flow that has two spatial variates.

In [10], Zhu studied a class of hyperbolic geometric flows defined on N-dimensional Riemannian manifolds. Later, Huo introduced three typical hyperbolic geometric flows on Riemann surface proposed by Kong et al in [11]: standard hyperbolic geometric flows, Einstein hyperbolic geometric flows and dissipative hyperbolic geometric flows.

Wang studied the model [12]:

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} = -Rg_{ij}, \tag{1.3}$$

she investigates group-invariant solutions to hyperbolic geometric flow on Riemann surfaces and discussed the blow up of the solution.

In this article, we will study a special case in eq (1.2). When $\beta = 1$, namely

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} - \frac{\partial g_{ij}}{\partial t}. \tag{1.4}$$

First of all, referring to a paper by Gao [13], we obtain a five-dimensional symmetric group of Lie points and a one-dimensional optimal system, and then further obtain the group-invariant solutions. After simplification and solution, many kinds of equations are obtained. It is found that the mixed equation is hyperbolic under limited conditions. Next, we consider the Cauchy problem of hyperbolic equation. Then the existence and uniqueness of the global solution is proved. Finally, blow up of the solution are investigates.

Suppose that the Riemann surface is the topological type R^2, let the initial metric be

$$t = 0: ds^2 = u_0(x)(dx^2 + dy^2), \tag{1.5}$$

in which $u_0(x) \in C^2$, with bounded C^2 norm and it satisfies

$$1 < H \leq u_0(x) \leq M < \infty, \tag{1.6}$$

in which H, M are positive constant.

Theorem 1.1 Suppose the initial metric (1.5) satisfies (1.6) and any smooth function $u_1(x)$ with bounded C^1 norm, if one of the following conditions holds:

(i)For any $x \in R$,

$$u_1(x)+\frac{u_0'(x)}{\sqrt{u_0(x)-1}}>0,\ u_1(x)-\frac{u_0'(x)}{\sqrt{u_0(x)-1}}<0\,, \tag{1.7}$$

$$\frac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x)-\frac{u_0'(x)}{\sqrt{u_0(x)-1}}\right)+\frac{2}{3}(u_0(x)-1)^{\frac{3}{4}}\geq 0. \tag{1.8}$$

(ii)For any $x \in R$,

$$u_1(x)+\frac{u_0'(x)}{\sqrt{u_0(x)-1}}<0,\ u_1(x)-\frac{u_0'(x)}{\sqrt{u_0(x)-1}}>0, \tag{1.9}$$

$$\frac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x)+\frac{u_0'(x)}{\sqrt{u_0(x)-1}}\right)+\frac{2}{3}(u_0(x)-1)^{\frac{3}{4}}\geq 0. \tag{1.10}$$

Then the Cauchy problem

$$\begin{cases}(e^w-1)w_{tt}-w_{xx}=-e^w(w_t^2+w_t),\\ t=0: e^w=u_0(x),\ (e^w)_t=u_1(x),\end{cases} \tag{1.11}$$

has a unique global classical solution on $t \geq 0$.

Theorem 1.2 Suppose (1.7) holds, and there is a point $x_0 \in R$, such that

$$\frac{1}{\sqrt[4]{u_0(x_0)-1}}\left(u_1(x_0)-\frac{u_0'(x)}{\sqrt{u_0(x_0)-1}}\right)+\frac{2}{3}(u_0(x_0)-1)^{\frac{3}{4}}<-\frac{4}{3}M^{\frac{3}{4}}, \tag{1.12}$$

or (1.9) holds, and there is a point $x_0 \in R$, which make

$$\frac{1}{\sqrt[4]{u_0(x_0)-1}}\left(u_1(x_0)+\frac{u_0'(x)}{\sqrt{u_0(x_0)-1}}\right)+\frac{2}{3}(u_0(x_0)-1)^{\frac{3}{4}}<-\frac{4}{3}M^{\frac{3}{4}}, \tag{1.13}$$

then the classical solution of the Cauchy problem (1.11) blows up on a local scale.

