Solutions to a three phase-field model for solidification

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Abstract: In this paper we present the phase-field models to describe nonisothermal solidification of ideal multicomponent and multiphase alloy systems. Governing equations are developed for the temporal and spatial variation of three phase-field functions, as well as the temperature field. The global existence of weak solutions to parabolic differential equations in three dimension was proved by the Galerkin method. The existence of a maximum theorem are also extensively studied.

Keywords: Nonlinear parabolic equation, Existence of solutions, Evolution of phase boundaries, Maximum theorem.

AMS subject classifications. 35K51; 74N20

1 Introduction

Multicomponent alloy system is a kind of important material, especially in technology application and technology. Therefore, it plays an important role in the formation of mechanical properties and microstructure of materials. The multi-component in the alloy combines with the appearance of multiphase, resulting in different phase transformation and different types of solidification. The solidification of binary alloy is the basis of studying the basic principle of solidification process. The solidification of multicomponent system can be analyzed by the solidification characteristics of binary system. The present model is a generalization of the one by Steinbach et al. in [2], which describes isothermal phase transitions of certain kinds of alloys. In this paper we study non-isothermal solidification of ideal multicomponent and multi-phase alloy systems. Thus we allow a temperature, which is assumed to be *a priori* given in the free energy functional $F[u, v, w, \theta]$. This means that the model not only considers the phase transformation caused by the difference of solute, but also considers the phase transformation caused by temperature change.

In [3] we have studied a phase-field model about martensitic phase transformations. In [4] we have studied a system of partial differential equations modeling the evolution of two phase boundary problems in sea-ice. In [5] we have investigated a system of partial differential equations modeling the evolution of three phase boundary problems

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in seawater-ice-snow and proved that in the case of one space dimension with an initial-boundary value problem to this system has global solutions.

In this paper we will prove the global solutions by the Galerkin method. And we conclude a maximum theorem.

According to the standard definition, the order parameter u in the phase transition problem represents the non-zero property of the system in a different region of the phase space and 0 otherwise. The model is a generalization of the one introduced by Steinbach et al. in [2], for isothermal solidification/melting process of certain kinds of alloys. Our model must satisfy the following system of partial differential equations

$$
\begin{aligned}
& u_t - k_1(w\Delta u - u\Delta w) - k_3(v\Delta u - u\Delta v) = -a_1 uw(w-u) \\
& \quad -a_3 uv(v-u) + \theta(l_1 uv + l_3 uw), && (1.1) \\
& v_t - k_2(w\Delta v - v\Delta w) - k_3(u\Delta v - v\Delta u) = -a_2 vw(w-v) \\
& \quad -a_3 vu(u-v) + \theta(-l_1 uv + l_2 vw), && (1.2) \\
& w_t - k_1(u\Delta w - w\Delta u) - k_2(v\Delta w - w\Delta v) = -a_1 wu(u-w) \\
& \quad -a_2 wv(v-w) + \theta(-l_3 uw - l_2 vw), && (1.3) \\
& \theta_t + (l_1 uv + l_3 uw)u_t + (-l_1 uv + l_2 vw)v_t + (-l_2 vw - l_3 uw)w_t = D\Delta\theta. && (1.4)
\end{aligned}
$$

for $(t,x) \in (0,T_e) \times \Omega$. The boundary and initial conditions are

$$
\begin{aligned}
u(t,x) = 0,\ v(t,x) = 0,\ w(t,x) = 0,\ (t,x) \in [0,T_e) \times \partial\Omega, \quad (1.5) \\
u(0,x) = u_0,\ v(0,x) = v_0(x),\ w(0,x) = w_0(x),\ x \in \Omega. \quad (1.6)
\end{aligned}
$$

The second law of thermodynamics holds for this system. Here, $\Omega \subset \mathbb{R}^3$ is a bounded open domain. The function θ is the temperature, and the phase-field functions u, v and w are the respective fractions of two different possible solid crystallization states and liquid state; thus, physically we must have $u+v+w=1$. For physical reasons, k_1, k_2, k_3, a_1, a_2, a_3, D are positive. In the free energy

$$
\begin{aligned}
F[u,v,w,\theta] = & \int_\Omega \Big\{ \frac{k_1}{2}|u\nabla w - w\nabla u|^2 + \frac{k_2}{2}|w\nabla v - v\nabla w|^2 \\
& + \frac{k_3}{2}|u\nabla v - v\nabla u|^2 + \widehat{\psi}(u,v,w) - \frac{1}{4}\theta\big(l_1(\frac{1}{3}u^3 + u^2 v - \frac{1}{3}v^3 - uv^2) \\
& + l_3(\frac{1}{3}u^3 + u^2 w - \frac{1}{3}w^3 - uw^2) + l_2(\frac{1}{3}v^3 + v^2 w - \frac{1}{3}w^3 - vw^2)\big)\Big\} dx,
\end{aligned}
$$

where

$$
\widehat{\psi}(u,v,w) = \frac{a_1}{2}u^2 w^2 + \frac{a_2}{2}v^2 w^2 + \frac{a_3}{2}u^2 v^2,
$$

we choose for $\widehat{\psi} \in C^2(\mathbb{R},[0,\infty))$ which represent the double well potential.

θ satisfies

$$
\begin{aligned}
\theta = & e - \frac{l_1}{4}(\frac{1}{3}u^3 + u^2 v - \frac{1}{3}v^3 - uv^2) - \frac{l_3}{4}(\frac{1}{3}u^3 + u^2 w - \frac{1}{3}w^3 - uw^2) \\
& - \frac{l_2}{4}(\frac{1}{3}v^3 + v^2 w - \frac{1}{3}w^3 - vw^2),
\end{aligned}
$$

where e is the local enthalpy, l_1, l_2, l_3 are the latent heat of fusion, respectively.

