GENERAL DECAY FOR A VON KARMAN PLATE SYSTEM WITH GENERAL TYPE OF RELAXATION FUNCTIONS ON THE BOUNDARY

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Abstract. In this paper, we consider a von Karman plate system with general type of relaxation functions on the boundary. Using some properties of the convex functions without the assumption that initial value $w_0 \equiv 0$ on the boundary, we study the general decay rate result.

Key words: von Karman plate; general decay; memory term; relaxation function; convexity; boundary condition
2010 MSC: 35B40; 35L05; 37L45; 74D99

1. Introduction

The purpose of this work is to investigate the general decay of the solutions to von Karman plate system with memory condition on the boundary:

$$w_{tt} + \Delta^2 w = [w, v] \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$\Delta^2 v = -[w, w] \quad \text{in } \Omega \times (0, \infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} + \int_0^t h_1(t-s)\Big(\mathcal{A}_1 w(s) + \alpha_1 \frac{\partial w(s)}{\partial \nu}\Big) ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.4}$$

$$w - \int_0^t h_2(t-s)\big(\mathcal{A}_2 w(s) - \alpha_2 w(s)\big) ds = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.5}$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x) \quad \text{in } \Omega \tag{1.6}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Γ, $x = (x_1, x_2)$, α_1 and α_2 are small positive constants. The von Karman bracket $[w, u]$ denotes the bilinear expression

$$[w, u] = w_{x_1x_1} u_{x_2x_2} - 2 w_{x_1x_2} u_{x_1x_2} + w_{x_2x_2} u_{x_1x_1}.$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal vector on Γ and by $\tau = (-\nu_2, \nu_1)$ the corresponding unit tangent vector. Denoting by the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\mathcal{A}_1 w = \Delta w + (1-\lambda) A_1 w, \quad \mathcal{A}_2 w = \frac{\partial \Delta w}{\partial \nu} + (1-\lambda)\frac{\partial A_2 w}{\partial \tau}$$

where

$$A_1 w = 2\nu_1\nu_2 w_{x_1x_2} - \nu_1^2 w_{x_2x_2} - \nu_2^2 w_{x_1x_1},$$

$$A_2 w = (\nu_1^2 - \nu_2^2) w_{x_1x_2} + \nu_1\nu_2 (w_{x_2x_2} - w_{x_1x_1})$$

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and the constant $\lambda \in (0, \frac{1}{2})$, represents Poisson's ratio. This system describes the transversal displacement w and the Airy-stress function v of a vibrating plate subjected to the boundary viscoelastic damping.

The stability of the solutions to a von Karman system was considered by several authors ([1-3]). The asymptotic behavior of the solutions to a von Karman plates with memory was studied by several authors ([4-6]). On the other hand, Rivera et al. [7] proved that the solution of system (1.1)-(1.6) decays exponentially provided the resolvent kernels satisfy

$$k_i(0) > 0,\ k_i'(t) \le -C_1 k_i(t),\ k_i''(t) \ge -C_2 k_i'(t),\ \forall t \ge 0,\ (i = 1, 2), \tag{1.7}$$

for some positive constants C_1 and C_2. Santos and Soufyane [8] improved the decay result of [7]. They assumed that the resolvent kernels satisfy

$$k_i(0) > 0,\ k_i(t) \ge 0,\ k_i'(t) \le 0,\ k_i''(t) \ge \eta_i(t)(-k_i'(t)),\quad (i = 1, 2), \tag{1.8}$$

where $\eta_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions

$$\eta_i(t) > 0,\ \eta_i'(t) \le 0,\ \int_0^{+\infty} \eta_i(t)dt = +\infty.$$

Kang [9] extended the results in [8] by considering general decay rates of the energy under $\alpha_1 = \alpha_2 = 0$ and the generalized conditions

$$k_i(0) > 0,\quad \lim_{t\to\infty} k_i(t) = 0,\quad k_i'(t) \le 0,\quad k_i''(t) \ge K(-k_i'(t)),\quad (i = 1, 2), \tag{1.9}$$

where K is a positive function, with $K(0) = K'(0) = 0$, and K is linear or it is strictly increasing and strictly convex on $(0, r]$, for some $0 < r < 1$. The inequality in (1.9) has been introduced for the first time in [10]. These are weaker conditions on H than those introduced in [10]. Later, Park [11] obtained the general decay of the solution for system (1.1)-(1.6) with $\alpha_1 = \alpha_2 = 0$ under the assumption (1.9) and $w_0 \neq 0$ on a part of the boundary. Moreover, the stability of the solutions to the viscoelastic problems with the memory on the boundary has been studied by many authors ([12-19]).

Motivated by their results, we prove the general decay of the solution for the system (1.1)-(1.6) when the initial data $w_0 \neq 0$ on Γ and the resolvent kernels k_i satisfy

$$k_i(0) > 0,\quad k_i'(t) \le 0,\quad k_i''(t) \ge \xi_i(t) G_i(-k_i'(t)),\quad (i = 1, 2), \tag{1.10}$$

where $\xi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive nonincreasing differential function and G_i is a positive function, with $G_i(0) = G_i'(0) = 0$, and G_i is a linear or it is strictly increasing and strictly convex on $(0, r]$, for some $0 < r < 1$. This is a more general condition than conditions (1.8) and (1.9). Recently, Feng and Soufyane [20] showed the general decay of the solution for system (1.1)-(1.6) with $\alpha_1 = \alpha_2 = 0$ under the assumption (1.10) and $w_0 = 0$ on a part of the boundary. The general stability result of viscoelastic equation, for relaxation function h satisfying $h'(t) \le -\xi(t)K(h(t))$, has been investigated in [21-23].

The paper is organized as follows. In Section 2 we present some notations and assumptions needed for our work. In Section 3 we prove the general decay of the solutions for the von Karman plate system with memory condition on the boundary.

2. Preliminaries

In this section, we present some material needed in the proof of our main result. Throughout this paper we denote $||\cdot||_{L^2(\Omega)}$ and $||\cdot||_{L^2(\Gamma)}$ by $||\cdot||$ and $||\cdot||_\Gamma$, respectively. Let us define the bilinear form

$$a(w,u)=\int_\Omega \big\{w_{x_1x_1}u_{x_1x_1}+w_{x_2x_2}u_{x_2x_2}+\mu(w_{x_1x_1}u_{x_2x_2}+w_{x_2x_2}u_{x_1x_1})+2(1-\mu)w_{x_1x_2}u_{x_1x_2}\big\}dx.$$

We assume that there exists $x_0\in\mathbb{R}^2$ such that

$$\Gamma=\{x\in\Gamma:\nu(x)\cdot(x-x_0)>0\}.$$

Denoting by $m(x)=x-x_0$, the compactness of Γ implies that there exists $\delta>0$ such that

$$m(x)\cdot\nu(x)\geq\delta>0,\quad\forall x\in\Gamma. \tag{2.1}$$

The following identity will be used later.

