POSITIVE SOLUTIONS FOR FRACTIONAL BOUNDARY VALUE PROBLEMS UNDER A GENERALIZED FRACTIONAL OPERATOR

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Abstract. The work reported here concerns with study a generalized nonlinear fractional boundary value problems involving ϑ- fractional derivative in the Riemann-Liouville sense. The existence and uniqueness of positive solutions to the problem at hand are proved. Our discussion relies on the properties of the Green's function, the upper and lower solutions method, and the classical fixed point theorems in a cone. Moreover, building upper and lower control functions have an effective role in the analysis. Some examples are offered to justify the validity of theoretical findings.

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1. Introduction

Fractional calculus (FC) is a generalization for standard calculus since it deals with fractional orders that exceed integer numbers, whether real or even complex. During the last four decades the theory of fractional calculus achieves great importance due to it applicability in many fields such as Control theory, Electrical networks, Artificial neural network, Physics, Mechanics, Electromagnetic theory and probability, Electrochemistry, Engineering, etc, see [10, 11, 16, 18, 21, 24, 26, 27, 28, 37] and references therein.

The fractional differential equations (FDEs) emerge in physics, biology, engineering, finance and economics. Recently, many mathematicians have studied FDEs using several definitions of a fractional derivative such as Riemann-Liouville (RL), Caputo, Hilfer, Hadamard, Katugampola, generalized Caputo, generalized Hilfer, we refer here to some of these famous operators in [6, 12, 14, 15, 17, 31].

On the other hand, there has been much more contemplation paid in developing the theory of existence and uniqueness of positive solutions for nonlinear FDEs, through standard fixed point techniques, for details see [4, 5, 8, 9, 19, 20, 25, 35, 36].

For instance, in [4] Abdo et.al., obtained the existence of positive solutions for the fractional BVP of the form

$$\begin{cases} {}^{C}D^{\upsilon}_{0+}\varsigma(t) = \mathfrak{P}(t,\varsigma(t)), & t \in [0,1] \\ \varsigma(0) = \lambda \int_0^1 \varsigma(s)ds + d, \end{cases}$$

where $0 < \upsilon < 1$, ${}^{C}D^{\upsilon}_{0+}$ is the Caputo operator, $\lambda \geq 0$, $d \in \mathbb{R}^+$, and $\mathfrak{P} : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous function.

On the advanced development of the generalized fractional calculus, some authors who are interested in this topic have presented many generalizations of fractional derivatives like ϑ-Caputo fractional derivative introduced by Almeida [6] and ϑ- Hilfer fractional derivative introduced by Sousa and Oliveira [31].

For recent papers about study the existence and uniqueness of solutions of FDEs involving ϑ-fractional derivative, has been few investigated, see but not limited to [1, 2, 3, 7, 13, 33, 23, 32, 34]. For example, Vivek et.al., in [34] studied the existence, uniqueness and stability results for the generalized fractional BVP of the form

$$\begin{cases} {}^{C}D^{\upsilon,\vartheta}_{0+}\varsigma(t) = \mathfrak{P}(t,\varsigma(t)), & t \in [0,T] \\ a\varsigma(0) + b\varsigma(T) = w, \end{cases}$$

where $0 < \upsilon < 1$, $a, b, w \in \mathbb{R}$ with $a + b \neq 0$, ${}^{C}D^{\upsilon}_{0+}$ is the ϑ-Caputo operator, and $\mathfrak{P} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous function.

Motivated by [4, 22, 33, 34], we investigate the existence and uniqueness of positive solution for nonlinear generalized fractional BVPs of the form

$$\begin{cases} D^{\upsilon,\vartheta}_{0+}\varsigma(t) + \mathfrak{P}(t,\varsigma(t)) = 0, & 0 < t < 1, \\ \varsigma(0) = \varsigma(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \upsilon \leq 2$, $D^{\upsilon,\vartheta}_{0+}$ is the generalized RL fractional derivative of order υ, $\mathfrak{P} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, and $\vartheta : [0,1] \to \mathbb{R}^+$ is a strictly increasing function such that $\vartheta \in C^2[0,1]$ with $\vartheta'(t) \neq 0$, for all $t \in [0,1]$.

Essentially, the researchers absorbed IVPs corresponding to mentioned fractional derivatives. But as far as we know that investigation of BVPs corresponding to FDEs is very scarcely considered by the use of upper and lower solutions method. Recently, Seemab et al., in [33], used the Leggett-Williams fixed point theorem to obtain the existence and multiplicity results of positive solutions for the problem (1.1).

Our aim is to study further results on the existence and uniqueness of positive solutions to the problem (1.1) involving ϑ-RL fractional derivative. Moreover, the results obtained are the first contribution through building the upper (lower) control functions of the nonlinear term that no need any monotone conditions except for the condition of continuity.

The remainder of the work is structured as follows: In section 2 we will fleetingly render some basic definitions and the axiom results that are used in the analysis. Section 3 deals with developing the Green function corresponding to the proposed problem and demonstrating some of its generalized properties related to ϑ, and through this, the problem (1.1) is converted into an equivalent fractional integral equation. Moreover, we obtain further results on the existence and uniqueness of positive solution by using the method of upper and lower solution, and some fixed point techniques. Some examples are offered in section 4. The conclusion is granted in the last section.

2. Preliminary results

Let $C[0,1]$ be the Banach space endowed with the norm $\|\varsigma\| = \sup\{|\varsigma(t)|\,;\, t \in [0,1]\}$ for $\varsigma \in C[0,1]$, and define the cone

$$K = \{\varsigma \in C[0,1] : \varsigma(t) \geq 0, \qquad t \in [0,1]\}.$$

The positive solution that we taking account in this work is such that $\varsigma(t) \geq 0$, $0 \leq t \leq 1$, $\varsigma \in C[0,1]$.

Definition 2.1. [18] *Let $\upsilon > 0$, $\rho : [a,b] \to \mathbb{R}$ be an integrable function and $\vartheta : [a,b] \to \mathbb{R}$ an increasing function with $\vartheta'(t) \neq 0$, for all $t \in [a,b]$. The ϑ- RL fractional integral of ρ of order υ is given by*

$$I_{a+}^{\upsilon,\vartheta}\rho(t) = \frac{1}{\Gamma(\upsilon)}\int_a^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\rho(s)ds.$$

Definition 2.2. [18] *Let $n-1 < \upsilon < n$, and $\rho, \vartheta \in C^n[a,b]$ such that ϑ is an increasing function with $\vartheta'(t) \neq 0$, for all $t \in [a,b]$. Then the ϑ-RL fractional derivative of ρ of order υ is given by*

$$D_{a+}^{\upsilon,\vartheta}\rho(t) = D^{n,\vartheta} I_{a+}^{n-\upsilon,\vartheta}\rho(t),$$

where $D^{n,\vartheta} = \left[\frac{1}{\vartheta'(t)}\frac{d}{dt}\right]^n$ and $n = [\upsilon] + 1$.