It is easy to see this in the case of dual phase systems, $u = 1 - v$, or $u = 1 - w$ or $v = 1 - w$, and $\frac{\partial u}{\partial v} = -1$, or $\frac{\partial u}{\partial w} = -1$, or $\frac{\partial v}{\partial w} = -1$.

We write $Q_T := (0, T_e) \times \Omega$, where T_e is a positive constant, and define

$$(v, \varphi)_{\mathbb{Z}} = \int_{\mathbb{Z}} v(y)\varphi(y)dy,$$

for $\mathbb{Z} = \Omega$ or $\mathbb{Z} = Q_{T_e}$.

Since we must have $u + v + w = 1$ and $u_t + v_t + w_t = 0$, the model can be reduced to

$$\begin{aligned}
&u_t - \big(k_1(1-v) + k_3 v\big)\Delta u - (k_1 - k_3)u\Delta v = -a_1 u(1-u-v)(1-2u-v) \\
&\quad -a_3 uv(v-u) + \theta\big(l_1 uv + l_3 u(1-u-v)\big), && (1.7)\\
&v_t - \big(k_2(1-u) + k_3 u\big)\Delta v - (k_2 - k_3)v\Delta u = -a_2 v(1-u-v)(1-u-2v) \\
&\quad -a_3 vu(u-v) + \theta\big(-l_1 uv + l_2 v(1-u-v)\big), && (1.8)\\
&\theta_t - D\Delta\theta = -\big(l_1 uv + l_2 v(1-u-v) + 2l_3 u(1-u-v)\big)u_t && (1.9)\\
&\quad -\big(-l_1 uv + 2l_2 v(1-u-v) + l_3 u(1-u-v)\big)v_t. && (1.10)
\end{aligned}$$

The boundary and initial conditions therefore are

$$u(t,x) = 0,\ v(t,x) = 0,\ \theta(t,x) = 0,\ (t,x) \in (0,T_e) \times \partial\Omega, \qquad (1.11)$$
$$u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ \theta(0,x) = \theta_0(x),\ x \in \Omega. \qquad (1.12)$$

Definition 1.1 *Let $(u_0, v_0, \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. A triple (u, v, θ) with*

$$u, v \in L^\infty(0,T_e; H_0^1(\Omega)) \cap L^2(0,T_e; H^2(\Omega)), \qquad (1.13)$$
$$\theta \in L^\infty(0,T_e; L^2(\Omega)) \cap L^2(0,T_e; H_0^1(\Omega)), \qquad (1.14)$$

is weak solution to the problem (1.7)-(1.10), *if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \Omega))$, there holds*

$$\begin{aligned}
0 \;=\;& (u, \varphi_t)_{Q_{T_e}} - \alpha(\nabla u, \nabla\varphi)_{Q_{T_e}} - \beta\big(\nabla v, \nabla(u\varphi)\big) - a_1\big(u(1-u-v)(1-2u-v), \varphi\big)_{Q_{T_e}} \\
& -a_3\big(uv(v-u), \varphi\big)_{Q_{T_e}} + (u_0, \varphi(0))_\Omega + \Big(\theta\big(l_1 uv + l_3 u(1-u-v)\big), \varphi\Big)_{Q_{T_e}}, \qquad (1.15)\\
0 \;=\;& (v, \varphi_t)_{Q_{T_e}} - \gamma(\nabla v, \nabla\varphi)_{Q_{T_e}} - \lambda\big(\nabla u, \nabla(v\varphi)\big) - a_2\big(v(1-u-v)(1-u-2v), \varphi\big)_{Q_{T_e}} \\
& -a_3\big(uv(u-v), \varphi\big)_{Q_{T_e}} + (v_0, \varphi(0))_\Omega + \Big(\theta\big(-l_1 uv + l_2 v(1-u-v)\big), \varphi\Big)_{Q_{T_e}}, \qquad (1.16)\\
0 \;=\;& (\theta, \varphi_t)_{Q_{T_e}} - (D\nabla\theta, \nabla\varphi)_{Q_{T_e}} + (\theta_0, \varphi(0))_\Omega - \Big(l_1(\frac{1}{3}u^3 + u^2 v - \frac{1}{3}v^3 - uv^2) \\
& +l_3(\frac{1}{3}u^3 + u^2(1-u-v) - \frac{1}{3}(1-u-v)^3 - u(1-u-v)^2) \\
& +l_2(\frac{1}{3}v^3 + v^2(1-u-v) - \frac{1}{3}(1-u-v)^3 - v(1-u-v)^2), \varphi_t\Big)_{Q_{T_e}} \\
& -\Big(l_1(\frac{1}{3}u_0^3 + u_0^2 v_0 - \frac{1}{3}v_0^3 - u_0 v_0^2) \\
& +l_3(\frac{1}{3}u_0^3 + u_0^2(1-u_0-v_0) - \frac{1}{3}(1-u_0-v_0)^3 - u_0(1-u_0-v_0)^2) \\
& +l_2(\frac{1}{3}v_0^3 + v_0^2(1-u_0-v_0) - \frac{1}{3}(1-u_0-v_0)^3 - v_0(1-u_0-v_0)^2), \varphi(0)\Big)_{Q_{T_e}}, \qquad (1.17)
\end{aligned}$$

where $\alpha = \big(k_1(1-v) + k_3 v\big) > 0$, $\beta = k_1 - k_3$, $\gamma = \big(k_2(1-u) + k_3 u\big) > 0$, $\lambda = k_2 - k_3$, $w_0 = 1 - u_0 - v_0$.

The main results of this article are as follows.