Lemma 2.1. ([24]) For any $w\in H^4(\Omega)$ and $u\in H^2(\Omega)$, we have

$$\int_\Omega(\Delta^2w)udx=a(w,u)+\int_\Gamma(\mathcal{A}_2w)u-(\mathcal{A}_1w)\frac{\partial u}{\partial\nu}d\Gamma, \tag{2.2}$$

$$\begin{aligned}\int_\Omega(m\cdot\nabla w)\Delta^2wdx=a(w,w)&+\int_\Gamma\Big[(\mathcal{A}_2w)(m\cdot\nabla w)-(\mathcal{A}_1w)\frac{\partial(m\cdot\nabla w)}{\partial\nu}\Big]d\Gamma\\&+\frac12\int_\Gamma(m\cdot\nu)\big[w^2_{x_1x_1}+w^2_{x_2x_2}+2\mu w_{x_1x_1}w_{x_2x_2}+2(1-\lambda)w^2_{x_1x_2}\big]d\Gamma.\end{aligned} \tag{2.3}$$

We state the relative results of the Airy stress function and von Karman bracket $[\cdot,\cdot]$.

Lemma 2.2. ([25]) Let w,u be functions in $H^2(\Omega)$ and v in $H^2_0(\Omega)$, where Ω is a open bounded and connected set of $\mathbb{R}^2$ with regular boundary. Then

$$\int_\Omega[w,v]udx=\int_\Omega[w,u]vdx. \tag{2.4}$$

We introduce the following binary operators

$$(h*w)(t)=\int_0^t h(t-s)w(s)ds,\quad(h\Box w)(t):=\int_0^t h(t-s)|w(t)-w(s)|^2ds$$

where $*$ is the convolution product. By differentiating the term $h\Box w$, we obtain the following lemma for the important property between these two operators.

Lemma 2.3. For $h,w\in C^1([0,\infty):\mathbb{R})$, we have

$$(h*w)w_t=-\frac12h(t)|w(t)|^2+\frac12h'\Box w-\frac12\frac{d}{dt}\Big[h\Box w-\Big(\int_0^t h(s)ds\Big)|w|^2\Big]. \tag{2.5}$$

Now, we use the boundary conditions (1.4) and (1.5) to estimate the terms $\mathcal{A}_1 w$ and $\mathcal{A}_2 w$. As shown in ([7-9]), differentiating (1.4) and (1.5) and applying the Volterra's inverse operator, we have

$$\mathcal{A}_1 w+\alpha_1 \frac{\partial w}{\partial \nu}=-\frac{1}{h_1(0)}\Big\{\frac{\partial w_t}{\partial \nu}+k_1 * \frac{\partial w_t}{\partial \nu}\Big\}, \quad \mathcal{A}_2 w-\alpha_2 w=\frac{1}{h_2(0)}\{w_t+k_2 * w_t\},$$

where the resolvent kernel $k_i, (i=1,2)$ satisfies

$$k_i+\frac{1}{h_i(0)} h_i' * k_i=-\frac{1}{h_i(0)} h_i'.$$

Denoting by $\gamma_1=\frac{1}{h_1(0)}$ and $\gamma_2=\frac{1}{h_2(0)}$, we get

$$\mathcal{A}_1 w=-\alpha_1 \frac{\partial w}{\partial \nu}-\gamma_1\Big\{\frac{\partial w_t}{\partial \nu}-k_1(t) \frac{\partial w_0}{\partial \nu}+k_1(0) \frac{\partial w}{\partial \nu}+k_1' * \frac{\partial w}{\partial \nu}\Big\}, \tag{2.6}$$

$$\mathcal{A}_2 w=\alpha_2 w+\gamma_2\{w_t-k_2(t) w_0+k_2(0) w+k_2' * w\}. \tag{2.7}$$

Thus, we use the boundary conditions (2.6) and (2.7) instead of (1.4) and (1.5).

As in [12, 20], we consider the following assumptions on k_i $(i=1,2)$.

(A) The resolvent kernel $k_i: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is twice differentiable functions such that

$$k_i(0)>0, \quad \lim_{t \rightarrow \infty} k_i(t)=0, \quad k_i'(t) \leq 0 \tag{2.8}$$

and there exists a positive function $G_i \in C^1(\mathbb{R}_+)$ and G_i is a linear or it is strictly increasing and strictly convex C^2 function on $(0, r],\ r<1$, with $G_i(0)=G_i'(0)=0$, such that

$$k_i''(t) \geq \xi_i(t) G_i(-k_i'(t)), \ \forall t>0. \tag{2.9}$$

where $\xi_i: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonincreasing differentiable function.

From (A), we easily see that there exists $t_0>0$ large enough such that

$$0<-k_i'(t_0) \leq-k_i'(t) \leq-k_i'(0), \text{ for } t \in[0, t_0] \tag{2.10}$$

and

$$\max\{k_i(t),-k_i'(t), k_i''(t)\}<\min\{r, G(r)\}, \text{ for } t \geq t_0, \tag{2.11}$$

where $G=\min\{G_1, G_2\}$.

As $\xi_i(t)$ and $-k_i'(t)$ are positive nonincreasing continuous functions and $G_i(t)$ is a positive continuous function, there exist positive constants a_i and b_i such that

$$a_i \leq \xi_i(t) G_i(-k_i'(t)) \leq b_i, \quad \text{for } t \in[0, t_0].$$

Therefore, for all $t \in[0, t_0]$, we obtain

$$k_i''(t) \geq \xi_i(t) G_i(-k_i'(t)) \geq a_i \frac{k_i'(t)}{k_i'(0)}=-c_i k_i'(t) \tag{2.12}$$

for some positive constant c_i.

The well-posedness of von Karman system plates with boundary conditions of memory type is given by the following theorem.