2. Group-invariant solutions of dissipative hyperbolic geometric flows [14]

For any surface $(\mathcal{M}^2, g)$, the scalar curvature $R=2K$, in which K is $Gauss$ curvature of the surface, since

$$R_{ij}=\frac{1}{2}Rg_{ij}.$$

The surface metric is locally conformal to the Euclidean metric,

$$g_{ij}=u(x,y,t)\delta_{ij},$$

in which $u(x,y,t)>0$ is the conformal factor of g_{ij}, δ_{ij} is $Kronecker$ symbol. So

$$R=-\frac{\Delta \ln u}{u}. \tag{2.1}$$

Thus, dissipative hyperbolic geometric flow eq (1.4) can become as follows:

$$u_{tt}+u_t=\Delta \ln u. \tag{2.2}$$

Make $w = \ln u$, eq (2.2) becomes

$$e^w w_{tt} + e^w w_t^2 + e^w w_t - w_{xx} - w_{yy} = 0. \tag{2.3}$$

Suppose the one-parameter group of infinitesimal transformations (x, y, t, w) is given by

$$\begin{gathered} x^* = x + \varepsilon\xi_1(x, y, t, w) + o(\varepsilon^2), \\ y^* = y + \varepsilon\xi_2(x, y, t, w) + o(\varepsilon^2), \\ t^* = t + \varepsilon\xi_3(x, y, t, w) + o(\varepsilon^2), \\ w^* = w + \varepsilon\eta(x, y, t, w) + o(\varepsilon^2), \end{gathered} \tag{2.4}$$

in which ε is a group parameter.

The decision equations of the equation (2.3) can be settled as follows:

$$\begin{cases} \xi_{3_t} = 0, \ \xi_{3_u} = 0, \ \xi_{3_x} = 0, \ \xi_{3_y} = 0, \\ \xi_{1_t} = 0, \ \xi_{1_u} = 0, \ \xi_{1_{xx}} = -\xi_{1_{yy}}, \\ \xi_{2_t} = 0, \ \xi_{2_u} = 0, \ \xi_{2_x} = -\xi_{1_y}, \ \xi_{2_y} = \xi_{1_x}, \\ \eta = -2\xi_{1_x}. \end{cases}$$

So, if we go further,

$$\begin{cases} \xi_1 = \xi_1(x, y), \ \xi_2 = \xi_2(x, y), \\ \xi_3 = c_1, \ \eta = -2\xi_{1_x}, \\ \xi_{1_x} - \xi_{2_y} = 0, \ \xi_{2_x} + \xi_{1_y} = 0. \end{cases}$$

From the last two equations we can see that $\xi_1 + i\xi_2 = F(x + iy)$. Here we let $F(x + iy) = k_1(x + iy) + k_2$ be linear, in which k_1, k_2 are complex constants, namely

$$\begin{cases} \xi_3 = c_1, \ \xi_1 = c_2 + c_3x + c_4y, \\ \xi_2 = c_5 - c_4x + c_3y, \ \eta = -2c_3, \end{cases} \tag{2.5}$$

in which c_1, c_2, c_3, c_4, c_5 are real constants. Then the vector field is known as:

$$\begin{aligned} V &= \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial w} \\ &= (c_2 + c_3x + c_4y)\frac{\partial}{\partial x} + (c_5 - c_4x + c_3y)\frac{\partial}{\partial y} + c_1\frac{\partial}{\partial t} + (-2c_3)\frac{\partial}{\partial w}. \end{aligned} \tag{2.6}$$

Hence the vector field of the Lie symmetric group of all vector fields in eq (2.3) is given:

$$\begin{cases} V_1 = \partial t, \\ V_2 = \partial x, \\ V_3 = \partial y, \\ V_4 = x\partial x + y\partial y - 2\partial w, \\ V_5 = y\partial x - x\partial y. \end{cases} \tag{2.7}$$

Theorem 2.1: Generators in (2.7) generate an optimal system S:

$$\{V_5 \pm V_1 \pm V_4,\ V_5 \pm V_4,\ V_5 \pm V_1, V_5,\ V_4 \pm V_1,\ V_4,\ V_2 \pm V_1,\ V_2,\ V_3 \pm V_1, V_3, V_1\}.$$

Next, we will consider the solution to the equation (2.3).