Lemma 2.1. [18] *Let $r \in \mathbb{R}$ with $r > n$. The ϑ-fractional integral and derivative of the function $\rho(t) = (\vartheta(t) - \vartheta(a))^{r-1}$ are*

$$I_{a+}^{\upsilon,\vartheta}\rho(t) = \frac{\Gamma(r)}{\Gamma(r+\upsilon)}(\vartheta(t)-\vartheta(a))^{\upsilon+r-1},$$

and

$$D_{a+}^{\upsilon,\vartheta}\rho(t) = \frac{\Gamma(r)}{\Gamma(r-\upsilon)}(\vartheta(t)-\vartheta(a))^{r-\upsilon-1}.$$

Lemma 2.2. [18] *Let $\upsilon, r > 0$ and $\rho : [a,b] \to \mathbb{R}$. Then we have $D_{a+}^{\upsilon,\vartheta} I_{a+}^{\upsilon,\vartheta}\rho(t) = \rho(t)$ and $I_{a+}^{\upsilon,\vartheta} I_{a+}^{r,\vartheta}\rho(t) = I_{a+}^{\upsilon+r,\vartheta}\rho(t)$.*

Lemma 2.3. *Let $\upsilon > 0$. If we suppose $\varsigma \in C(0,1) \cap L(0,1)$, then the FDE $D_{a+}^{\upsilon,\vartheta}\varsigma(t) = 0$ has a unique solution*

$$\varsigma(t) = c_1[\vartheta(t)-\vartheta(a)]^{\upsilon-1} + c_2[\vartheta(t)-\vartheta(a)]^{\upsilon-2} + ... + c_n[\vartheta(t)-\vartheta(a)]^{\upsilon-n},$$

Moreover, if $\varsigma, D_{a+}^{\upsilon,\vartheta}\varsigma \in C(0,1) \cap L(0,1)$,then

$$I_{a+}^{\upsilon,\vartheta} D_{a+}^{\upsilon,\vartheta}\varsigma(t) = \varsigma(t) + c_1[\vartheta(t)-\vartheta(a)]^{\upsilon-1} + c_2[\vartheta(t)-\vartheta(a)]^{\upsilon-2} + ... + c_n[\vartheta(t)-\vartheta(a)]^{\upsilon-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Definition 2.3. *A function $\varsigma \in C[0,1] \cap L[0,1]$ be a solution of (1.1) if ς satisfies $D_{0^+}^{\upsilon,\vartheta}\varsigma(t) + \mathfrak{P}(t, \varsigma(t)) = 0$, $t \in (0,1)$ with $\varsigma(0) = 0$ and $\varsigma(1) = 0$.*

Definition 2.4. *A function $\varsigma \in C[0,1]$ is a positive solution of (1.1) if $\varsigma(t) \geq 0$ for all $t \in [0,1]$ and ς satisfies (1.1).*

Theorem 2.1. [37] *(Banach). Let X be a Banach space with a contraction mapping $T : X \to X$. Then, T has a unique fixed-point x in X.*

Theorem 2.2. [37] *(Schauder). Let X be a Banach space and let S a closed, convex, bounded subset of X. If $T : S \longrightarrow S$ is a continuous map such that the set $\{Ts : s \in S\}$ is relatively compact in X. Then T has at least one fixed point.*

3. Main results

This section is dedicated to demonstrating developed Green function corresponding to problem (1.1) and proving the existence and uniqueness of positive solutions to a problem (1.1).

Lemma 3.1. *Let $1 < \upsilon \leq 2$ and $\phi : [0,1] \to \mathbb{R}^+$ is a continuous. Then*

$$\begin{aligned} &D_{0+}^{\upsilon,\vartheta}\varsigma(t) + \phi(t) = 0, \quad 0 < t < 1, \\ &\varsigma(0) = \varsigma(1) = 0, \end{aligned} \tag{3.1}$$

has a unique solution $\varsigma \in C[0,1]$ given by

$$\varsigma(t) = \int_0^1 \vartheta'(s) G(t,s) \phi(s) ds. \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s) - \mathcal{Z}_\vartheta^\upsilon(1,0)\mathcal{Z}_\vartheta^\upsilon(t,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}, & 0 \leq s \leq t \leq 1, \\ \frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.3}$$

Here $G(t,s)$ means the Green function of fractional BVP (3.1) and the given notation is adopted for easiness

$$\mathcal{Z}_\vartheta^\upsilon(t,s) = [\vartheta(t) - \vartheta(s)]^{\upsilon-1}.$$

Proof. By applying Lemma 2.3 on first equation of (3.1), we obtain

$$\begin{aligned} \varsigma(t) &= -\frac{1}{\Gamma(\upsilon)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\upsilon-1}\phi(s) ds \\ &\quad + c_1[\vartheta(t) - \vartheta(0)]^{\upsilon-1} + c_2[\vartheta(t) - \vartheta(0)]^{\upsilon-2}, \text{ for some } c_1, c_2 \in \mathbb{R}. \end{aligned} \tag{3.4}$$

From the second equation of (3.1), we get $c_2 = 0$ and

$$c_1 = \frac{[\vartheta(1) - \vartheta(0)]^{1-\upsilon}}{\Gamma(\upsilon)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{\upsilon-1}\phi(s) ds.$$

Substitute the values of c_1 and c_2 in (3.4), we get

$$\begin{aligned} \varsigma(t) &= -\frac{1}{\Gamma(\upsilon)} \int_0^t \vartheta'(s)(\vartheta(t) - \vartheta(s))^{\upsilon-1}\phi(s) ds \\ &\quad + \left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{\upsilon-1} \frac{1}{\Gamma(\upsilon)} \int_0^1 \vartheta'(s)(\vartheta(1) - \vartheta(s))^{\upsilon-1}\phi(s) ds. \end{aligned}$$

Hence

$$\varsigma(t) = -\frac{1}{\Gamma(\upsilon)} \int_0^t \vartheta'(s)\mathcal{Z}_\vartheta^\upsilon(t,s)\phi(s) ds$$

$$
\begin{aligned}
& +\frac{\mathcal{Z}_\vartheta^\upsilon(t,0)}{\mathcal{Z}_\vartheta^\upsilon(1,0)}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)\mathcal{Z}_\vartheta^\upsilon(1,s)\phi(s)ds \\
= & -\int_0^t \vartheta'(s)\frac{\mathcal{Z}_\vartheta^\upsilon(t,s)}{\Gamma(\upsilon)}\phi(s)ds \\
& +\frac{\mathcal{Z}_\vartheta^\upsilon(t,0)}{\mathcal{Z}_\vartheta^\upsilon(1,0)}\int_0^t \vartheta'(s)\frac{\mathcal{Z}_\vartheta^\upsilon(1,s)}{\Gamma(\upsilon)}\phi(s)ds \\
& +\frac{\mathcal{Z}_\vartheta^\upsilon(t,0)}{\mathcal{Z}_\vartheta^\upsilon(1,0)}\frac{1}{\Gamma(\upsilon)}\int_t^1 \vartheta'(s)\mathcal{Z}_\vartheta^\upsilon(1,s)\phi(s)ds \\
= & \int_0^t \vartheta'(s)\frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s)-\mathcal{Z}_\vartheta^\upsilon(1,0)\mathcal{Z}_\vartheta^\upsilon(t,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\phi(s)ds \\
& +\int_t^1 \vartheta'(s)\frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\phi(s)ds \\
= & \int_0^1 \vartheta'(s)G(t,s)\phi(s)ds.
\end{aligned}
$$