Theorem 2.1. ([7]) Let $k_i(i=1,2)\in C^2(\mathbb{R}_+)$ be such that $k_i,\ -k_i',\ k_i''\geq 0$. If the initial data $(w_0,w_1)\in (H^4(\Omega)\cap H^2(\Omega))\times H^2(\Omega)$ satisfy the conditions

$$\mathcal{A}_1 w_0+\alpha_1\frac{\partial w_0}{\partial\nu}+\gamma_1\frac{\partial w_1}{\partial\nu}=0,\quad \mathcal{A}_2 w_0-\alpha_2 w_0-\gamma_2 w_1=0\ \text{ on }\Gamma,$$

then the solution of (1.1)-(1.6) has the following regularity

$$w\in C^1([0,T]:H^2(\Omega))\cap C^0([0,T]:H^4(\Omega)).$$

The energy function of system (1.1)-(1.6) is given by

$$\begin{aligned}E(t)=&\frac{1}{2}||w_t||^2+\frac{1}{2}a(w,w)+\frac{1}{4}||\Delta v||^2+\frac{\alpha_1}{2}\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2+\frac{\gamma_1}{2}k_1(t)\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2\\&+\frac{\alpha_2}{2}||w||_\Gamma^2+\frac{\gamma_2}{2}k_2(t)||w||_\Gamma^2-\frac{\gamma_1}{2}\int_\Gamma k_1'\Box\frac{\partial w}{\partial\nu}d\Gamma-\frac{\gamma_2}{2}\int_\Gamma k_2'\Box w d\Gamma.\end{aligned}\tag{2.13}$$

To get a general stability result, the following is needed.

Remark 2.1. 1. If G_i is a strictly convex on $(0,r]$ and $G_i(0)=0$, then

$$G_i(\theta x)\leq\theta G_i(x),\quad x\in(0,r]\ \text{ and } 0\leq\theta\leq 1.\tag{2.14}$$

2. Let G^* be the convex conjugate of G in the sense of Young (see [26]); then

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)]\leq s(G')^{-1}(s),\quad\text{if } s\in(0,G'(r)]\tag{2.15}$$

and G^* satisfies the following Young's inequality

$$ab\leq G^*(a)+G(b),\quad\text{if } a\in(0,G'(r)],\ b\in(0,r].\tag{2.16}$$

3. Let F be a convex function on $[c,d]$, $\varrho:\Omega\to[c,d]$ and p are integrable functions on Ω such that $p(x)\geq 0$ and $\int_\Omega p(x)dx=p_0>0$, then Jensen's inequality holds that

$$F\Big(\frac{1}{p_0}\int_\Omega\varrho(x)p(x)dx\Big)\leq\frac{1}{p_0}\int_\Omega F\big(\varrho(x)\big)p(x)dx.\tag{2.17}$$

3. General decay

In this section, we study the asymptotic behavior of the solutions for the system (1.1)-(1.6). To show the general decay property, we first prove the dissipative property. Multiplying (1.1) by w_t and using (2.2), (2.5), Young's inequality and the boundary conditions (2.6) and (2.7), we obtain the following.

Lemma 3.1. ([7]) The energy function $E(t)$ satisfies

$$\begin{aligned}E'(t)\leq&-\frac{\gamma_2}{2}||w_t||_\Gamma^2+\frac{\gamma_2}{2}k_2'(t)||w||_\Gamma^2-\frac{\gamma_2}{2}\int_\Gamma k_2''\Box w d\Gamma+\frac{\gamma_2}{2}k_2^2(t)||w_0||_\Gamma^2\\&-\frac{\gamma_1}{2}\Big|\Big|\frac{\partial w_t}{\partial\nu}\Big|\Big|_\Gamma^2+\frac{\gamma_1}{2}k_1'(t)\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2-\frac{\gamma_1}{2}\int_\Gamma k_1''\Box\frac{\partial w}{\partial\nu}d\Gamma+\frac{\gamma_1}{2}k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2.\end{aligned}\tag{3.1}$$

Since $w_0\neq 0$ on Γ, Lemma 3.1 says that $E(t)$ may not be nonincreasing. So, we introduce the modified energy functional $\mathcal{E}(t)$ by

$$\mathcal{E}(t)=E(t)+\frac{\gamma_2}{2}||w_0||_\Gamma^2\int_t^\infty k_2^2(s)ds+\frac{\gamma_1}{2}\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\int_t^\infty k_1^2(s)ds.\tag{3.2}$$

Then from (3.1), we have

$$\mathcal{E}'(t) = E'(t) - \frac{\gamma_2}{2}k_2^2(t)||w_0||_\Gamma^2 - \frac{\gamma_1}{2}k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 \le -\frac{\gamma_2}{2}\int_\Gamma k_2''\Box w d\Gamma - \frac{\gamma_1}{2}\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma \le 0. \qquad (3.3)$$

For suitable choice of N_1 and N_2, let us introduce the Lyapunov functional

$$L(t) := N_1 E(t) + N_2 \Upsilon(t)$$

where

$$\Upsilon(t) := \int_\Omega \Big(m\cdot\nabla w + \frac{1}{2}w\Big)w_t dx.$$

It is not difficult to see that $L(t)$ satisfies $q_0 E(t) \le L(t) \le q_1 E(t)$, for some positive constants q_0 and q_1.

Lemma 3.2. Under the assumption (A), the functional $\Upsilon(t)$ satisfies

$$\begin{aligned}
\Upsilon'(t) \le\ & \frac{1}{2}\int_\Gamma (m\cdot\nu)|w_t|^2 d\Gamma - \frac{1}{2}||w_t||^2 - ||\Delta v||^2 - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2 d\Gamma \\
& -\Big(\frac{3}{2} - \epsilon\lambda_0\Big)a(w,w) - \alpha_2\Big(\frac{1}{4} - \alpha_2 C_\epsilon - \epsilon\lambda_0\Big)||w||_\Gamma^2 - \alpha_1\Big(\frac{1}{4} - \alpha_1 C_\epsilon - \epsilon\lambda_0\Big)\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2 \\
& -\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)\int_\Gamma (m\cdot\nu)\big[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2\big]d\Gamma \\
& +4\gamma_2^2 C_\epsilon\Big(||w_t||_\Gamma^2 + k_2^2(t)||w||_\Gamma^2 + k_2^2(t)||w_0||_\Gamma^2 + C(\delta_2)\int_\Gamma g_2\Box w d\Gamma\Big) \\
& +4\gamma_1^2 C_\epsilon\Big(\Big|\Big|\frac{\partial w_t}{\partial\nu}\Big|\Big|_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2 + C(\delta_1)\int_\Gamma g_1\Box\frac{\partial w}{\partial\nu}d\Gamma\Big), \qquad (3.4)
\end{aligned}$$

for any $0 < \delta_i < 1$ $(i = 1, 2)$, where

$$C(\delta_i) = \int_0^\infty \frac{(-k_i'(s))^2}{g_i(s)}ds \quad \text{and} \quad g_i(t) = k_i''(t) - \delta_i k_i'(t) > 0. \qquad (3.5)$$