(1) $V = V_5 + V_1 + V_4 = (x + y)\partial x + (y - x)\partial y + \partial t - 2\partial w.$

The corresponding characteristic equations are

$$\frac{dx}{x+y}=\frac{dy}{y-x}=\frac{dt}{1}=\frac{dw}{-2},$$

by solving the above equations, we get the invariances

$$z_1=2\arctan\left(\frac{y}{x}\right)+\ln(x^2+y^2),\ \ z_2=w+2t,$$

the invariant solution is

$$w=-2t+\ln\big(h(z_1)\big),$$

then eq (2.3) can be reduced as

$$2he^{-2t}-\frac{8}{x^2+y^2}\frac{(hh''-h'^2)}{h^2}=0.$$

(2) $V=V_5+V_4=(x+y)\partial x+(y-x)\partial y-2\partial w.$

The corresponding characteristic equations are

$$\frac{dx}{x+y}=\frac{dy}{y-x}=\frac{dt}{0}=\frac{dw}{-2},$$

the invariances are

$$z=2\arctan\left(\frac{y}{x}\right)+\ln(x^2+y^2),\ \ t,$$

the invariant solution is given by

$$w=-2\ln(x+y)+\ln\big(h(z,t)\big),$$

then eq (2.3) can be reduced as

$$\frac{1}{(x+y)^2}(h_{tt}+h_t)-\frac{8}{x+y}-\frac{8}{x^2+y^2}\frac{(hh_{zz}-h_z^2)}{h^2}=0.$$

(3) $V=V_5=y\partial x-x\partial y.$

The corresponding characteristic equations are

$$\frac{dx}{y}=\frac{dy}{-x}=\frac{dt}{0}=\frac{dw}{0},$$

the invariances are

$$z=x^2+y^2,\ \ t,$$

the invariant solution is

$$w=w(z,t),$$

then eq (2.3) can be reduced as

$$e^w(w_{tt}+w_t^2+w_t)-2z^2w_{zz}-4w_z=0.$$

(4) $V=V_4+V_1=x\partial x+y\partial y+\partial t-2\partial w.$

The corresponding characteristic equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{1}=\frac{dw}{-2},$$

the invariances are

$$z_1=xy^{-1},\ \ z_2=w+2t,$$

the invariant solution is

$$w = -2t + \ln\big(h(z_1)\big),$$

then eq (2.3) can be reduced as

$$2he^{-2t} - \frac{hh'' - h'^2}{y^2h^2} + \frac{-x^2hh'' - 2xyhh' + x^2h'^2}{y^4h^2} = 0.$$

(5) $V = V_2 + V_1 = \partial x + \partial t.$

The corresponding characteristic equations are

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{1} = \frac{dw}{0},$$

the invariances are

$$y,\ \ z = t - x,$$

the invariant solution is

$$w = w(y, z),$$

then eq (2.3) can be reduced as

$$w_{zz} + w_{yy} = e^w(w_{zz} + w_z^2 + w_z). \qquad (2.8)$$

Look for travelling wave solution to eq (2.8), suppose $\sigma = z + \delta y$, and $\lambda = 1 + \delta^2$, then eq (2.8) becomes

$$(1 + \delta^2)w_{\sigma\sigma} = e^w(w_{\sigma\sigma} + w_\sigma^2 + w_\sigma),$$

namely

$$\lambda w_{\sigma\sigma} = (e^w)_{\sigma\sigma} + e^w{}_\sigma.$$

Let's integrate above equation twice with respect to σ, the implicit solution of eq (2.8) is given by

$$\lambda w = e^w + \int e^w d\sigma + c_1\sigma + c_2,$$

where c_1, c_2 are arbitrary constants.

(6) $V = V_3 - V_1 = \partial y - \partial t.$

The corresponding characteristic equations are

$$\frac{dx}{0} = \frac{dy}{1} = -\frac{dt}{1} = \frac{dw}{0},$$

the invariances are

$$z = y + t,\ \ x,$$

the invariant solution is

$$w = w(z, x),$$

then eq (2.3) can be reduced as

$$e^w(w_{zz} + w_z^2 + w_z) = w_{xx} + w_{zz}. \qquad (2.9)$$

Look for travelling wave solution to the above equation, suppose $\rho = z + \delta x$, and $\lambda = 1 + \delta^2$, then equation becomes

$$e^w\left(w_{\rho\rho} + w_\rho^2 + w_\rho\right) = \lambda w_{\rho\rho},$$

namely

$$(e^w)_{\rho\rho} + (e^w)_\rho = \lambda w_{\rho\rho}.$$

Let's integrate above equation twice with respect to ρ, the solution of eq (2.9) satisfies

$$e^w + \int e^w d\rho = \lambda w + c_1\rho + c_2,$$

in which c_1, c_2 are arbitrary constants.