□

Lemma 3.2. *For all $\upsilon \in (1,2]$. The Green function given by (3.3) satisfies the following properties:*

(i): *$G(t,s)$ is continuous on $[0,1]\times[0,1]$.*
(ii): *$G(t,s) > 0, \quad 0 < t, s < 1$.*
(iv): *For $s \in (0,1)$*

$$\Gamma(\upsilon)\max_{t\in[0,1]} G(t,s) \le \frac{\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)}. \tag{3.5}$$

Proof. Let us assume

$$
\begin{aligned}
G_1(t,s) &= \frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s)-\mathcal{Z}_\vartheta^\upsilon(1,0)\mathcal{Z}_\vartheta^\upsilon(t,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)} \\
&= \frac{[\vartheta(t)-\vartheta(0)]^{\upsilon-1}[\vartheta(1)-\vartheta(s)]^{\upsilon-1}}{[\vartheta(1)-\vartheta(0)]^{\upsilon-1}\Gamma(\upsilon)} - \frac{[\vartheta(t)-\vartheta(s)]^{\upsilon-1}}{\Gamma(\upsilon)}, \ 0 \le s \le t \le 1,
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(t,s) &= \frac{\mathcal{Z}_\vartheta^\upsilon(t,0)\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)} \\
&= \frac{[\vartheta(t)-\vartheta(0)]^{\upsilon-1}[\vartheta(1)-\vartheta(s)]^{\upsilon-1}}{[\vartheta(1)-\vartheta(0)]^{\upsilon-1}\Gamma(\upsilon)}, \quad 0 \le t \le s \le 1.
\end{aligned}
$$

Since $\vartheta \in C^2[0,1]$, one can check easily that $G_1(t,s)$ and $G_2(t,s)$ are continuous on $[0,1]\times[0,1]$. Thus (i) holds. Now, we show that (ii) is satisfied.

Case (1) For $0 \le t \le s \le 1$:
Since ϑ is a strictly increasing function, we have

$$
\begin{aligned}
\vartheta(1) > \vartheta(s) \Rightarrow \vartheta(1)-\vartheta(s) > 0 \quad &\text{whenever} \quad s < 1, \\
\vartheta(t) > \vartheta(0) \Rightarrow \vartheta(t)-\vartheta(0) > 0 \quad &\text{whenever} \quad 0 < t.
\end{aligned}
$$

which implies $G_2(t,s) > 0$.

Case (2) For $0 \leq s \leq t \leq 1$:

We consider $\vartheta(1) - \vartheta(t) > 0$ whenever $t < 1$. Multiplying both sides by $\vartheta(s) - \vartheta(0) > 0$, it follows that

$$[\vartheta(1) - \vartheta(t)]\,[\vartheta(s) - \vartheta(0)] > 0,$$

which leads

$$\vartheta(1)\vartheta(s) + \vartheta(0)\vartheta(t) > \vartheta(0)\vartheta(1) + \vartheta(t)\vartheta(s).$$

i.e.

$$-\vartheta(0)\vartheta(1) - \vartheta(t)\vartheta(s) > -\vartheta(1)\vartheta(s) - \vartheta(0)\vartheta(t).$$

Adding to $\vartheta(1)\vartheta(t) + \vartheta(0)\vartheta(s)$ both sides,we get

$$[\vartheta(t) - \vartheta(0)]\,[\vartheta(1) - \vartheta(s)] > [\vartheta(t) - \vartheta(s)]\,[\vartheta(1) - \vartheta(0)]\,,$$

For $1 < \upsilon \leq 2$,we have

$$[\vartheta(t) - \vartheta(0)]^{\upsilon-1}\,[\vartheta(1) - \vartheta(s)]^{\upsilon-1} > [\vartheta(t) - \vartheta(s)]^{\upsilon-1}\,[\vartheta(1) - \vartheta(0)]^{\upsilon-1}\,,$$

Dividing by $[\vartheta(t) - \vartheta(0)]^{\upsilon-1}$, we obtain

$$\frac{[\vartheta(t) - \vartheta(0)]^{\upsilon-1}}{[\vartheta(1) - \vartheta(0)]^{\upsilon-1}}\,[\vartheta(1) - \vartheta(s)]^{\upsilon-1} > [\vartheta(t) - \vartheta(s)]^{\upsilon-1}\,.$$

This shows that

$$\frac{\mathcal{Z}_{\vartheta}^{\upsilon}(t,0)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)}\mathcal{Z}_{\vartheta}^{\upsilon}(1,s) > \mathcal{Z}_{\vartheta}^{\upsilon}(t,s)$$

Hence, we conclude that $G_1(t,s) > 0$.

Finally, we prove that (iii) holds. For this, we have

Case (i) For $0 \leq s \leq t \leq 1$:

Since ϑ is a strictly increasing function, we have

$$\vartheta(t) > \vartheta(s) \Rightarrow \vartheta(t) - \vartheta(s) > 0 \quad \text{whenever} \quad s < t,$$

$$\vartheta(t) > \vartheta(0) \Rightarrow \vartheta(t) - \vartheta(0) > 0 \quad \text{whenever} \quad 0 < t.$$

Hence

$$\begin{aligned} G(t,s) &= \frac{\mathcal{Z}_{\vartheta}^{\upsilon}(t,0)\mathcal{Z}_{\vartheta}^{\upsilon}(1,s) - \mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\mathcal{Z}_{\vartheta}^{\upsilon}(t,s)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\Gamma(\upsilon)} \\ &= \frac{[\vartheta(t) - \vartheta(0)]^{\upsilon-1}}{[\vartheta(1) - \vartheta(0)]^{\upsilon-1}}\frac{[\vartheta(1) - \vartheta(s)]^{\upsilon-1}}{\Gamma(\upsilon)} - \frac{[\vartheta(t) - \vartheta(s)]^{\upsilon-1}}{\Gamma(\upsilon)} \\ &\leq \frac{[\vartheta(1) - \vartheta(s)]^{\upsilon-1}}{[\vartheta(1) - \vartheta(0)]^{\upsilon-1}\Gamma(\upsilon)} = \frac{\mathcal{Z}_{\vartheta}^{\upsilon}(1,s)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\Gamma(\upsilon)}. \end{aligned}$$

Case (ii) For $0 \leq t \leq s \leq 1$:

Since ϑ is a strictly increasing function, we have $\vartheta(t) - \vartheta(0) > 0$ whenever $0 < t$. Thus

$$\begin{aligned} G(t,s) &= \frac{\mathcal{Z}_{\vartheta}^{\upsilon}(t,0)\mathcal{Z}_{\vartheta}^{\upsilon}(1,s)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\Gamma(\upsilon)} \\ &= \frac{[\vartheta(t) - \vartheta(0)]^{\upsilon-1}}{[\vartheta(1) - \vartheta(0)]^{\upsilon-1}}\frac{[\vartheta(1) - \vartheta(s)]^{\upsilon-1}}{\Gamma(\upsilon)} \end{aligned}$$

$$\leq \frac{[\vartheta(1)-\vartheta(s)]^{\upsilon-1}}{[\vartheta(1)-\vartheta(0)]^{\upsilon-1}\Gamma(\upsilon)} = \frac{\mathcal{Z}_{\vartheta}^{\upsilon}(1,s)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\Gamma(\upsilon)}.$$