Proof. According to [7-9], from (2.3) and (2.4), we obtain

$$\begin{aligned}
\Upsilon'(t) =\ & \frac{1}{2}\int_\Gamma (m\cdot\nu)|w_t|^2 d\Gamma - \frac{1}{2}||w_t||^2 - ||\Delta v||^2 - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2 d\Gamma \\
& -\frac{3}{2}a(w,w) - \int_\Gamma (\mathcal{A}_2 w)\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma + \int_\Gamma (\mathcal{A}_1 w)\frac{\partial}{\partial\nu}\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma \\
& -\frac{1}{2}\int_\Gamma (m\cdot\nu)\big[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2\big]d\Gamma. \qquad (3.6)
\end{aligned}$$

Applying Young's inequality we get

$$\Big| - \int_\Gamma (\mathcal{A}_2 w)\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma\Big| \le \epsilon||m\cdot\nabla w||_\Gamma^2 + C_\epsilon||\mathcal{A}_2 w - \alpha_2 w||_\Gamma^2 - \Big(\frac{\alpha_2}{4} - \alpha_2^2 C_\epsilon\Big)||w||_\Gamma^2, \qquad (3.7)$$

$$\Big|\int_\Gamma (\mathcal{A}_1 w)\frac{\partial}{\partial\nu}\Big(m\cdot\nabla w + \frac{1}{2}w\Big)d\Gamma\Big| \le \epsilon\Big|\Big|\frac{\partial}{\partial\nu}(m\cdot\nabla w)\Big|\Big|_\Gamma^2 + C_\epsilon\Big|\Big|\mathcal{A}_1 w + \alpha_1\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2 - \Big(\frac{\alpha_1}{4} - \alpha_1^2 C_\epsilon\Big)\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2, \qquad (3.8)$$

where ϵ is a positive constant. On the other hand, by the bilinear form $a(w,u) + \int_\Gamma \big(\alpha_1\frac{\partial w}{\partial\nu}\frac{\partial u}{\partial\nu} + \alpha_2 wu\big)d\Gamma$ is strictly coercive on $H^2(\Omega)$ and (2.1), we obtain

$$\begin{aligned}
||m\cdot\nabla w||_\Gamma^2 + \Big|\Big|\frac{\partial}{\partial\nu}(m\cdot\nabla w)\Big|\Big|_\Gamma^2 \le\ & \lambda_0\Big(a(w,w) + \alpha_2||w||_\Gamma^2 + \alpha_1\Big|\Big|\frac{\partial w}{\partial\nu}\Big|\Big|_\Gamma^2\Big) \\
& + \frac{\lambda_0}{\delta}\int_\Gamma (m\cdot\nu)\big[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2\big]d\Gamma, \qquad (3.9)
\end{aligned}$$

where λ_0 is a positive constant. Noting that

$$(k_2' * w)(t) = w(t)[k_2(t) - k_2(0)] - \int_0^t k_2'(t-s)(w(t) - w(s))ds,$$

the boundary condition (2.7) can be written as

$$\mathcal{A}_2 w - \alpha_2 w = \gamma_2\{w_t + k_2(t)w - k_2(t)w_0 - \int_0^t k_2'(t-s)(w(t) - w(s))ds\}. \tag{3.10}$$

Similarly, we can show that

$$\mathcal{A}_1 w + \alpha_1 \frac{\partial w}{\partial \nu} = -\gamma_1 \Big\{ \frac{\partial w_t}{\partial \nu} + k_1(t)\frac{\partial w}{\partial \nu} - k_1(t)\frac{\partial w_0}{\partial \nu} - \int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(s)}{\partial \nu}\Big)ds\Big\}. \tag{3.11}$$

Using (3.6)-(3.11), we arrive at

$$\begin{aligned}
\Upsilon'(t) \le\ & \frac{1}{2}\int_\Gamma (m\cdot\nu)|w_t|^2 d\Gamma - \frac{1}{2}||w_t||^2 - ||\Delta v||^2 - \frac{1}{2}\int_\Gamma (m\cdot\nu)|\Delta v|^2 d\Gamma \\
& -\Big(\frac{3}{2} - \epsilon\lambda_0\Big)a(w,w) - \alpha_2\Big(\frac{1}{4} - \alpha_2 C_\epsilon - \epsilon\lambda_0\Big)||w||_\Gamma^2 - \alpha_1\Big(\frac{1}{4} - \alpha_1 C_\epsilon - \epsilon\lambda_0\Big)\Big|\Big|\frac{\partial w}{\partial \nu}\Big|\Big|_\Gamma^2 \\
& -\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)\int_\Gamma (m\cdot\nu)\big[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2\big]d\Gamma \\
& +4\gamma_1^2 C_\epsilon\Big(\Big|\Big|\frac{\partial w_t}{\partial \nu}\Big|\Big|_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w}{\partial \nu}\Big|\Big|_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 + \Big|\Big| - \int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(s)}{\partial \nu}\Big)ds\Big|\Big|_\Gamma^2\Big) \\
& +4\gamma_2^2 C_\epsilon\Big(||w_t||_\Gamma^2 + k_2^2(t)||w||_\Gamma^2 + k_2^2(t)||w_0||_\Gamma^2 + \Big|\Big| - \int_0^t k_2'(t-s)(w(t)-w(s))ds\Big|\Big|_\Gamma^2\Big).
\end{aligned} \tag{3.12}$$

Using Cauchy-Schwarz inequality and (3.5), we have(see details in [21, 23])

$$\begin{aligned}
& \Big|\Big| - \int_0^t k_2'(t-s)(w(t)-w(s))ds\Big|\Big|_\Gamma^2 \\
& \le \int_0^t \frac{(-k_2'(s))^2}{g_2(s)}ds \int_\Gamma\int_0^t \big(k_2''(t-s) - \delta_2 k_2'(t-s)\big)|w(t)-w(s)|^2 ds d\Gamma \le C(\delta_2)\int_\Gamma g_2\Box w d\Gamma,
\end{aligned} \tag{3.13}$$

and

$$\Big|\Big| - \int_0^t k_1'(t-s)\Big(\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(s)}{\partial \nu}\Big)ds\Big|\Big|_\Gamma^2 \le C(\delta_1)\int_\Gamma g_1\Box\frac{\partial w}{\partial \nu}d\Gamma. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we have (3.4). $\Box$