3. Global solution and blow up

3.1 Preliminaries

Eq (2.8) is equivalent to $(e^w - 1)w_{zz} - w_{yy} = -e^w(w_z^2 + w_z)$, Let's replace z with t, and replace y with x, we have

$$(e^w - 1)w_{tt} - w_{xx} = -e^w(w_t^2 + w_t), \tag{3.1}$$

if $e^w - 1 > 0$, eq (3.1) is hyperbolic; if $e^w - 1 < 0$, eq (3.1) is elliptic. Next, Let's talk about the hyperbolic case. Make $v = w_t$, $h = w_x$, then eq (3.1) is able to become to a first order quasilinear equations set:

$$\begin{cases} w_t = v, \\ h_t - v_x = 0, \\ v_t - \dfrac{1}{e^w - 1} h_x = -\dfrac{e^w}{e^w - 1}(v^2 + v). \end{cases} \tag{3.2}$$

The eigenvalue of eq (3.2) can be easily calculated as $\lambda_1 = -\lambda,\ \lambda_2 = 0,\ \lambda_3 = \lambda,$ in which $\lambda = \sqrt{\frac{1}{e^w - 1}}$, the matrix $L(U)(U = (w, h, v)^T)$ of left eigenvectors and the matrix $R(U)$ of right eigenvectors are respectively,

$$L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & -\lambda \end{pmatrix},$$

$$R(U) = \left(r_1(U), r_2(U), r_3(U)\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \lambda & 0 & -\lambda \end{pmatrix}.$$

Since $\nabla\lambda_i(U) r_i(U) \equiv 0\ (i = 1,2,3)$, so the system (3.2) is a linearly degenerate hyperbolic system. Set $p = v + \lambda h$, $q = v - \lambda h$, and $\mu = -\frac{e^w}{4(e^w - 1)}$, then we have the following lemma.

Lemma 3.1 p and q satisfy

$$\begin{cases} w_t = \dfrac{p + q}{2}, \\ p_t - \lambda p_x = \mu[(p + 3q + 2)p + 2q], \\ q_t + \lambda q_x = \mu[(q + 3p + 2)q + 2p]. \end{cases} \tag{3.3}$$

Because of Cauchy problem

$$\begin{cases}(e^w-1)w_{tt}-w_{xx}=-e^w(w_t^2+w_t),\\ t=0: e^w=u_0(x),\ \ (e^w)_t=u_1(x),\end{cases}\tag{3.4}$$

is equivalent to

$$\begin{cases}w_t=\dfrac{p+q}{2},\\ p_t-\lambda p_x=\mu[(p+3q+2)p+2q],\\ q_t+\lambda q_x=\mu[(q+3p+2)q+2p],\\ t=0: w=\ln\big(u_0(x)\big),\ \ p=p_0(x),\ \ q=q_0(x),\end{cases}\tag{3.5}$$

in which

$$\begin{cases}p_0(x)=\dfrac{u_1(x)}{u_0(x)}+\sqrt{\dfrac{1}{u_0(x)-1}\dfrac{u_0'(x)}{u_0(x)}},\\ q_0(x)=\dfrac{u_1(x)}{u_0(x)}-\sqrt{\dfrac{1}{u_0(x)-1}\dfrac{u_0'(x)}{u_0(x)}},\\ \lambda=\sqrt{\dfrac{1}{e^w-1}},\ \ \mu=-\dfrac{e^w}{4(e^w-1)}.\end{cases}\tag{3.5'}$$

Lemma 3.2 Let $r=p_x,\ \ s=q_x$, then r and s satisfy

$$\begin{cases}r_t-\lambda r_x=A_1r+A_2s+h_1,\\ s_t+\lambda s_x=B_1s+B_2r+h_2,\end{cases}\tag{3.6}$$

in which

$$A_1=\mu(3p+2q+2),\ \ A_2=\mu(3p+2),\ \ h_1=-\frac{\mu\lambda}{2}(p-q)[(p+3q+2)p+2q],$$

$$B_1=\mu(3q+2p+2),\ \ B_2=\mu(3q+2),\ \ h_2=-\frac{\mu\lambda}{2}(p-q)[(q+3p+2)q+2p].$$