Therefore, we conclude that

$$\max_{t\in[0,1]} G(t,s) \leq \frac{\mathcal{Z}_{\vartheta}^{\upsilon}(1,s)}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)\Gamma(\upsilon)}, \quad \text{for all } s \in (0,1).$$

□

Definition 3.1. *Let $a, b \in \mathbb{R}^+ (b > a)$. For any $\varsigma \in [a,b]$, we say that $\mathfrak{P}(t,\cdot)$ is the upper-control function if $\overline{\mathfrak{P}}(t,\varsigma) = \sup_{a\leq\eta\leq\varsigma} \mathfrak{P}(t,\eta)$, and is the lower-control function if $\underline{\mathfrak{P}}(t,\varsigma) = \inf_{\varsigma\leq\eta\leq b} \mathfrak{P}(t,\eta)$, Certainly, $\overline{\mathfrak{P}}(t,\varsigma)$ and $\underline{\mathfrak{P}}(t,\varsigma)$ are nondecreasing on ς and*

$$\underline{\mathfrak{P}}(t,\varsigma) \leq \mathfrak{P}(t,\varsigma) \leq \overline{\mathfrak{P}}(t,\varsigma).$$

Definition 3.2. *Let $\overline{\varsigma}(t), \underline{\varsigma}(t) \in K$ and $a \leq \underline{\varsigma}(t) \leq \overline{\varsigma}(t) \leq b$ satisfy*

$$-D_{0+}^{\upsilon;\vartheta}\overline{\varsigma}(t) \geq \overline{\mathfrak{P}}(t,\overline{\varsigma}(t)), \quad \text{or } \overline{\varsigma}(t) \geq \int_0^1 \vartheta'(s)G(t,s)\overline{\mathfrak{P}}(s,\overline{\varsigma}(s))ds, \quad 0 \leq t \leq 1,$$

and

$$-D_{0+}^{\upsilon;\vartheta}\underline{\varsigma}(t) \leq \underline{\mathfrak{P}}(t,\underline{\varsigma}(t)), \quad \text{or } \ \underline{\varsigma}(t) \leq \int_0^1 \vartheta'(s)G(t,s)\underline{\mathfrak{P}}(s,\underline{\varsigma}(s))ds, \quad 0 \leq t \leq 1,$$

Then, $\overline{\varsigma}(t)$ and $\underline{\varsigma}(t)$ are upper and lower solutions, respectively of problem (1.1).

Theorem 3.1. *Suppose $\mathfrak{P} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Then there exists at least one positive solution $\varsigma(t)$ of (1.1). Moreover,*

$$\underline{\varsigma}(t) \leq \varsigma(t) \leq \overline{\varsigma}(t), \quad t \in [0,1].$$

where $\overline{\varsigma}(t), \underline{\varsigma}(t)$ are upper and lower solutions of (1.1).

Proof. Define $Q : K \to K$ by

$$\left(Q\varsigma\right)(t) = \int_0^1 \vartheta'(s)G(t,s)\mathfrak{P}(s,\varsigma(s))ds. \tag{3.6}$$

Lemma 3.1 shows that fixed points of Q are solutions of (1.1). Since $\mathfrak{P}(s,\varsigma)$ and $G(t,s)$ are nonnegative and continuous, $Q : K \to K$ is continuous. Define the ball

$$\mathcal{B}_r = \{\varsigma \in K : \|\varsigma\| \leq r\} \subset K,$$

and set $L := \max_{(t,\varsigma)\in[0,1]\times[0,r]} |\mathfrak{P}(t,\varsigma)| + 1$. Then for any $\varsigma \in \mathcal{B}_r$ we get

$$\begin{aligned}
|\left(Q\varsigma\right)(t)| &\leq \int_0^1 \vartheta'(s)G(t,s)\left|\mathfrak{P}(s,\varsigma(s))\right| ds \\
&\leq \int_0^1 \vartheta'(s)\max_{t\in[0,1]} G(t,s)\left|\mathfrak{P}(s,\varsigma(s))\right| ds \\
&\leq \frac{L}{\mathcal{Z}_{\vartheta}^{\upsilon}(1,0)}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)\mathcal{Z}_{\vartheta}^{\upsilon}(1,s)ds \\
&= \frac{L}{[\vartheta(1)-\vartheta(0)]^{\upsilon-1}}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)[\vartheta(1)-\vartheta(s)]^{\upsilon-1}ds
\end{aligned}$$

$$\leq \quad L\frac{[\vartheta(1)-\vartheta(0)]}{\Gamma(\upsilon+1)}.$$

This show that $(Q\mathcal{B}_r)$ is uniformly bounded.

Now, we prove that Q is equicontinuous. Let $\varsigma \in \mathcal{B}_r$. Then for $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\left|(Q\varsigma)(t_2)-(Q\varsigma)(t_1)\right| \leq \int_0^1 \vartheta'(s)\left|G(t_2,s)-G(t_1,s)\right|\left|\mathfrak{P}(s,\varsigma(s))\right| ds. \tag{3.7}$$

Consider $\Delta := \left|G(t_2,s)-G(t_1,s)\right|$. Thus

$$\begin{aligned}
\Delta &= \Bigg|\frac{\mathcal{Z}_\vartheta^\upsilon(t_2,0)\mathcal{Z}_\vartheta^\upsilon(1,s)-\mathcal{Z}_\vartheta^\upsilon(1,0)\mathcal{Z}_\vartheta^\upsilon(t_2,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}+\frac{\mathcal{Z}_\vartheta^\upsilon(t_2,0)\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)} \\
&\quad -\frac{\mathcal{Z}_\vartheta^\upsilon(t_1,0)\mathcal{Z}_\vartheta^\upsilon(1,s)-\mathcal{Z}_\vartheta^\upsilon(1,0)\mathcal{Z}_\vartheta^\upsilon(t_1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}-\frac{\mathcal{Z}_\vartheta^\upsilon(t_1,0)\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\Bigg| \\
&= \left|\frac{2\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\left[\mathcal{Z}_\vartheta^\upsilon(t_2,0)-\mathcal{Z}_\vartheta^\upsilon(t_1,0)\right]+\frac{1}{\Gamma(\upsilon)}\left[\mathcal{Z}_\vartheta^\upsilon(t_1,s)-\mathcal{Z}_\vartheta^\upsilon(t_2,s)\right]\right| \\
&\leq \left|\frac{2\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\left[[\vartheta(t_2)-\vartheta(0)]^{\upsilon-1}-[\vartheta(t_1)-\vartheta(0)]^{\upsilon-1}\right]\right| \\
&\quad +\frac{1}{\Gamma(\upsilon)}\left|[\vartheta(t_1)-\vartheta(s)]^{\upsilon-1}-[\vartheta(t_2)-\vartheta(s)]^{\upsilon-1}\right|.
\end{aligned}$$