Next, we define the functionals

$$K_1(t) = \int_0^t f_1(t-s)\Big|\Big|\frac{\partial w(s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds \quad \text{and} \quad K_2(t) = \int_0^t f_2(t-s)||w(s)||_\Gamma^2 ds$$

where $f_i(t) = \int_t^\infty (-k_i'(s))ds,\ i = 1, 2.$

Lemma 3.3. Under the assumption (A), the functionals $K_1(t)$ and $K_2(t)$ satisfy the estimates

$$K_1'(t) \le 3k_1(0)\Big|\Big|\frac{\partial w}{\partial \nu}\Big|\Big|_\Gamma^2 + \frac{1}{2}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma, \tag{3.15}$$

$$K_2'(t) \le 3k_2(0)||w||_\Gamma^2 + \frac{1}{2}\int_\Gamma k_2'\Box w d\Gamma. \tag{3.16}$$

Proof. Taking the derivative of the functional $K_2(t)$ and using the fact $f_2'(t) = k_2'(t)$, we find that

$$K_2'(t) = f_2(0)||w||_\Gamma^2 + \int_0^t k_2'(t-s)||w(s)||_\Gamma^2 ds$$
$$= \int_0^t k_2'(t-s)||w(s)-w(t)||_\Gamma^2 ds + 2\int_\Gamma w(t)\int_0^t k_2'(t-s)\big(w(s)-w(t)\big)dsd\Gamma + k_2(t)||w||_\Gamma^2. \quad (3.17)$$

Using Young's inequality and (2.8), we obtain

$$2\int_\Gamma w(t)\int_0^t k_2'(t-s)\big(w(s)-w(t)\big)dsd\Gamma$$
$$\le 2k_2(0)||w||_\Gamma^2 + \frac{\int_0^t -k_2'(s)ds}{2k_2(0)}\int_\Gamma\int_0^t(-k_2'(t-s))|w(s)-w(t)|^2 dsd\Gamma$$
$$\le 2k_2(0)||w||_\Gamma^2 - \frac{1}{2}\int_\Gamma k_2'\Box w d\Gamma. \quad (3.18)$$

From (3.17) and (3.18), we get the estimate (3.16). Similarly, we can obtain the estimate (3.15). □

Lemma 3.4. Suppose that the assumption (A) holds. Then, for $N_1, N_2 > 0$ large enough, there exist positive constants β_1 and β_2 such that

$$L'(t) \le -\beta_1\big(||w_t||^2 + a(w,w) + ||\Delta v||^2\big) + \beta_2\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big)$$
$$-4\gamma_2 k_2(0)||w||_\Gamma^2 - 4\gamma_1 k_1(0)\Big|\Big|\frac{\partial w}{\partial \nu}\Big|\Big|_\Gamma^2 - \frac{\gamma_2}{4}\int_\Gamma k_2'\Box w d\Gamma - \frac{\gamma_1}{4}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma, \text{ for } t \ge t_0. \quad (3.19)$$

where t_0 was introduced in (2.10).

Proof. Combining (3.1), (3.4) and (3.5), we see that

$$L'(t) \le -\frac{N_2}{2}||w_t||^2 - N_2||\Delta v||^2 - \gamma_2\Big(\frac{N_1}{2} - 4\gamma_2 C_\epsilon N_2 - \frac{RN_2}{2\gamma_2}\Big)||w_t||_\Gamma^2 - \gamma_1\big(\frac{N_1}{2} - 4\gamma_1 C_\epsilon N_2\big)\Big|\Big|\frac{\partial w_t}{\partial \nu}\Big|\Big|_\Gamma^2$$
$$-\Big(\frac{3}{2} - \epsilon\lambda_0\Big)N_2 a(w,w) - \Big(\alpha_2\big(\frac{1}{4} - \epsilon\lambda_0 - \alpha_2 C_\epsilon\big)N_2 - 4\gamma_2^2 C_\epsilon k_2^2(t)N_2\Big)||w||_\Gamma^2$$
$$-\Big(\alpha_1\big(\frac{1}{4} - \epsilon\lambda_0 - \alpha_1 C_\epsilon\big)N_2 - 4\gamma_1^2 C_\epsilon k_1^2(t)N_2\Big)\Big|\Big|\frac{\partial w}{\partial \nu}\Big|\Big|_\Gamma^2 - \frac{\gamma_2\delta_2 N_1}{2}\int_\Gamma k_2'\Box w d\Gamma - \frac{\gamma_1\delta_1 N_1}{2}\int_\Gamma k_1'\Box\frac{\partial w}{\partial \nu}d\Gamma$$
$$-\gamma_2\Big(\frac{N_1}{2} - 4\gamma_2 C_\epsilon C(\delta_2)N_2\Big)\int_\Gamma g_2\Box w d\Gamma - \gamma_1\Big(\frac{N_1}{2} - 4\gamma_1 C_\epsilon C(\delta_1)N_2\Big)\int_\Gamma g_1\Box\frac{\partial w}{\partial \nu}d\Gamma$$
$$-\Big(\frac{1}{2} - \frac{\epsilon\lambda_0}{\delta}\Big)N_2\int_\Gamma(m\cdot\nu)\big[w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1}w_{x_2x_2} + 2(1-\lambda)w_{x_1x_2}^2\big]d\Gamma$$
$$+k_2^2(t)\Big(\frac{\gamma_2 N_1}{2} + 4\gamma_2^2 C_\epsilon N_2\Big)||w_0||_\Gamma^2 + k_1^2(t)\Big(\frac{\gamma_1 N_1}{2} + 4\gamma_1^2 C_\epsilon N_2\Big)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2,$$

where $R = \max\{m(x)\cdot\nu(x) : x \in \Gamma\}$. We first fix $\epsilon > 0$ small such that

$$\frac{1}{4} - \epsilon\lambda_0 > 0 \text{ and } \frac{1}{2} - \frac{\epsilon\lambda_0}{\delta} > 0,$$

and then take α_1 and α_2 small such that

$$\frac{1}{4} - \epsilon\lambda_0 - \alpha_1 C_\epsilon > 0 \text{ and } \frac{1}{4} - \epsilon\lambda_0 - \alpha_2 C_\epsilon > 0.$$