Theorem 3.1 In the existence domain $G(T)$ of the classical solutions of Cauchy problems (3.5) and (3.5′), if there is a constant $M_0>0$, such that

$$|p(x,t)|\le M_0,\ \ |q(x,t)|\le M_0,\tag{3.7}$$

so, in the region $G(T)$,

$$|u(x,t)|\le N(T),\ \ |u_x(x,t)|\le N(T),\ \ |u_{xx}|\le N(T),\tag{3.8}$$

in which $N(T)$ is a positive constant that depends on T, $G(T)=\{(x,t)\lfloor x\in R, 0\le t\le T, \forall T>0\}$.Thus, from the local existence results of classical solutions of quasilinear hyperbolic equations [14], the Cauchy problem (3.4) has a unique global solution for all $t\ge 0$.

Proof: At any point (t,x), denote

$$x=x_1(t,\beta_1),\ \ x=x_2(t,\beta_2),\ \ x=x_3(t,\beta_3),$$

they are the characteristic lines of λ_1, λ_2 and λ_3 respectively, and satisfy

$$\frac{dx_1}{dt}=\lambda_1=-\lambda,\ \ \frac{dx_2}{dt}=\lambda_2=0,\ \ \frac{dx_3}{dt}=\lambda_3=\lambda,$$

$$x_1(0,\beta_1)=\beta_1,\ \ x_2(0,\beta_2)=\beta_2,\ \ x_3(0,\beta_3)=\beta_3.$$

It is calculated that

$$w_x=\frac{p-q}{2\lambda}, \tag{3.9}$$

$$w_{xx}=\tfrac{1}{2\lambda}(r-s)-\tfrac{\mu}{2\lambda^2}(p-q)^2. \tag{3.10}$$

Thus, by the first formula of (3.5), (3.6) and (3.7), we can get

$$\ln H-M_0t\le w\le \ln M+M_0t, \tag{3.11}$$

$$-M_0\sqrt{He^{-M_0t}-1}\le w_x\le M_0\sqrt{Me^{M_0t}-1}, \tag{3.12}$$

$$|A_i|\le N_1,\ \ |B_i|\le N_1, i=1,2,$$

in which N_1 is a positive constant. From **Theorem 2.3** in [15],

$$|r|\le M(T),\ \ |s|\le M(T). \tag{3.13}$$

From (3.10) - (3.13), we have

$$|w_{xx}|\le M(T). \tag{3.14}$$

By $u=e^w$, from (3.11) - (3.12) and (3.14), we can prove (3.8).

3.2 Proof of Theorem 1.1 and Theorem 1.2

In this section, we study the brow up of the classical solutions to the initial-value problem for $(3.5)-(3.5')$.

If want to prove **Theorem 1.1**, in term of the local existence theorem of classical solutions of quasilinear hyperbolic equations [14], it is only necessary to prove that C^2 norm of $u(x,t)$ has a consistent prior estimation in the existence domain of the smooth solutions.

We know from the previous, $w=\ln u$ and

$$-\frac{3}{4}pq=-\frac{3}{4}pL_1w=-\frac{3}{4}qL_2w, \tag{3.15}$$

in which

$$L_1=\frac{\partial}{\partial t}-\lambda\frac{\partial}{\partial x},\ \ L_2=\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial x}.$$

Let

$$\bar p=(u-1)^{\frac{3}{4}}p,\ \ \bar q=(u-1)^{\frac{3}{4}}q, \tag{3.16}$$

so

$$\begin{cases} L_1\bar p=\bar p_t-\lambda\bar p_x=\dfrac{\mu}{(u(x)-1)^{\frac{3}{4}}}\bar p^2+2k(\bar p+\bar q),\\ L_2\bar q=\bar q_t+\lambda\bar q_x=\dfrac{\mu}{(u(x)-1)^{\frac{3}{4}}}\bar q^2+2k(\bar p+\bar q).\end{cases} \tag{3.17}$$