By applying the mean value theorem, then for $a, b \in (t_1, t_2)$

$$\begin{aligned}
\Delta &\leq \frac{2\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\left|t_2-t_1\right|\theta_1'(a)+\frac{1}{\Gamma(\upsilon)}\left|t_2-t_1\right|\theta_2'(b) \\
&= \left|t_2-t_1\right|\left[\frac{2\mathcal{Z}_\vartheta^\upsilon(1,s)}{\mathcal{Z}_\vartheta^\upsilon(1,0)\Gamma(\upsilon)}\theta_1'(a)+\frac{1}{\Gamma(\upsilon)}\theta_2'(b)\right],
\end{aligned}$$

The estimation of (3.7) becomes

$$\begin{aligned}
\left|(Q\varsigma)(t_2)-(Q\varsigma)(t_1)\right| &\leq \left|t_2-t_1\right|\left[\frac{2\theta_1'(a)}{\mathcal{Z}_\vartheta^\upsilon(1,0)}\frac{L}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)[\vartheta(1)-\vartheta(s)]^{\upsilon-1}ds\right. \\
&\quad \left.+\frac{\theta_2'(b)L}{\Gamma(\upsilon)}\int_0^1\vartheta'(s)ds\right] \\
&= \left|t_2-t_1\right|\left[\frac{2\theta_1'(a)}{\Gamma(\upsilon+1)}[\vartheta(1)-\vartheta(0)]+\frac{\theta_2'(b)}{\Gamma(\upsilon)}[\vartheta(1)-\vartheta(0)]\right]L \\
&= \left|t_2-t_1\right|\left[\frac{2\theta_1'(a)}{\Gamma(\upsilon+1)}+\frac{\theta_2'(b)}{\Gamma(\upsilon)}\right][\vartheta(1)-\vartheta(0)]L.
\end{aligned}$$

As $t_2 - t_1 \to 0$, $|(Q\varsigma)(t_2)-(Q\varsigma)(t_1)| \to 0$, which means that $(Q\mathcal{B}_r)$ is equicontinuous. So by Arzela-Ascoli theorem, we conclude that Q is completely continuous.

To apply Theorem 2.2, we need only to prove $Q : \Lambda \to \Lambda$, where

$$\Lambda = \{w(t) : w(t) \in K,\ \underline{\varsigma}(t) \leq w(t) \leq \overline{\varsigma}(t),\ t \in [0,1]\},$$

and $\|w\| = \max\{|w(t)| \leq b;\ t \in [0,1]\}$. Certainly, Λ is a bounded, closed and convex subset of $C[0,1]$. For any $w(t) \in \Lambda$, then $\underline{\varsigma}(t) \leq w(t) \leq \overline{\varsigma}(t)$, it follows from the Definitions 3.1, 3.2 that

$$\left(Qw\right)(t) = \int_0^1 \vartheta'(s)G(t,s)\mathfrak{P}(s,w(s))ds \leq \int_0^1 \vartheta'(s)G(t,s)\overline{\mathfrak{P}}(s,w(s))ds$$

$$\begin{aligned} &\leq \int_0^1 \vartheta'(s)G(t,s)\overline{\mathfrak{P}}(s,\overline{\varsigma}(s))ds \\ &\leq \overline{\varsigma}(t), \end{aligned}$$

and

$$\begin{aligned} Qw(t) &= \int_0^1 \vartheta'(s)G(t,s)\mathfrak{P}(s,w(s))ds \geq \int_0^1 \vartheta'(s)G(t,s)\underline{\mathfrak{P}}(s,w(s))ds \\ &\geq \int_0^1 \vartheta'(s)G(t,s)\underline{\mathfrak{P}}(s,\underline{\varsigma}(s))ds \\ &\geq \underline{\varsigma}(t). \end{aligned}$$

Thus $Qw(t) \in \Lambda$, due to $\underline{\varsigma}(t) \leq Qw(t) \leq \overline{\varsigma}(t)$, $t \in [0,1]$. Hence $Q : \Lambda \to \Lambda$. According to Theorem 2.2, Q has at least one fixed point $\varsigma(t) \in \Lambda$ for $t \in [0,1]$. Therefore, the problem (1.1) has at least one positive solution $\varsigma(t) \in C[0,1]$ and $\underline{\varsigma}(t) \leq \varsigma(t) \leq \overline{\varsigma}(t)$, $t \in [0,1]$. □

Corollary 3.1. *Let $\mathfrak{P} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and there exist two constants $L_1(\geq 0)$ and $L_2(\geq 0)$ such that*

$$L_1 \leq \mathfrak{P}(t,\kappa) \leq L_2, \quad (t,\kappa) \in [0,1] \times \mathbb{R}^+. \tag{3.8}$$

Then the problem (1.1) has at least one positive solution $\varsigma(t) \in C[0,1]$. Moreover, for each $t \in (0,1)$,

$$\varsigma(t) \geq L_1 \left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)} \right)^{-1} - 1 \right) \frac{(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}. \tag{3.9}$$

and.

$$\varsigma(t) \leq L_2 \left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)} \right)^{-1} - 1 \right) \frac{(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}. \tag{3.10}$$

Proof. Thanks to Definition 3.1 and (3.8), we have

$$L_1 \leq \underline{\mathfrak{P}}(t,\kappa) \leq \overline{\mathfrak{P}}(t,\kappa) \leq L_2, \quad (t,\kappa) \in [0,1] \times \mathbb{R}^+. \tag{3.11}$$

Consider the following FDE

$$\begin{aligned} -D_{0+}^{\upsilon;\vartheta}\overline{\varsigma}(t) &= L_2, \qquad 0 < t < 1, \\ \overline{\varsigma}(0) &= \overline{\varsigma}(1) = 0, \end{aligned} \tag{3.12}$$

Certainly, (3.12) has a positive solution

$$\begin{aligned} \overline{\varsigma}(t) &= -\frac{L_2}{\Gamma(\upsilon)} \int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}ds \\ &\quad + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)} \right)^{\upsilon-1} \frac{L_2}{\Gamma(\upsilon)} \int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}ds \\ &= -\frac{L_2(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)} + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)} \right)^{\upsilon-1} \frac{L_2(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)} \\ &= \left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)} \right)^{-1} - 1 \right) \frac{L_2\,(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}. \end{aligned} \tag{3.13}$$

Taking into account (3.11), we can find that

$$\begin{aligned}\overline{\varsigma}(t) &\geq -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\overline{\mathfrak{P}}(s,\overline{\varsigma}(s))ds \\ &\quad + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\overline{\mathfrak{P}}(s,\overline{\varsigma}(s))ds.\end{aligned}$$

Consequently, $\overline{\varsigma}$ is the upper solution of (1.1). Also, we consider the following FDE

$$\begin{aligned} -D_{0+}^{\upsilon;\vartheta}\underline{\varsigma}(t) = L_1, &\qquad 0<t<1, \\ \underline{\varsigma}(0)=\underline{\varsigma}(1)=0. \end{aligned}\tag{3.14}$$