Next, applying the fact $\lim_{t\to\infty} k_i(t) = 0$ $(i = 1, 2)$, we choose N_2 large enough so that

$$\alpha_i\big(\frac{1}{4} - \epsilon\lambda_0 - \alpha_i C_\epsilon\big)N_2 - 4\gamma_i^2 C_\epsilon k_i^2(t)N_2 > 4\gamma_i k_i(0), \ i = 1, 2.$$

From (2.8), (2.9) and (3.5), we have

$$-\delta_i k_i'(t) \le k_i''(t) - \delta_i k_i'(t) = g_i(t) \ \Rightarrow \ \frac{-\delta_i k_i'(t)}{g_i(t)} \le 1 \ \Rightarrow \ \frac{\delta_i(-k_i'(t))^2}{g_i(t)} \le -k_i'(t), \ i = 1, 2. \tag{3.20}$$

Integrating (3.20) and using (2.8), we obtain

$$\delta_i C(\delta_i) = \delta_i \int_0^\infty \frac{(-k_i'(s))^2}{g_i(s)} ds \le k_i(0), \ i = 1, 2.$$

By the Lebesgue dominated convergence theorem, we find that $\delta_i C(\delta_i) \to 0$ as $\delta_i \to 0$. Then there exists $0 < \delta_0 < 1$ such that if $\delta_i < \delta_0$, then

$$\max\{4\delta_1\gamma_1 C_\epsilon C(\delta_1)N_2, \ 4\delta_2\gamma_2 C_\epsilon C(\delta_2)N_2\} < \frac{1}{8}.$$

Finally, taking N_1 large enough so that

$$N_1 > \max\Big\{8\gamma_1 C_\epsilon N_2, \ \Big(8\gamma_2 C_\epsilon + \frac{R}{\gamma_2}\Big)N_2\Big\}$$

and choosing $\delta_i = \frac{1}{2N_1} < \delta_0$ $(i = 1, 2)$, we have the estimate (3.19). □

Now, we are now ready to prove our main result.

Theorem 3.1. Suppose that the assumption (A) holds. Then there exist positive constants $\epsilon_0, \sigma_1, \sigma_2, \kappa_1$ and κ_2 such that the energy functional satisfies, for all $t \ge t_0$,

$$E(t) \le \sigma_1\Big\{1 + \int_{t_0}^t e^{\sigma_2 \int_{t_0}^s \xi(\eta)d\eta}\Big(k_2^2(s)||w_0||_\Gamma^2 + k_1^2(s)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big)ds\Big\}e^{-\sigma_2\int_{t_0}^t \xi(s)ds}, \quad \text{if } G \text{ is linear}, \tag{3.21}$$

$$\begin{aligned} E(t) \le \kappa_1 G_1^{-1}\Big(&\frac{\kappa_2\Big(1 + ||w_0||_\Gamma^2 \int_{t_0}^t G(k_2(s))\xi(s)ds + \big|\big|\frac{\partial w_0}{\partial \nu}\big|\big|_\Gamma^2 \int_{t_0}^t G(k_1(s))\xi(s)ds\Big)}{t\xi(t)}\Big) \\ &-\frac{\gamma_2}{2}||w_0||_\Gamma^2 \int_t^\infty k_2^2(s)ds - \frac{\gamma_1}{2}\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 \int_t^\infty k_1^2(s)ds, \quad \text{if } G \text{ is nonlinear}, \end{aligned} \tag{3.22}$$

where $G_1(t) = tG'(\epsilon_0 t)$, $G = \min\{G_1, G_2\}$ and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$.

Proof. From (2.13) and (3.19), there exist positive constants β_3 and β_4 such that for $t \ge t_0$,

$$L'(t) \le -\beta_3 E(t) - \beta_4\Big(\int_\Gamma k_2' \Box w d\Gamma + \int_\Gamma k_1' \Box \frac{\partial w}{\partial \nu} d\Gamma\Big) + \beta_2\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big). \tag{3.23}$$

Applying (2.12) and (3.1), we see that, for all $t \ge t_0$,

$$\begin{aligned} &\beta_4 \int_\Gamma \int_0^{t_0} \Big(- k_2'(s)|w(t) - w(t-s)|^2 - k_1'(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2\Big)dsd\Gamma \\ &\le \frac{\beta_4}{c_0}\int_\Gamma \int_0^{t_0} \Big(k_2''(s)|w(t) - w(t-s)|^2 + k_1''(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2\Big)dsd\Gamma \\ &\le \frac{\beta_4}{c_0\gamma_0}\Big(\gamma_2 k_2^2(t)||w_0||_\Gamma^2 + \gamma_1 k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 - 2E'(t)\Big), \end{aligned} \tag{3.24}$$

where $c_0 = \min\{c_1, c_2\}$ and $\gamma_0 = \min\{\gamma_1, \gamma_2\}$. Let $\Phi(t) = L(t) + \frac{2\beta_4}{c_0\gamma_0}E(t)$, which is equivalent to $E(t)$. Using (3.23) and (3.24), we obtain for all $t \ge t_0$,

$$\begin{aligned} \Phi'(t) \le &-\beta_3 E(t) + \beta_5\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big) \\ &-\beta_4\Big(\int_\Gamma \int_{t_0}^t k_2'(s)|w(t) - w(t-s)|^2 dsd\Gamma + \int_\Gamma \int_{t_0}^t k_1'(s)\Big|\frac{\partial w(t)}{\partial \nu} - \frac{\partial w(t-s)}{\partial \nu}\Big|^2 dsd\Gamma\Big), \end{aligned} \tag{3.25}$$

where $\beta_5 = \max\{\beta_2 + \frac{\beta_4\gamma_1}{c_0\gamma_0},\ \beta_2 + \frac{\beta_4\gamma_2}{c_0\gamma_0}\}$.

We consider the following two cases.