Theorem 1.1 *For all $(u_0, v_0, \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique weak solution (u, v, θ) of the problem* (1.7)-(1.12)*, which in addition to* (1.13)-(1.14) *satisfies*

$$u_t \in L^2(Q_{T_e}),\ u \in L^4(Q_{T_e}),\ v_t \in L^2(Q_{T_e}),\ v \in L^4(Q_{T_e}),\ \theta_t \in L^2(0, T_e; H^{-1}(\Omega)). \tag{1.18}$$

Theorem 1.2 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then the semigroup $S(t)$ associated with the system* (1.1)-(1.6) *possesses a maximal attractor $\mathcal{A}$ which is bounded in $H^2(\Omega) \times H_0^1(\Omega)$, compact and connected in $H_0^1(\Omega) \times L^2(\Omega)$, and attracts the bounded sets of $H_0^1(\Omega) \times L^2(\Omega)$.*

Notation. In the following sections, we employ the letter C to denote any positive constants that can be explicitly computed in terms of known quantities and may change from line to line. The $L^2(\Omega)$-norm is denoted by $\| \cdot \|$.

The outline of this paper is as follows. In section 2, we will prove the existence of solutions of the initial-boundary value problem for the nonlinear equations (1.7)-(1.12) by the Galerkin method.

In section 3 we shall show that a maximum theorem holds.

2 Existence of solutions

In this section, we construct the approximate solutions by the Galerkin method, and derive the *a priori* estimates, then we propose to send $m \to \infty$ and to show that a subsequence of our solutions u_m, v_m, θ_m converges to a weak solution of (1.7)-(1.12).

2.1 Construction of approximate solutions

Let $\{\omega_k\}_{k=1}^{\infty}$ be a basis in $H_0^1(\Omega)$. And ω_k is a solution to eigen-problem

$$\begin{cases} -\Delta\omega_k = \lambda_k \omega_k,\ in\ \Omega \\ \omega_k = 0, \quad k = 1, \cdots,\ on\ \partial\Omega. \end{cases}$$

For a positive integer m. We will look for approximate solutions u_m, v_m, e_m of the form

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t)\omega_k,\ v_m(t) = \sum_{k=1}^{m} g_m^k(t)\omega_k,\ e_m(t) = \sum_{k=1}^{m} h_m^k(t)\omega_k,\ (k = 1, 2, ..., m), \tag{2.1}$$

where we select the coefficients $d_m^k(t), g_m^k(t), h_m^k(t)$ so that

$$d_m^k(0) = (u_0, \omega_k) = \delta_m^k,\ g_m^k(0) = (v_0, \omega_k) = \eta_m^k,\ h_m^k(0) = (\theta_0, \omega_k) = \zeta_m^k, \tag{2.2}$$

and

$$
\begin{aligned}
&(u_{mt},\omega_j)-\alpha(\Delta u_m,\omega_j)-\beta(u_m\Delta v_m,\omega_j)= \\
&\quad -a_1\big(u_m(1-u_m-v_m)(1-2u_m-v_m),\omega_j\big)-a_3\big(u_mv_m(v_m-u_m),\omega_j\big) \\
&\quad +\Big(\big(e_m-\frac{l_1}{4}(\frac{1}{3}u_m^3+u_m^2v_m-\frac{1}{3}v_m^3-u_mv_m^2) \\
&\quad -\frac{l_3}{4}(\frac{1}{3}u_m^3+u_m^2w_m-\frac{1}{3}w_m^3-u_mw_m^2) \\
&\quad -\frac{l_2}{4}(\frac{1}{3}v_m^3+v_m^2w_m-\frac{1}{3}w_m^3-v_mw_m^2)\big)\big(l_1u_mv_m+l_3u_mw_m\big),\omega_j\Big), \qquad (2.3) \\
&(v_{mt},\omega_j)-\gamma(\Delta v_m,\omega_j)-\lambda(v_m\Delta u_m,\omega_j)= \\
&\quad -a_2\big(v_m(1-u_m-v_m)(1-u_m-2v_m),\omega_j\big)-a_3\big(v_mu_m(u_m-v_m),\omega_j\big) \\
&\quad +\Big(\big(e_m-\frac{l_1}{4}(\frac{1}{3}u_m^3+u_m^2v_m-\frac{1}{3}v_m^3-u_mv_m^2) \\
&\quad -\frac{l_3}{4}(\frac{1}{3}u_m^3+u_m^2w_m-\frac{1}{3}w_m^3-u_mw_m^2) \\
&\quad -\frac{l_2}{4}(\frac{1}{3}v_m^3+v_m^2w_m-\frac{1}{3}w_m^3-v_mw_m^2)\big)\big(-l_1u_mv_m+l_2v_mw_m\big),\omega_j\Big), \qquad (2.4) \\
&(e_{mt},\omega_j)=D\Delta\big(\theta_m-\frac{l_1}{4}(\frac{1}{3}u_m^3+u_m^2v_m-\frac{1}{3}v_m^3-u_mv_m^2) \\
&\quad -\frac{l_3}{4}(\frac{1}{3}u_m^3+u_m^2w_m-\frac{1}{3}w_m^3-u_mw_m^2) \\
&\quad -\frac{l_2}{4}(\frac{1}{3}v_m^3+v_m^2w_m-\frac{1}{3}w_m^3-v_mw_m^2),\omega_j\big),\ (j=1,\dots,m). \qquad (2.5)
\end{aligned}
$$

where $w_m=1-u_m-v_m$. (2.3)-(2.5) are a system of nonlinear ordinary differential equations whose the nonlinear terms are local Lipschitz continuous.