Certainly, (3.14) has a positive solution

$$\begin{aligned}\underline{\varsigma}(t) &= -\frac{L_1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}ds \\ &\quad + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{L_1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}ds \\ &= -\frac{L_1(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)} + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{L_1(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)} \\ &= \left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{-1}-1\right)\frac{L_1\left(\vartheta(t)-\vartheta(0)\right)^{\upsilon}}{\Gamma(\upsilon+1)}\end{aligned}$$

Taking into account (3.11), we have

$$\begin{aligned}\underline{\varsigma}(t) &\leq -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\underline{\mathfrak{P}}(s,\underline{\varsigma}(s))ds \\ &\quad + \left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\underline{\mathfrak{P}}(s,\underline{\varsigma}(s))ds.\end{aligned}\tag{3.15}$$

Therefore, $\underline{\varsigma}$ is the lower solution of (1.1). So, Theorem 3.1 yields that (1.1) has at least one positive solution $\varsigma(t)\in C[0,1]$ which satisfies the inequality (3.9) and (3.10). □

Theorem 3.2. *Let a is a positive constant. Assume that $\mathfrak{P}(t,\varsigma):[0,1]\times\mathbb{R}^+\to[a,+\infty)$ is continuous. If*

$$a<\lim_{\varsigma\to+\infty}\max_{0\leq t\leq 1}\frac{\mathfrak{P}(t,\varsigma)}{\varsigma}<+\infty,\tag{3.16}$$

then the problem (1.1) has at least one positive solution $\varsigma(t)\in C[0,\sigma]$ where $0<\sigma<1$.

Proof. According to assumption (3.16), there there exist $A_{\mathfrak{P}}>0$ and $B_{\mathfrak{P}}>0$ such that for any $\varsigma(t)\in X$, we have

$$\mathfrak{P}(t,\varsigma(t))\leq A_{\mathfrak{P}}\varsigma(t)+B_{\mathfrak{P}}.$$

Definition 3.1 gives

$$\overline{\mathfrak{P}}(t,\varsigma(t))\leq A_{\mathfrak{P}}\varsigma(t)+B_{\mathfrak{P}}.\tag{3.17}$$

On the other hand, we consider the following FDE

$$-D_{0+}^{\upsilon;\vartheta}\varsigma(t)=A_{\mathfrak{P}}\varsigma(t)+B_{\mathfrak{P}},\quad 1<\upsilon\leq 2,\quad 0<t<1.\tag{3.18}$$

According to Lemma 3.1, the equation (3.18) has the following equivalent solution

$$\begin{aligned}\varsigma(t) &= -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma(s)+B_{\mathfrak{P}}\right]ds \\ &\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma(s)+B_{\mathfrak{P}}\right]ds.\end{aligned}$$

Let $\Phi^*: K\longrightarrow K$ an operator defined by

$$\begin{aligned}\Phi^*\varsigma(t) &= -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma(s)+B_{\mathfrak{P}}\right]ds \\ &\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma(s)+B_{\mathfrak{P}}\right]ds.\end{aligned}$$

Now, we show that $\Phi^*: K\longrightarrow K$ is compact.

Let $\{\varsigma_n\}$ be a secuence in $\mathcal{E}$ such $\varsigma_n\to\varsigma$ as $n\to\infty$. For $t\in[0,\sigma]$, we have

$$\begin{aligned}&\left|\Phi^*\varsigma_n(t)-\Phi^*\varsigma(t)\right| \\ &\leq \frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}A_{\mathfrak{P}}\left|\varsigma_n(s)-\varsigma(s)\right|ds \\ &\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}A_{\mathfrak{P}}\left|\varsigma_n(s)-\varsigma(s)\right|ds \\ &\leq \frac{2\left(\vartheta(1)-\vartheta(0)\right)^{\upsilon-1}}{\Gamma(\upsilon+1)}A_{\mathfrak{P}}\left\|\varsigma_n-\varsigma\right\|_\infty\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Thus, $\Phi^*: K\longrightarrow K$ is continuous. Next, let

$$\mathcal{S}_r=\{\varsigma\in K:\|\varsigma\|\leq\lambda\}\subset K.$$

Then, for any $\varsigma\in\mathcal{S}_r$ and $t\in[0,\sigma]$, we have

$$\begin{aligned}\left|\left(\Phi^*\varsigma\right)(t)\right| &\leq \frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\left|\varsigma(s)\right|+B_{\mathfrak{P}}\right]ds \\ &\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\left|\varsigma(s)\right|+B_{\mathfrak{P}}\right]ds \\ &\leq \frac{(\vartheta(\sigma)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\lambda+B_{\mathfrak{P}}\right]+\left(\frac{\vartheta(\sigma)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\lambda+B_{\mathfrak{P}}\right] \\ &\leq \frac{2(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\lambda+B_{\mathfrak{P}}\right]:=\ell.\end{aligned}$$

Thus, $\|\Phi^*\varsigma\|\leq\ell$. Hence, $\Phi^*(\mathcal{S}_r)$ is uniformly bounded. Finally, we prove that $\Phi^*(\mathcal{S}_r)$ is equicontinuous. For each $t\in[0,\sigma]$ and using (3.10), we can estimate the operator derivative as

$$\begin{aligned}&\left|\left(\Phi^*\varsigma\right)'(t)\right| \\ &\leq \frac{1}{\Gamma(\upsilon-1)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-2}\left[A_{\mathfrak{P}}\left|\varsigma(s)\right|+B_{\mathfrak{P}}\right]ds \\ &\quad+\left|\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right|^{\upsilon-1}\frac{1}{\Gamma(\upsilon-1)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-2}\left[A_{\mathfrak{P}}\left|\varsigma(s)\right|+B_{\mathfrak{P}}\right]ds \\ &\quad+(\upsilon-1)\left|\vartheta'(t)\right|\left|\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right|^{\upsilon-2}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\left|\varsigma(s)\right|+B_{\mathfrak{P}}\right]ds\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{2(\vartheta(t)-\vartheta(0))^{\upsilon-1}}{\Gamma(\upsilon)}\left[A_{\mathfrak{P}}\|\varsigma\|+B_{\mathfrak{P}}\right] \\
& +(\upsilon-1)\left|\vartheta'(t)\right|\left|\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right|^{\upsilon-2}\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\|\varsigma\|+B_{\mathfrak{P}}\right].
\end{aligned}
$$

Since $\vartheta \in C^1[0,\sigma]$, there exists a constant ν such that $\sup_{0\leq t\leq\sigma}|\vartheta'(t)| \leq \nu$ and from fact that

$$
\left|\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right|^{\upsilon-2}<1,\ \text{ for all } 1\leq\upsilon\leq 2.
$$

Then

$$
\begin{aligned}
\left|(\Phi^*\varsigma)'(t)\right| &\leq \frac{2(\vartheta(t)-\vartheta(0))^{\upsilon-1}}{\Gamma(\upsilon)}\left[A_{\mathfrak{P}}\|\varsigma\|+B_{\mathfrak{P}}\right] \\
&\quad +(\upsilon-1)\,\nu\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\|\varsigma\|+B_{\mathfrak{P}}\right] \\
&= \left(\frac{2(\vartheta(t)-\vartheta(0))^{\upsilon-1}}{\Gamma(\upsilon)}+(\upsilon-1)\,\nu\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\right)\left[A_{\mathfrak{P}}\|\varsigma\|+B_{\mathfrak{P}}\right].
\end{aligned}
$$