(1) G is linear: Multiplying (3.25) by the nonincreasing function $\xi(t)$ and using (2.9) and (3.1), we have

$$\begin{aligned}\xi(t)\Phi'(t) &\leq -\beta_3\xi(t)E(t) + \beta_5\xi(t)\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big)\\ &\quad+\beta_4\Big(\int_\Gamma\int_{t_0}^t k_2''(s)|w(t)-w(t-s)|^2 dsd\Gamma + \int_\Gamma\int_{t_0}^t k_1''(s)\Big|\frac{\partial w(t)}{\partial\nu} - \frac{\partial w(t-s)}{\partial\nu}\Big|^2 dsd\Gamma\Big)\\ &\leq -\beta_3\xi(t)E(t) + \beta_6\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big) - \frac{2\beta_4}{\gamma_0}E'(t),\ \forall t\geq t_0,\end{aligned}$$

where $\beta_6 = \max\{\beta_5\xi_0 + \frac{\beta_4\gamma_1}{\gamma_0},\ \beta_5\xi_0 + \frac{\beta_4\gamma_2}{\gamma_0}\}$ and $\xi(t)\leq\xi_0$. This gives

$$(\xi(t)\Phi(t) + \frac{2\beta_4}{\gamma_0}E(t))' \leq -\beta_3\xi(t)E(t) + \beta_6\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big),\ \forall t\geq t_0.$$

Hence, using the fact that $I(t) = \xi(t)\Phi(t) + \frac{2\beta_4}{\gamma_0}E(t) \sim E(t)$, we deduce that

$$I'(t) \leq -\beta_7\xi(t)I(t) + \beta_6\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big),\ \forall t\geq t_0, \tag{3.26}$$

where β_7 is a positive constant. We introduce

$$J(t) = I(t) - \beta_6 e^{-\beta_7\int_{t_0}^t\xi(s)ds}\Big(\int_{t_0}^t k_2^2(s)e^{\beta_7\int_{t_0}^s\xi(\eta)d\eta}ds||w_0||_\Gamma^2 + \int_{t_0}^t k_1^2(s)e^{\beta_7\int_{t_0}^s\xi(\eta)d\eta}ds\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big). \tag{3.27}$$

From (3.26), we have

$$J'(t) \leq -\beta_7\xi(t)J(t),\ \forall t\geq t_0.$$

Integrating this over (t_0,t), we obtain

$$J(t) \leq J(t_0)e^{-\beta_7\int_{t_0}^t\xi(s)ds},\ \forall t\geq t_0.$$

Using the fact that $I(t)\sim E(t)$ and (3.27), we get the estimate (3.21).

(2) G is nonlinear: First, we construct the functional

$$\Psi(t) = L(t) + \gamma_1K_1(t) + \gamma_2K_2(t),$$

which is nonnegative. From (2.13), (3.15), (3.16) and (3.19), we obtain

$$\Psi'(t) \leq -\rho_0E(t) + \beta_2\Big(k_2^2(t)||w_0||_\Gamma^2 + k_1^2(t)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\Big),$$

where ρ_0 is some positive constant. Integrating this over (t_0,t), we arrive at

$$\rho_0\int_{t_0}^t E(s)ds \leq \Psi(t_0) + \beta_2\Big(k_2(0)||w_0||_\Gamma^2\int_{t_0}^t k_2(t)ds + k_1(0)\Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2\int_{t_0}^t k_1(t)ds\Big).$$

Therefore, from (2.8), we conclude that

$$\int_{t_0}^t E(s)ds < \infty.$$

Then, we define $\eta_1(t)$ and $\eta_2(t)$ by, for constants θ_1 and $\theta_2\in(0,1)$,

$$\eta_1(t) := \theta_1\int_{t_0}^t\Big|\Big|\frac{\partial w(t)}{\partial\nu} - \frac{\partial w(t-s)}{\partial\nu}\Big|\Big|_\Gamma^2 ds,\quad \eta_2(t) := \theta_2\int_{t_0}^t||w(t)-w(t-s)||_\Gamma^2 ds \in (0,1).$$

Using (2.9), (2.14), (2.17) and the fact that $\xi_1(t)$ is a positive nonincreasing, we find that

$$\begin{aligned}
&-\int_{t_0}^{t} k_1'(s)\Big|\Big|\frac{\partial w(t)}{\partial \nu}-\frac{\partial w(t-s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds \leq \int_{t_0}^{t} G_1^{-1}\Big(\frac{k_1''(s)}{\xi_1(s)}\Big)\Big|\Big|\frac{\partial w(t)}{\partial \nu}-\frac{\partial w(t-s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds\\
&\leq \frac{\eta_1(t)}{\theta_1}G_1^{-1}\Big(\theta_1\int_{t_0}^{t}\frac{k_1''(s)}{\xi_1(s)\eta_1(t)}\Big|\Big|\frac{\partial w(t)}{\partial \nu}-\frac{\partial w(t-s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds\Big)\\
&\leq \frac{1}{\theta_1}G_1^{-1}\Big(\theta_1\int_{t_0}^{t}\frac{k_1''(s)}{\xi_1(s)}\Big|\Big|\frac{\partial w(t)}{\partial \nu}-\frac{\partial w(t-s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds\Big)\\
&\leq \frac{1}{\theta_1}G_1^{-1}\Big(\frac{1}{\xi_1(t)}\int_{t_0}^{t}k_1''(s)\Big|\Big|\frac{\partial w(t)}{\partial \nu}-\frac{\partial w(t-s)}{\partial \nu}\Big|\Big|_\Gamma^2 ds\Big)\\
&\leq \frac{1}{\theta_1}G_1^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma\Big), \qquad (3.28)
\end{aligned}$$

where $\xi(t)=\min\{\xi_1(t),\xi_2(t)\}$. Similarly, we can prove that

$$-\int_{t_0}^{t} k_2'(s)||w(t)-w(t-s)||_\Gamma^2 ds \leq \frac{1}{\theta_2}G_2^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_2''\Box w d\Gamma\Big). \qquad (3.29)$$