Assuming u_m,v_m,e_m have the structure (2.1), we note that

$$(u_m(t),\omega_j)=(\sum_{k=1}^m {d_m^k}'(t)\omega_k,\omega_j)=\int_\Omega\sum_{k=1}^m {d_m^k}'(t)\omega_k\omega_j dx=\sum_{k=1}^m {d_m^k}'(t)\int_\Omega \omega_k\omega_j dx, \quad (2.6)$$

$$(v_m(t),\omega_j)=(\sum_{k=1}^m {g_m^k}'(t)\omega_k,\omega_j)=\int_\Omega\sum_{k=1}^m {g_m^k}'(t)\omega_k\omega_j dx=\sum_{k=1}^m {g_m^k}'(t)\int_\Omega \omega_k\omega_j dx, \quad (2.7)$$

$$(e_m(t),\omega_j)=(\sum_{k=1}^m {h_m^k}'(t)\omega_k,\omega_j)=\int_\Omega\sum_{k=1}^m {h_m^k}'(t)\omega_k\omega_j dx=\sum_{k=1}^m {h_m^k}'(t)\int_\Omega \omega_k\omega_j dx, \quad (2.8)$$

We deal with other terms in the same way.

We introduce the vectors

$$D_m=D_m(t)=\begin{pmatrix} d_m^1(t)\\ \vdots\\ d_m^m(t)\end{pmatrix}, G_m=G_m(t)=\begin{pmatrix} g_m^1(t)\\ \vdots\\ g_m^m(t)\end{pmatrix}, H_m=H_m(t)=\begin{pmatrix} h_m^1(t)\\ \vdots\\ h_m^m(t)\end{pmatrix}$$

and $F_1(D_m^T,G_m^T,H_m^T), F_2(D_m^T,G_m^T,H_m^T), F_3(D_m^T,G_m^T,H_m^T)$ denote the nonlinear terms. Thus, we obtain a system of ordinal differential equations.

$$
\begin{aligned}
&BD_m'+F_1(D_m^T,G_m^T,H_m^T)=0, && (2.9)\\
&BG_m'+F_2(D_m^T,G_m^T,H_m^T)=0, && (2.10)\\
&BH_m'+F_3(D_m^T,G_m^T,H_m^T)=0, && (2.11)
\end{aligned}
$$

where

$$B=(b_{ij})=\begin{bmatrix}(\omega_1,\omega_1) & \dots & (\omega_1,\omega_m)\\ \dots & \dots & \dots\\ (\omega_m,\omega_1) & \dots & (\omega_m,\omega_m)\end{bmatrix}=\int\begin{bmatrix}\omega_1\\ \vdots\\ \omega_m\end{bmatrix}\begin{bmatrix}\omega_1 & \dots & \omega_m\end{bmatrix}dx. \tag{2.12}$$

Then, if we choose a vector $\overrightarrow{X}=(x_1,\dots,x_m)$ that is not equal to zero, there hold for $(\overrightarrow{X})^T B\overrightarrow{X}=\int_\Omega (x_1\omega_1+\dots+x_m\omega_m)^2dx>0$, otherwise, invoking that $\omega_1,\dots,\omega_m$ are linearly independent, we obtain $\overrightarrow{X}=0$. B is a positive-definite matric.

For the initial date, we make a smooth approximation

$$u_{0m}=\sum_{k=1}^m \delta_m^k\omega_k\to u_0,\ \textit{strongly in } H_0^1(\Omega), \tag{2.13}$$

$$v_{0m}=\sum_{k=1}^m \eta_m^k\omega_k\to v_0,\ \textit{strongly in } H_0^1(\Omega), \tag{2.14}$$

$$e_m=\sum_{k=1}^m \zeta_m^k\omega_k\to e_0,\ \textit{strongly in } H_0^1(\Omega). \tag{2.15}$$

According to the existence theorem of local solutions to ordinary differential equations, there exists the solutions for a.e. $0\le t\le t_m$. We extend $t_m=T_e$. Then, we obtain the global solutions. But we need show that the *a priori* estimates holds.

2.2 *A priori* estimates

Theorem 2.1 *There exists a constant C, depending on Ω, T_e, such that*

$$\begin{aligned}&\|u_m\|^2_{H_0^1(\Omega)}+\|v_m\|^2_{H_0^1(\Omega)}+\|e_m\|^2+\int_0^{T_e}\|u_m\|^4_{L^4(\Omega)}d\tau+\int_0^{T_e}\|v_m\|^4_{L^4(\Omega)}d\tau\\ &+\int_0^{T_e}(\|u_m\|^2_{H^2(\Omega)}+\|v_m\|^2_{H^2(\Omega)}+\|e_m\|^2_{H_0^1(\Omega)})d\tau\le C.\end{aligned} \tag{2.16}$$

for $m=1,2,\dots$ where $\|u_m\|_{L^\infty(\Omega)},\|v_m\|_{L^\infty(\Omega)},\|w_m\|_{L^\infty(\Omega)}$ are suitably small.

Proof. Multiplying (2.3)-(2.5) by $d_m^k(t), g_m^k(t), h_m^k(t)$ respectively, summing up over $j=1,\dots,m$, integrating by parts with respect to x over Ω, adding and recalling (2.1) to find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u_m\|^2+\|v_m\|^2+\frac{1}{D}\|e_m\|^2)+\frac{1}{2\gamma}(\gamma\alpha-\lambda^2)\|\nabla u_m\|^2\\ &+\frac{1}{2\alpha}(\gamma\alpha-\beta^2)\|\nabla v_m\|^2+\frac{1}{2}\|\nabla e_m\|^2\\ &+\frac{1}{64}(128a_1-33|3a_1-a_3|-|3a_2-a_3|-64\epsilon)\|u_m\|^4_{L^4(\Omega)}\\ &+\frac{1}{64}(128a_2-33|3a_2-a_3|-|3a_1-a_3|-64\epsilon)\|v_m\|^4_{L^4(\Omega)}\\ &+\frac{1}{32}(32a_1+64a_3+32a_2-15|3a_1-a_3|-15|3a_2-a_3|)\|u_mv_m\|^2\\ &\le C(\|u_m\|^2+\|v\|^2+\frac{1}{D}\|e_m\|^2)+C(\Omega),\end{aligned} \tag{2.17}$$