Hence, for each $t_1,t_2\in[0,\sigma]$ with $0<t_1<t_2<\sigma$ and for $\varsigma\in\mathcal{S}_r$, we get

$$
\begin{aligned}
& |\Phi^*\varsigma(t_2)-\Phi^*\varsigma(t_1)| \\
=\ & \int_{t_1}^{t_2}\left|(\Phi^*\varsigma)'(s)\right|ds \\
\leq\ & \left(\frac{2(\vartheta(\sigma)-\vartheta(0))^{\upsilon-1}}{\Gamma(\upsilon)}+(\upsilon-1)\,\nu\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\right)\left[A_{\mathfrak{P}}\lambda+B_{\mathfrak{P}}\right]|t_2-t_1|.
\end{aligned}
$$

So, we can deduce that $|\Phi^*\varsigma(t_2)-\Phi^*\varsigma(t_1)|\to 0$ as $t_2\to t_1$, that is, the family $\{\Phi^*\varsigma:\varsigma\in\mathcal{S}_r\}$ is equicontinuous. The Arzela-Ascoli Lemma implies that Φ^* is compact.

To apply Theorem 2.2, we need to verify that $\Phi^*B_\zeta\subset B_\zeta$ where

$$
B_\zeta=\left\{\varsigma(t)\in K,\left\|\varsigma-\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}\right\|\leq\zeta<+\infty\right\},
$$

with $\zeta\geq\frac{\Lambda_1}{1-\Lambda_2}$ and

$$
\Lambda_1:=2A_{\mathfrak{P}}\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(1)-\vartheta(0))^{2\upsilon}}{(\Gamma(\upsilon+1))^2}B_{\mathfrak{P}}
$$

and

$$
\Lambda_2:=\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}A_{\mathfrak{P}}<\frac{1}{2}. \tag{3.19}
$$

Clearly, B_ζ is bounded, convex, and closed subset of $C[0,\sigma]$. Then for any $\varsigma\in B_\zeta$, we have

$$
\begin{aligned}
\|\varsigma\| &\leq \frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}+\zeta \\
&\leq \frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(\sigma)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}+\zeta \\
&\leq \frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}+\zeta.
\end{aligned}
$$

Thus

$$
\left\|\Phi^*\varsigma-\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}\right\|
$$

$$\begin{aligned}
&\leq \frac{(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\left(\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}+\zeta\right)\right]\\
&\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\left[A_{\mathfrak{P}}\left(\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}B_{\mathfrak{P}}+\zeta\right)\right]\\
&\leq \left[2A_{\mathfrak{P}}\frac{(\vartheta(1)-\vartheta(0)-1)(\vartheta(1)-\vartheta(0))^{2\upsilon}}{(\Gamma(\upsilon+1))^{2}}B_{\mathfrak{P}}\right]+\left[2A_{\mathfrak{P}}\frac{(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}\right]\zeta\\
&= \Lambda_1+\Lambda_2\zeta\leq\zeta.
\end{aligned}$$

Hence, Theorem 2.2 assures Φ^* has at least one fixed point, and then (3.18) has at least one positive solution $\varsigma^*(t)$, where $0<t<\sigma$. Thus,

$$\begin{aligned}
\varsigma^*(t) &= -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma^*(s)+B_{\mathfrak{P}}\right]ds\\
&\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\left[A_{\mathfrak{P}}\varsigma^*(s)+B_{\mathfrak{P}}\right]ds. \quad (3.20)
\end{aligned}$$

Combining (3.20) and (3.17) gives

$$\begin{aligned}
\varsigma^*(t) &\geq -\frac{1}{\Gamma(\upsilon)}\int_0^t \vartheta'(s)(\vartheta(t)-\vartheta(s))^{\upsilon-1}\overline{\mathfrak{P}}(t,\varsigma^*(t))ds\\
&\quad+\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}\overline{\mathfrak{P}}(t,\varsigma^*(t))ds.
\end{aligned}$$

Undoubtedly, $\varsigma^*(t)$ is the upper solution of (1.1), and we have

$$\begin{aligned}
\varsigma_*(t) &= \left[\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}-1\right]\mathcal{I}_{0^+}^{\upsilon;\vartheta}(a)\\
&= \left[\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{\upsilon-1}-1\right]\frac{a(\vartheta(1)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}>0
\end{aligned}$$

is the lower solution of (1.1). By Theorem 3.1, the problem (1.1) has at least one positive solution $\varsigma(t)\in C[0,\sigma]$, where $0<\sigma<1$ and $\varsigma_*(t)\leq\varsigma(t)\leq\varsigma^*(t)$. □

Corollary 3.2. *Suppose $\mathfrak{P}:[0,1]\times\mathbb{R}^+\to[a,+\infty)$ is continuous, where $a>0$. If*

$$a<\lim_{\varsigma\to+\infty}\mathfrak{P}(t,\varsigma)<+\infty. \quad (3.21)$$

then the problem (1.1) has at least one positive solution.

Proof. By hypothesis (3.21), there exist $\mathcal{N}_1,\mathcal{N}_2>0$ such that if $\varsigma>\mathcal{N}_2$, we have $\mathfrak{P}(t,\varsigma)<\mathcal{N}_1$.

Let

$$M=\max_{\substack{0\leq t\leq 1\\ 0\leq\varsigma\leq\mathcal{N}_2}}\mathfrak{P}(t,\varsigma).$$

Then

$$a\leq\mathfrak{P}(t,\varsigma)\leq\mathcal{N}_1+M,\quad \text{for } 0<\varsigma<+\infty.$$

According to Corollary 3.1, the problem (1.1) has at least one positive solution $\varsigma(t)\in C[0,1]$. Moreover, for each $t\in(0,1)$,

$$\varsigma(t)\geq a\left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{-1}-1\right)\frac{(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}.$$

and.

$$\varsigma(t) \leq (\mathcal{N}_1 + M)\left(\left(\frac{\vartheta(t)-\vartheta(0)}{\vartheta(1)-\vartheta(0)}\right)^{-1} - 1\right)\frac{(\vartheta(t)-\vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}.$$

□

The following result is based on the Theorem 2.1.

Theorem 3.3. *Suppose* $\mathfrak{P} : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and there exists a constant* $\wp > 0$ *such that*

$$|\mathfrak{P}(t,\varsigma_1) - \mathfrak{P}(t,\varsigma_2)| \leq \wp\,|\varsigma_1 - \varsigma_2|\,, \ \forall t \in (0,1), \ \varsigma_1, \varsigma_2 \in \mathbb{R}^+$$

If

$$\Omega := \frac{\wp\,[\vartheta(1)-\vartheta(0)]}{\Gamma(\upsilon+1)} < 1, \tag{3.22}$$

then the problem (1.1) has a unique positive solution $\varsigma(t) \in C[0,1]$.