By calculation,

$$\begin{cases} L_1(u(x)-1)^{\frac{3}{4}} = -3\mu(u(x)-1)^{\frac{3}{4}}q, \\ L_2(u(x)-1)^{\frac{3}{4}} = -3\mu(u(x)-1)^{\frac{3}{4}}p. \end{cases} \tag{3.18}$$

Lemma 3.3 For any $x \in R$, if

$$\bar{p}_0(x) < 0,\ \ \bar{q}_0(x) > 0, \tag{3.19}$$

or

$$\bar{p}_0(x) > 0,\ \ \bar{q}_0(x) < 0, \tag{3.20}$$

then in the existence domain of the classical solution to Cauchy problem (3.17) with initial condition (3.21), for any $(x,t) \in R \times R^+$, the following inequalities

$$t=0: \begin{cases} \bar{p} = \bar{p}_0(x) = \dfrac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x) + \dfrac{u_0'(x)}{\sqrt{u_0(x)-1}}\right), \\ \bar{q} = \bar{q}_0(x) = \dfrac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x) - \dfrac{u_0'(x)}{\sqrt{u_0(x)-1}}\right). \end{cases} \tag{3.21}$$

$$\bar{p}(x,t) < 0,\ \ \bar{q}(x,t) > 0, \tag{3.22}$$

or

$$\bar{p}(x,t) > 0,\ \ \bar{q}(x,t) < 0. \tag{3.23}$$

Proof: If (3.19) or (3.20) holds, then by the continuity of $\bar{p}(x,t)$ and $\bar{q}(x,t)$, there exists $\delta > 0$, such that for any $(x,t) \in R \times [0,\delta]$, (3.22) or (3.23) holds.

Let's prove that (3.22) or (3.23) holds for any $(x,t) \in R \times R^+$. If (3.22) or (3.23) does not hold for $(x,t) \in R \times R^+$, then there is a $(x_0,t_0)(t_0 > 0)$ such that

$$\begin{cases} \bar{p}(x,t) < 0,\ \ \bar{q}(x,t) > 0,\ \ t \in R \times [0,t_0), \\ \bar{p}(x_0,t_0) = 0,\ \ \bar{q}(x_0,t_0) > 0, \end{cases} \tag{3.24}$$

or

$$\begin{cases} \bar{p}(x,t) < 0,\ \ \bar{q}(x,t) > 0,\ \ t \in R \times [0,t_0), \\ \bar{p}(x_0,t_0) < 0,\ \ \bar{q}(x_0,t_0) = 0. \end{cases} \tag{3.25}$$

If (3.24) holds, then by the first one in (3.17), we have,

$$L_1\bar{p}(x_0,t_0) = -\frac{u(x_0)}{2(u(x_0)-1)}\bar{q}(x_0,t_0) < 0,$$

however, because of $\bar{p}(x,t) < 0\ \big(t \in R \times [0,t_0)\big)$ and $\bar{p}(x_0,t_0) = 0$, so $L_1\bar{p}(x_0,t_0) \geq 0$, this is a contradiction.

Similarly, we can get (3.23) holds for any $(x,t) \in R \times R^+$. Therefore, the proof is completed.

Denote

$$m = (u(x)-1)^{\frac{3}{4}}\left(p + \frac{2}{3}\right),\ \ n = (u(x)-1)^{\frac{3}{4}}\left(q + \frac{2}{3}\right), \tag{3.26}$$

the following lemma is readily available.

Lemma 3.4 m and n satisfy

$$\begin{cases} L_1 m = \frac{\mu}{(u(x)-1)^{\frac{3}{4}}}(m-m_1)(m-m_2), \\ L_2 n = \frac{\mu}{(u(x)-1)^{\frac{3}{4}}}(n-n_1)(n-n_2), \end{cases} \tag{3.27}$$

in which

$$m_1 = n_1 = -\frac{4}{3}(u(x)-1)^{\frac{3}{4}} < 0, m_2 = n_2 = \frac{2}{3}(u(x)-1)^{\frac{3}{4}} > 0. \tag{3.28}$$

Lemma 3.5 [16,17] If there is

$$\begin{cases} z'(t) = -h(t)\big(z - z_1(t)\big)\big(z - z_2(t)\big), \\ z(0) = z_0, \end{cases} \tag{3.29}$$

in which $h(t),\ z_1(t),\ z_2(t) \in C[0,+\infty),\ h(t) > 0,$

$$z_1(t) < 0 < z_2(t), \tag{3.30}$$

(1) If $z_0 \geq 0$, then the continuous solution $z(t)$ of the problem (3.29) satisfies

$$0 \leq z(t) \leq \sup_{t\geq 0} z_2(t);$$

(2) If

$$z_0 < \inf_{t\geq 0} z_1(t),$$

then there is a finite $\tau > 0$, such that as $t \to \tau^+$, the continuous solution $z(t)$ to the problem (3.29) satisfies $z(t) \to -\infty$.