where $\alpha\gamma - \lambda^2 > 0, \alpha\gamma - \beta^2, 128a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| - 64\epsilon > 0$, $128a_2 - 33|3a_2 - a_3| - |3a_1 - a_3| - 64\epsilon > 0, 32a_1 + 64a_3 + 32a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3| > 0$. Using the Gronwall inequality, we obtain

$$
\begin{aligned}
&(\|u_m\|^2 + \|v_m\|^2 + \frac{1}{D}\|e_m\|^2) + \frac{1}{2\gamma}(\gamma\alpha - \lambda^2)\int_0^{T_e} \|\nabla u_m\|^2 d\tau \\
&+\frac{1}{2\alpha}(\gamma\alpha - \beta^2)\int_0^{T_e} \|\nabla v_m\|^2 d\tau + \frac{1}{2}\int_0^{T_e} \|\nabla e_m\|^2 d\tau \\
&+\frac{1}{64}(128a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| - 64\epsilon)\int_0^{T_e} \|u_m\|^4_{L^4(\Omega)} d\tau \\
&+\frac{1}{64}(128a_2 - 33|3a_2 - a_3| - |3a_1 - a_3| - 64\epsilon)\int_0^{T_e} \|v_m\|^4_{L^4(\Omega)} d\tau \\
&+\frac{1}{32}(32a_1 + 64a_3 + 32a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|)\int_0^{T_e} \|u_m v_m\|^2 d\tau \\
&\leq C(T_e). \qquad (2.18)
\end{aligned}
$$

Multiplying (2.3)-(2.4) by $\lambda_j d_m^k(t), \lambda_j g_m^k(t)$ respectively, summing up over $j = 1, \ldots, m$. Formally, integrating by parts with respect to x over Ω and adding them, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u_m\|^2 + \|\nabla v_m\|^2) + \frac{1}{2\gamma}(\alpha\gamma - \lambda^2 - \gamma\epsilon)\|\Delta u_m\|^2 \\
&+\frac{1}{2\alpha}(\gamma\alpha - \beta^2 - \alpha\epsilon)\|\Delta v_m\|^2 \\
&+\frac{1}{2}(6a_1 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_2|^2)\|u_m \nabla u_m\|^2 \\
&+\frac{1}{2}(2a_1 + 2a_3 - |3a_2 - a_3| - |5a_1 - 3a_3 - a_2|^2)\|v_m \nabla u_m\|^2 \\
&+\frac{1}{2}(6a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2)\|v_m \nabla v_m\|^2 \\
&+\frac{1}{2}(2a_2 + 2a_3 - |3a_1 - a_3| - |5a_2 - 3a_3 - a_1|^2)\|u_m \nabla v_m\|^2 \\
&\leq C(\|\nabla u_m\|^2 + \|\nabla v_m\|^2 + \|\nabla e_m\|^2 + 1), \qquad (2.19)
\end{aligned}
$$

where $\alpha\gamma - \lambda^2 - \gamma\epsilon > 0$, $\gamma\alpha - \beta^2 - \alpha\epsilon > 0$, $6a_1 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_2|^2 > 0$, $2a_1 + 2a_3 - |3a_2 - a_3| - |5a_1 - 3a_3 - a_2|^2 > 0$, $2a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2 > 0$, $2a_2 + 2a_3 - |3a_1 - a_3| - |5a_2 - 3a_3 - a_1|^2 > 0$.
Using the gronwall inequality to yield

$$
\begin{aligned}
&\|\nabla u_m\|^2 + \|\nabla v_m\|^2 + \frac{1}{2\gamma}(\alpha\gamma - \lambda^2 - \gamma\epsilon)\int_0^{T_e} \|\Delta u_m\|^2 d\tau \\
&+\frac{1}{2\alpha}(\gamma\alpha - \beta^2 - \alpha\epsilon)\int_0^{T_e} \|\Delta v_m\|^2 d\tau \\
&\leq C(T_e). \qquad (2.20)
\end{aligned}
$$

It follows from (2.18) and (2.20) that

$$
\|u_{mt}\|_{L^2(Q_{T_e}) + L^{\frac{4}{3}}(Q_{T_e})} + \|v_{mt}\|_{L^2(Q_{T_e}) + L^{\frac{4}{3}}(Q_{T_e})} + \|e_{mt}\|_{L^2(0,T_e;H^{-1}(\Omega))} \leq C(T_e). \qquad (2.21)
$$

2.3 Existence of weak solutions

Next we pass to limits as $m \to \infty$, to build weak solutions of our initial-boundary value problem (1.7)-(1.12).

Theorem 2.2 *(Aubin-Lions) Let B_0 be a normed linear space imbedded compactly into another normed linear space B, which is continuously imbedded into a Hausdorff locally convex space B_1, and $1 \le p < +\infty$. If $v, v_i \in L^p(0,t;B_0), i \in \mathbb{N}$, the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to v in $L^p(0,t;B_0)$, and $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0,t;B_1)$, then v_i converges to v strongly in $L^p(0,t;B)$.*

Lemma 2.1 *Let $(0,t)\times\Omega$ be an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose functions g_n, g are in $L^q((0,t)\times\Omega$ for any given $1 < q < \infty$, which satisfy*

$$\|g_n\|_{L^q((0,t)\times\Omega)} \le C, \quad g_n \to g \quad a.e. \ in \ (0,t)\times\Omega.$$

Then g_n converges to g weakly in $L^q((0,t)\times\Omega)$.