Combining (3.2), (3.25), (3.28) and (3.29), we see that for all $t\geq t_0$,

$$\begin{aligned}
\Phi'(t) \leq& -\beta_3\mathcal{E}(t)+\beta_8\Big(k_2(t)||w_0||_\Gamma^2+k_1(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big)\\
&+\frac{\beta_4}{\theta_2}G_2^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_2''\Box w d\Gamma\Big)+\frac{\beta_4}{\theta_1}G_1^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma\Big), \qquad (3.30)
\end{aligned}$$

where $\beta_8=\max\{\beta_5 k_1(0)+\frac{\beta_3\gamma_1}{2}\int_t^\infty k_1(s)ds,\ \beta_5 k_2(0)+\frac{\beta_3\gamma_2}{2}\int_t^\infty k_2(s)ds\}$. Now, for $\epsilon_0<r$, we define the functional

$$R(t):=\Phi(t)G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big),$$

where $\mathcal{E}(t)$ is the modified energy given in (3.2). Using (2.15), (2.16), (3.3), (3.30) and the fact that $\mathcal{E}'\leq 0,\ G'>0$ and $G''>0$, we obtain

$$\begin{aligned}
R'(t) \leq& -\beta_3 G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\mathcal{E}(t)+\beta_8 G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\Big(k_2(t)||w_0||_\Gamma^2+k_1(t)\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2\Big)\\
&+\frac{\beta_4}{\theta_2}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)G_2^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_2''\Box w d\Gamma\Big)+\frac{\beta_4}{\theta_1}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)G_1^{-1}\Big(\frac{1}{\xi(t)}\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma\Big)\\
\leq& -\Big[\beta_3\mathcal{E}(0)-\big(\beta_8||w_0||_\Gamma^2+\beta_8\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2+\frac{2\beta_4}{\theta_0}\big)\epsilon_0\Big]\frac{\mathcal{E}(t)}{\mathcal{E}(0)}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\\
&+\beta_8||w_0||_\Gamma^2 G(k_2(t))+\beta_8\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 G(k_1(t))+\frac{\beta_4}{\theta_0\xi(t)}\Big(\int_\Gamma k_2''\Box w d\Gamma+\int_\Gamma k_1''\Box\frac{\partial w}{\partial \nu}d\Gamma\Big)\\
\leq& -\Big[\beta_3\mathcal{E}(0)-\big(\beta_8||w_0||_\Gamma^2+\beta_8\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2+\frac{2\beta_4}{\theta_0}\big)\epsilon_0\Big]\frac{\mathcal{E}(t)}{\mathcal{E}(0)}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\\
&+\beta_8||w_0||_\Gamma^2 G(k_2(t))+\beta_8\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 G(k_1(t))-\frac{2\beta_4}{\theta_0\gamma_0\xi(t)}\mathcal{E}'(t),
\end{aligned}$$

where $\theta_0=\min\{\theta_1,\ \theta_2\}$ and $\gamma_0=\min\{\gamma_1,\ \gamma_2\}$. Choosing ϵ_0 such that $\rho_1=\beta_3\mathcal{E}(0)-\big(\beta_8||w_0||_\Gamma^2+\beta_8\big|\big|\frac{\partial w_0}{\partial \nu}\big|\big|_\Gamma^2+\frac{2\beta_4}{\theta_0}\big)\epsilon_0>0$, we have

$$R'(t)\leq -\rho_1\frac{\mathcal{E}(t)}{\mathcal{E}(0)}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)+\beta_8||w_0||_\Gamma^2 G(k_2(t))+\beta_8\Big|\Big|\frac{\partial w_0}{\partial \nu}\Big|\Big|_\Gamma^2 G(k_1(t))-\frac{2\beta_4}{\theta_0\gamma_0\xi(t)}\mathcal{E}'(t).$$

Then, multiplying this by $\xi(t)$, we get

$$\xi(t)R'(t) \le -\rho_1\xi(t)\frac{\mathcal{E}(t)}{\mathcal{E}(0)}G'\big(\epsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big) + \beta_8\Big(||w_0||_\Gamma^2 G(k_2(t)) + \Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2 G(k_1(t))\Big)\xi(t) - \frac{2\beta_4}{\theta_0\gamma_0}\mathcal{E}'(t). \quad (3.31)$$

Taking $\mathcal{F}(t) = \xi(t)R(t) + \frac{2\beta_4}{\theta_0\gamma_0}\mathcal{E}(t)$ and using (3.31), we arrive at

$$\mathcal{F}'(t) \le -\rho_1\xi(t)G_1\big(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big) + \beta_8\Big(||w_0||_\Gamma^2 G(k_2(t)) + \Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2 G(k_1(t))\Big)\xi(t), \ \forall t \ge t_0, \quad (3.32)$$

where $G_1 = tG'(\epsilon_0 t)$. Applying (3.32) and the fact that $\xi' \le 0,\ G_1' \ge 0$ and $\mathcal{E}' \le 0$, we find that

$$\begin{aligned}\Big[t\xi(t)G_1\big(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\Big]' &\le \xi(t)G_1\big(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big)\\ &\le -\frac{1}{\rho_1}\mathcal{F}'(t) + \frac{\beta_8}{\rho_1}\Big(||w_0||_\Gamma^2 G(k_2(t)) + \Big|\Big|\frac{\partial w_0}{\partial\nu}\Big|\Big|_\Gamma^2 G(k_1(t))\Big)\xi(t), \ \forall t \ge t_0.\end{aligned}$$

Integrating this over (t_0, t), we see that

$$\begin{aligned}t\xi(t)G_1\big(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\big) &\le t_0\xi(t_0)G_1\big(\frac{\mathcal{E}(t_0)}{\mathcal{E}(0)}\big) + \frac{1}{\rho_1}\mathcal{F}(t_0) + \frac{\beta_8}{\rho_1}\int_{t_0}^t \Big(||w_0||_\Gamma^2 G(k_2(s)) + ||\frac{\partial w_0}{\partial\nu}||_\Gamma^2 G(k_1(s))\Big)\xi(s)ds\\ &\le \rho_2\Big(1 + \int_{t_0}^t \big(||w_0||_\Gamma^2 G(k_2(s)) + ||\frac{\partial w_0}{\partial\nu}||_\Gamma^2 G(k_1(s))\big)\xi(s)ds\Big),\end{aligned}$$

where $\rho_2 = \max\{t_0\xi(t_0)G_1\big(\frac{\mathcal{E}(t_0)}{\mathcal{E}(0)}\big) + \frac{1}{\rho_1}\mathcal{F}(t_0),\ \frac{\beta_8}{\rho_1}\}$. Therefore, we conclude that

$$\mathcal{E}(t) \le \mathcal{E}(0)G_1^{-1}\Big(\frac{\rho_2\Big(1 + \int_{t_0}^t \big(||w_0||_\Gamma^2 G(k_2(s)) + ||\frac{\partial w_0}{\partial\nu}||_\Gamma^2 G(k_1(s))\big)\xi(s)ds\Big)}{t\xi(t)}\Big), \ \forall t \ge t_0.$$

Hence, applying (3.2), (3.22) is established.

□

Competing interests

The author declares that they have no competing interests.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2021R1I1A3042239).

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