Proof. Theorem 3.1 assures that (1.1) has at least one positive solution in K given by

$$\varsigma(t) = \int_0^1 \vartheta'(s)G(t,s)\mathfrak{P}(s,\varsigma(s))ds. \tag{3.23}$$

Hence, we need only to show that $Q : C[0,1] \to C[0,1]$ defined by

$$(Q\varsigma)(t) = \int_0^1 \vartheta'(s)G(t,s)\mathfrak{P}(s,\varsigma(s))ds. \tag{3.24}$$

is contraction in $C[0,1]$. For the end, let $\varsigma_1, \varsigma_2 \in C[0,1]$. Then by our assumption and (3.5), we have

$$\begin{aligned}
|(Q\varsigma_1)(t) - (Q\varsigma_2)(t)| &\leq \max_{t\in[0,1]} |(Q\varsigma_1)(t) - (Q\varsigma_2)(t)| \\
&\leq \max_{t\in[0,1]} \int_0^1 \vartheta'(s)\,|G(t,s)|\,|\mathfrak{P}(s,\varsigma_1(s)) - \mathfrak{P}(s,\varsigma_2(s))|\,ds \\
&\leq \frac{\wp}{\mathcal{Z}_\vartheta^{\upsilon}(1,0)}\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)\mathcal{Z}_\vartheta^{\upsilon}(1,s)\,|\varsigma_1(s) - \varsigma_2(s)|\,ds \\
&\leq \frac{\wp}{(\vartheta(1)-\vartheta(0))^{\upsilon-1}}\frac{1}{\Gamma(\upsilon)}\,\|\varsigma_1 - \varsigma_2\|\,\frac{1}{\Gamma(\upsilon)}\int_0^1 \vartheta'(s)(\vartheta(1)-\vartheta(s))^{\upsilon-1}ds \\
&\leq \frac{\wp\,[\vartheta(1)-\vartheta(0)]}{\Gamma(\upsilon+1)}\,\|\varsigma_1 - \varsigma_2\| = \Omega\,\|\varsigma_1 - \varsigma_2\|\,.
\end{aligned}$$

Since $\Omega < 1$, Q is contraction. Hence, Theorem 2.1, concludes the problem (1.1) has a unique positive solution $\varsigma(t) \in C[0,1]$. □

4. Examples

This section gives some examples to illuminate obtained results.

Example 4.1. *Consider the fractional BVP*

$$\begin{aligned}
&-D_{0^+}^{\frac{5}{3},\sin t}\varsigma(t) = 1 + \tfrac{\varsigma(t)}{6+\sin(\varsigma(t))}, \qquad 0 < t < 1, \\
&\varsigma(0) = \varsigma(1) = 0,
\end{aligned} \tag{4.1}$$

where $\upsilon = \frac{5}{3}$, $\vartheta(t) = \sin t$, *and* $\mathfrak{P}(t,\varsigma) = 1 + \frac{\varsigma}{6+\sin(\varsigma)}$. *It is easy to see that* $\mathfrak{P}(t,\varsigma)$ *is nonnegative and continuous function for all* $t \in [0,1]$ *and* $\varsigma \in [0,\infty)$. *It is clear that*

$$|\mathfrak{P}(\cdot,\varsigma) - \mathfrak{P}(\cdot,v)| \leq \frac{1}{7}|\varsigma - v| = \wp|\varsigma - v|, \ \forall \varsigma, v \in [0,\infty).$$

Moreover, by some computations, we get

$$\Omega := \frac{\wp[\vartheta(1) - \vartheta(0)]}{\Gamma(\upsilon+1)} = \frac{\frac{1}{7}[\sin(1) - \sin(0)]}{\Gamma(\frac{5}{3}+1)} \approx 0.08 < 1.$$

All suppositions of Theorem 3.3 hold. So, Theorem 3.3 guarantees that (4.1) has a unique positive solution $\varsigma(t) \in C[0,1]$.

Observe that $\mathfrak{P} : [0,1] \times \mathbb{R}^+ \to [1,\infty)$ *is continuous and*

$$1 < \lim_{\varsigma \to +\infty} \mathfrak{P}(t,\varsigma) < 2.$$

Thus, since all the suppositions in Corollary 3.2 are fulfilled with $a = 1$, *Corollary 3.2 can be applied to the problem (4.1).*

Example 4.2. *To apply Corollary 3.1, we consider* $\mathfrak{P}(t,\varsigma)$ *as in Example 4.1. It follows that*

$$1 \leq \mathfrak{P}(t,\sigma) \leq \frac{8}{7}, \quad (t,\sigma) \in [0,1] \times \mathbb{R}^+.$$

Hence

$$1 \leq \underline{\mathfrak{P}}(t,\sigma) \leq \overline{\mathfrak{P}}(t,\sigma) \leq \frac{8}{7}.$$

Here $L_1 = 1$ *and* $L_2 = \frac{8}{7}$. *Let the fractional BVP*

$$\begin{aligned} -D_{0^+}^{\upsilon,\vartheta}\overline{\varsigma}(t) &= L_2, \quad 0 < t < 1, \\ \overline{\varsigma}(0) &= \overline{\varsigma}(1) = 0. \end{aligned}$$

which has a positive solution

$$\overline{\varsigma}(t) = \frac{8}{7}\left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right)\frac{(\vartheta(t) - \vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}.$$

Similarly, the fractional BVP

$$\begin{aligned} -D_{0^+}^{\upsilon,\vartheta}\underline{\varsigma}(t) &= L_1, \quad 0 < t < 1, \\ \underline{\varsigma}(0) &= \underline{\varsigma}(1) = 0, \end{aligned}$$

has a positive solution

$$\underline{\varsigma}(t) = \left(\left(\frac{\vartheta(t) - \vartheta(0)}{\vartheta(1) - \vartheta(0)}\right)^{-1} - 1\right)\frac{(\vartheta(t) - \vartheta(0))^{\upsilon}}{\Gamma(\upsilon+1)}.$$

In particular, if $\vartheta(t) = t$, *then* $\overline{\varsigma}(t) = \frac{8}{7}\frac{t^{\upsilon}}{\Gamma(\upsilon+1)}\left(\frac{1}{t} - 1\right)$ *and* $\underline{\varsigma}(t) = \frac{t^{\upsilon}}{\Gamma(\upsilon+1)}\left(\frac{1}{t} - 1\right)$.

Thus, the functions $\overline{\varsigma}(t)$ *and* $\underline{\varsigma}(t)$ *are upper and lower solution of (4.1), respectively. By Corollary 3.1, we get that (4.1) has at least one positive solution* $\varsigma(t) \in C[0,1]$, *which produces the inequalities (3.9) and (3.10).*

5. Conclusion

The research of generalized FC has become a novel field of investigation. Through some fixed point theorem, properties of Green functions, and upper and lower control function, a further of existence results of positive solutions for the generalized problem are obtained. Two examples are offered to illustrate the fundamental results. The epilogue obtained in this work will be very advantageous in the applications. Also, we anticipate finding some applications in further nonlinear problems.

Competing interests

The authors declare that they have no competing interests.

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