If (1.7) or (1.9) holds, then (3.19) or (3.20) holds. And (3.22) or (3.23) is satisfied. Therefore, according to **Lemma 3.3**, (1.6), (3.16) and (3.18), we have

$$(H-1)^{\frac{3}{4}} \leq (u(x)-1)^{\frac{3}{4}} \leq (M-1)^{\frac{3}{4}}. \tag{3.31}$$

From (3.28) and (3.31), we get

$$-\frac{4}{3}(M-1)^{\frac{3}{4}} \leq m_1,\ \ n_1 \leq -\frac{4}{3}(H-1)^{\frac{3}{4}}, \tag{3.32}$$

$$\frac{2}{3}(M-1)^{\frac{3}{4}} \leq m_2,\ \ n_2 \leq \frac{2}{3}(M-1)^{\frac{3}{4}}. \tag{3.33}$$

Proof of Theorem 1.1: It is easily known that at $t = 0$,

$$\begin{cases} m = m_0(x) = \frac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)-1}}\right) + \frac{2}{3}(u_0(x)-1)^{\frac{3}{4}}, \\ n = n_0(x) = \frac{1}{\sqrt[4]{u_0(x)-1}}\left(u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)-1}}\right) + \frac{2}{3}(u_0(x)-1)^{\frac{3}{4}}. \end{cases} \tag{3.34}$$

Therefore, based on the assumptions, if (1.7) holds, then we have

$$m_0(x) \geq \frac{2}{3}(u_0(x)-1)^{\frac{3}{4}} \geq 0.$$

According to (1.7), (1.8), **Lemma 3.3** and **Lemma 3.5**, we have

$$\frac{2}{3}(H-1)^{\frac{3}{4}} \leq m \leq \frac{2}{3}(M-1)^{\frac{3}{4}},\ \ 0 \leq n \leq \frac{2}{3}(M-1)^{\frac{3}{4}}. \tag{3.35}$$

By (3.16) and (3.26), we get

$$p = (u(x)-1)^{-\frac{3}{4}}m - \frac{2}{3},\ \ q = (u(x)-1)^{-\frac{3}{4}}n - \frac{2}{3}, \tag{3.36}$$

Then we have

$$0 < p \le \frac{2}{3}\left[\left(\frac{M-1}{H-1}\right)^{\frac{3}{4}} - 1\right],\ \ -\frac{2}{3} \le q < 0. \tag{3.37}$$

Similarly, if condition (1.9) holds, then

$$0 < q \le \frac{2}{3}\left[\left(\frac{M-1}{H-1}\right)^{\frac{3}{4}} - 1\right],\ \ -\frac{2}{3} \le p < 0. \tag{3.38}$$

Based on **Theorem 3.1**, **Theorem 1.1** proved.

Proof of Theorem 1.2: If (1.12) holds, then

$$n_0(x_0) < \inf_{t\ge 0} n_1(t),$$

and from **Lemma 3.5**, there is a finite $\tau > 0$, as $t \to \tau^+$,we have $n(x,t) \to -\infty$.

Similarly, if (1.13) holds, there is also a finite $\tau > 0$, when $t \to \tau^+$, $m(x,t) \to -\infty$.

4. Conclusion

This paper includes three parts: the first part recommended the background knowledge of dissipative hyperbolic geometric flow; In the second part, symmetric group of dissipative hyperbolic flows, one-dimensional optimal system and exact solutions are given. Finally, we proved the main results.

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Competing interests

The authors declare that they have no competing interest.

Authors' contributions

The reduction equations were derived from symmetries and their solutions were obtained by the second author Zenggui Wang, the other part of the work is done by the author Fang Gao. All authors read and approved the final manuscript.

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