Theorem 2.2 is a general version of the Aubin-Lions lemma valid under the weak assumption $\partial_t v_i \in L^1(0,t;B_1)$. This version, which we need here, is proved in [1]. A proof of Lemma 2.1 can be found in [1].

Lemma 2.2 *Problems (1.7)-(1.12) has at least one weak solution (u,v,e) in the sense of definition 1.1. Each of the weak solutions satisfies: $e_t \in L^2(0,T_e;H^{-1}(\Omega))$ and $(u_t,v_t) \in L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e})$.*

Proof of Lemma 2.2 According to the energy estimates (2.16), we see sequence $\{u_m\}_{m=1}^{\infty}$, $\{v_m\}_{m=1}^{\infty}, \{e_m\}_{m=1}^{\infty}$ are bounded in $L^\infty(0,T_e;H_0^1(\Omega)) \cap L^2(0,T_e;H^2(\Omega))$, $L^\infty(0,T_e;L^2(\Omega)) \cap L^2(0,T_e;H_0^1(\Omega))$, respectively, and $(\{u_{mt}\}_{m=1}^{\infty}, \{v_{mt}\}_{m=1}^{\infty}), \{e_{mt}\}_{m=1}^{\infty}$ are bounded in $L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e}), L^2(0,T_e;H^{-1}(\Omega))$, respectively.

Consequently there exists subsequences $\{u_{m_l}\}_{l=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$, $\{v_{m_l}\}_{l=1}^{\infty} \subset \{v_m\}_{m=1}^{\infty}$, $\{e_{m_l}\}_{l=1}^{\infty} \subset \{e_m\}_{m=1}^{\infty}$ and functions $(u,v) \in L^\infty(0,T_e;H_0^1(\Omega)) \cap L^2(0,T_e;H^2(\Omega))$, $e \in L^2(0,T_e;H_0^1(\Omega)) \cap L^\infty(0,T_e;L^2(\Omega))$, with $(u_t,v_t) \in L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e})$, $e_t \in L^2(0,T_e;H^{-1}(\Omega))$, such that

$$\begin{cases} u_{m_l} \rightharpoonup u, \ weakly \ in \ L^2(0,T_e;H^2(\Omega)) \\ u_{m_l} \overset{*}{\rightharpoonup} u, \ weakly \ in \ L^\infty(0,T_e;H_0^1(\Omega)) \\ u_{m_l t} \rightharpoonup u_t, \ weakly \ in \ L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e}) \\ v_{m_l} \rightharpoonup v, \ weakly \ in \ L^2(0,T_e;H^2(\Omega)) \\ v_{m_l} \overset{*}{\rightharpoonup} v, \ weakly \ in \ L^\infty(0,T_e;H_0^1(\Omega)) \\ v_{m_l t} \rightharpoonup v_t, \ weakly \ in \ L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e}). \\ e_{m_l} \overset{*}{\rightharpoonup} e, \ weakly \ in \ L^\infty(0,T_e;L^2(\Omega)) \\ e_{m_l} \rightharpoonup e, \ weakly \ in \ L^2(0,T_e;H_0^1(\Omega)) \\ e_{m_l t} \rightharpoonup e_t, \ weakly \ in \ L^2(0,T_e;H^{-1}(\Omega)) \end{cases}$$

There exists functions ξ, ϕ such that

$$u_m^3 \rightharpoonup \xi, \ weakly \ in \ L^{\frac{4}{3}}(Q_{T_e}) \tag{2.22}$$

$$v_m^3 \rightharpoonup \phi, \ weakly \ in \ L^{\frac{4}{3}}(Q_{T_e}). \tag{2.23}$$

It remains to show that $\xi = u^3, \phi = v^3$. To this end, we show first that $u_m \to u, v_m \to v$ in $L^2(0,T_e;H^1(\Omega))$ by applying Theorem 2.2. To apply this theorem, u_t, v_t have estimtes $u_t \in L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e}) \subset L^{\frac{4}{3}}(Q_{T_e}) \subset L^1(0,T_e;L^{\frac{4}{3}}(\Omega))$, $v_t \in L^2(0,T_e;L^2(\Omega)) + L^{\frac{4}{3}}(Q_{T_e}) \subset L^{\frac{4}{3}}(Q_{T_e}) \subset L^1(0,T_e;L^{\frac{4}{3}}(\Omega))$, $e_t \in L^2(0,T_e;H^{-1}(\Omega))$.

Applying Theorem 2.2 with $B_0 = H^2(\Omega), B = H^1(\Omega), B_1 = L^{\frac{4}{3}}(\Omega)$, and $p = 2$ we obtain

$$\begin{aligned} u_m &\to u, \ \textit{strongly in } L^2(0,T_e;H^1(\Omega)) \\ v_m &\to v, \ \textit{strongly in } L^2(0,T_e;H^1(\Omega)). \end{aligned}$$

Consequently, from these sequences we select the other sequences, denoted in the same way, which converges almost everywhere in Q_{T_e}. This implies the convergence $u_m^3 \to u^3, v_m^3 \to v^3$ almost everywhere in Q_{T_e}. Using the embedding $H^1 \Subset L^4(\Omega)$ and applying the Lemma 2.1, we obtain $\xi = u^3, \phi = v^3$. Then, we obtain the others in the same way. Equation (1.15) follows from these relations if we show that

$$(u_{0m}, \varphi(0))_\Omega \to (u_0, \varphi(0)), \tag{2.24}$$

$$(u_m, \varphi_t)_{Q_{T_e}} \to (u, \varphi_t)_{Q_{T_e}}, \tag{2.25}$$

$$(\nabla u_m, \nabla \varphi)_{Q_{T_e}} \to (\nabla u, \nabla \varphi)_{Q_{T_e}}, \tag{2.26}$$

$$(\nabla v_m, \nabla(u_m \varphi))_{Q_{T_e}} \to (\nabla v_m, \nabla(u\varphi))_{Q_{T_e}}, \tag{2.27}$$

$$\big(u_m v_m (v_m - u_m), \varphi\big)_{Q_{T_e}} \to \big(uv(v-u), \varphi)_{Q_{T_e}}, \tag{2.28}$$

$$(\theta_m u_m v_m, \varphi)_{Q_{T_e}} \to (\theta uv, \varphi)_{Q_{T_e}}. \tag{2.29}$$

for $m \to \infty$. Now, the relation (2.24) follows from (2.13), the relation (2.25) is a consequence of $u_{mt} \in L^{\frac{4}{3}}(Q_{T_e})$, the relation (2.26) is consequence of $u_m \in L^2(0,T_e;H^2(\Omega))$, the relation (2.27) is a consequence of $u_m \to u, \nabla u_m \to \nabla u$, $\nabla v_m \to \nabla v$ in $L^2(Q_{T_e})$, the relation (2.28) is consequence of $u_m v_m \in L^2(Q_{T_e})$ and $u_m \to u, v_m \to v$ in $L^2(Q_{T_e})$, and the relation (2.29) is obtained from $u_m v_m \in L^2(Q_{T_e})$ and $\theta_m \to \theta$ in $L^2(Q_{T_e})$. We obtain other terms in the same way as above.

2.4 Uniqueness

In this subsection we show uniqueness of the solution of (u, v, e) that it obtained in the Subsections 2.2 and 2.3.

Theorem 2.3 *Let* $(u_{0i}, v_{0i}) \in H_0^1(\Omega), e_{0i} \in L^2(\Omega)$, $i = 1, 2$ *be given functions. Let* (u_i, v_i, e_i) *be weak solutions of problem* (1.7)-(1.12) *with* $(u_i, v_i) \in L^\infty(0,T_e;H_0^1(\Omega)) \cap L^2(0,T_e;H^2(\Omega))$, *and* $e_i \in L^2(0,T : H_0^1(\Omega)) \cap L^\infty(0,T_e;L^2(\Omega))$, $i = 1, 2$. *Then*

$$\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + \|\widetilde{e}\|^2 \le C(1 + T_e e^{C_1 T_e})(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2 + \|\widetilde{e}_0\|^2). \tag{2.30}$$

where $\widetilde{u} = u_1 - u_2, \widetilde{v} = v_1 - v_2$ *and* $\widetilde{e} = e_1 - e_2$. *The constant* C *is independent of* $u_i, v_i, e_i, u_{0i}, v_{0i}, e_{0i}$.

Proof. Multiplying the difference of (1.7) for u_1 and u_2 by $\widetilde{u}$. We obtain the others in the same way. Then $\widetilde{u}, \widetilde{v}$ and $\widetilde{e}$ satisfy the inequality

$$\begin{aligned}
&\frac{1}{2}(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + \frac{1}{D}\|\widetilde{e}\|^2) + \frac{1}{2\gamma}(\gamma\alpha - \lambda^2)\|\nabla\widetilde{u}\|^2 + \dots \\
&+\frac{1}{2\alpha}(\gamma\alpha - \beta^2)\|\nabla\widetilde{v}\|^2 + \frac{1}{4}\|\nabla\widetilde{e}\|^2 \\
&\leq C\{(\|\widetilde{u}_0\|^2 + \|\widetilde{v}_0\|^2 + \|\widetilde{e}_0\|^2) + \int_0^t \int_\Omega (\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + \|\widetilde{e}\|^2)\}. \qquad (2.31)
\end{aligned}$$

Using the Gronwall inequality yields

$$\|\overline{u}\|^2 + \|\overline{v}\|^2 + \|\overline{e}\|^2 \leq C(1 + T_e e^{C_1 T_e})(\|\overline{u}_0\|^2 + \|\overline{v}_0\|^2 + \|\overline{e}_0\|^2). \qquad (2.32)$$

In particular, for $\overline{u}_0 = \overline{v}_0 = \overline{e}_0 = 0$, we obtain the uniqueness of the solution.

3 Absorbing set

In what follows, we prove the existence of an absorbing set for u, v, θ. Multiplying (1.1)-(1.4) by u_t, v_t, w_t, e respectively, adding and integrating yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big\{\frac{k_1}{2}|u\nabla w - w\nabla u|^2 + \frac{k_2}{2}|w\nabla v - v\nabla w|^2 + \frac{k_3}{2}|u\nabla v - v\nabla u|^2 \\
&+\widehat{\psi}(u,v,w) + \frac{1}{2}\|e\|^2\Big\}dx + \|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + D\|\nabla e\|^2 = 0. \qquad (3.1)
\end{aligned}$$

Let

$$\begin{aligned}
V(t) = &\int_\Omega \Big\{\frac{k_1}{2}|u\nabla w - w\nabla u|^2 + \frac{k_2}{2}|w\nabla v - v\nabla w|^2 \\
&+\frac{k_3}{2}|u\nabla v - v\nabla u|^2 + \widehat{\psi}(u,v,w) + \frac{1}{2}\|e\|^2\Big\}dx. \qquad (3.2)
\end{aligned}$$

From this we get $\|\nabla u\|^2 + \|\nabla v\|^2 + \|e\|^2 \leq C$. Therefore, we only need to prove that $\limsup_{t\to\infty} V(t) \leq C$.

Multiplying (1.1)-(1.3) by u, v, w respectively, adding and integrating yields

$$\begin{aligned}
&\int_\Omega \Big\{\frac{k_1}{2}|u\nabla w - w\nabla u|^2 + \frac{k_2}{2}|w\nabla v - v\nabla w|^2 \\
&+\frac{k_3}{2}|u\nabla v - v\nabla u|^2 + \widehat{\psi}(u,v,w) - \epsilon\|e\|^2\Big\}dx \\
&\leq \|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + C, \qquad (3.3)
\end{aligned}$$

where we have used $\|u\|_{L^\infty(\Omega)} < \epsilon, \|v\|_{L^\infty(\Omega)} < \epsilon, \|w\|_{L^\infty(\Omega)} < \epsilon$. Choosing $\frac{\alpha\gamma-\lambda^2}{\gamma} = k_3, \frac{\alpha\gamma-\beta^2}{\alpha} = k_2$. By Poincaré's inequality, we have

$$\|e\|^2 \leq C\|\nabla e\|^2, \qquad (3.4)$$

with $C > 0$ depending only on the domain Ω. Choosing $\frac{D}{C} - \epsilon = \frac{1}{2}$, adding with (3.1) yields

$$\frac{dV(t)}{dt} + V(t) \leq C'. \qquad (3.5)$$

It follows from (3.5) that

$$V(t) \leq e^{-t}V(0) + C'. \tag{3.6}$$

Notice that

$$\begin{aligned}V(0) = \int_\Omega \Big\{ & \frac{k_1}{2}|u_0\nabla w_0 - w_0\nabla u_0|^2 + \frac{k_2}{2}|w_0\nabla v_0 - v_0\nabla w_0|^2 \\ & + \frac{k_3}{2}|u_0\nabla v_0 - v_0\nabla u_0|^2 + \widehat{\psi}(u_0, v_0, w_0) + \frac{1}{2}\|e_0\|^2 \Big\} dx. \end{aligned} \tag{3.7}$$

is bounded if $|u_0\nabla w_0 - w_0\nabla u_0|^2 + |w_0\nabla v_0 - v_0\nabla w_0|^2 + |u_0\nabla v_0 - v_0\nabla u_0|^2 + \|e_0\|^2$ is bounded. The inequality (3.6) implies the existence of an absorbing set.

Proof of Theorem 1.2. The semigroup $S(t)$ is defined

$$S(t): \ (u_0, v_0, \theta_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \big(u(t,\cdot), v(t,\cdot), \theta(t,\cdot)\big). \tag{3.8}$$

In Theorem 1.1 $(u_t, v_t) \in L^2(Q_{T_e}), \theta \in L^2(0, T_e; H^1(\Omega))$ are proved, which implies that $S(t)$ is continuous in $H_0^1(\Omega) \times L^2(\Omega)$. Theorem 1.1 claims that any bounded set $B \subset H_0^1(\Omega) \times L^2(\Omega)$, $\cup\{S(t)B: \ t > 0\}$ is relatively compact in $H_0^1(\Omega) \times L^2(\Omega)$. The existence of an absorbing set has been proved in the above.

Theorem 3.1 *(uniqueness) Assume the u, v and θ are the weak solution of* (1.7)-(1.12) *for $(t, x) \in (0, T_e) \times \Omega$. Then the weak solution is unique.*

Proof. If u_1, v_1, w_1, θ_1 and u_2, v_2, w_2, θ_2 are two solutions, write $\widetilde{u} = u_1 - u_2, \widetilde{v} = v_1 - v_2, \widetilde{w} = w_1 - w_2, \widetilde{\theta} = \theta_1 - \theta_2$. We replace in (1.15)-(1.17) by $\varphi = u_1 - u_2, \varphi = v_1 - v_2, \varphi = \theta_1 - \theta_2$ and integrating by parts in Ω, using Young's inequality. We find

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2) + (\alpha - \epsilon)\|\widetilde{u}_x\|^2 + (\gamma - \epsilon)\|\widetilde{v_x}\|^2 \\ & \leq C(\|\widetilde{u}\|^2 + \|\widetilde{v}\|^2)(1 + \|\theta_x\|^2 + \|u_{1xx}\|^2 + \|v_{1xx}\|^2), \end{aligned} \tag{3.9}$$

$$\frac{d}{dt}\|\widetilde{\theta}\| + \|\widetilde{\theta}_x\|^2 \leq 0. \tag{3.10}$$

Using Gronwall's inequality, we thus conclude $u_1 = u_2$, $v_1 = v_2$, $\theta_1 = \theta_2$ for almost everywhere Q_{T_e}.

4 A maximum theorem

If we want that the model is physically meaningful it is necessary to show that the order parameter u, v, w remain confined within the interval $[0, 1]$ during their evolution. We will study a maximum theorem.

Theorem 4.1 *Let u, v, w, θ be solution of system* (1.7)-(1.12) *and $u_0 \in [0, 1], v_0 \in [0, 1], w_0 \in [0, 1]$ for each $x \in \Omega$. Then such solution satisfies $u \in [0, 1], v \in [0, 1], w \in [0, 1]$ and $u + v + w = 1$ for each $x \in \Omega$ and $t \in \mathbb{R}^+$.*

Proof. We only analyze the equation (1.7). (1.8) is a similar way. To prove that $u(t,x) \geq 0$, let us define

$$u^- = \begin{cases} -u, & \text{if } u < 0, \\ 0, & \text{if } u \geq 0, \end{cases}$$

so that $u^- \geq 0$ and satisfies boundary and initial conditions

$$\begin{aligned} u^-(t,x) &= 0,\ t \geq 0,\ x \in \partial\Omega, \\ u^-(0,x) &= 0,\ x \in \Omega. \end{aligned}$$

Multiplying (1.7) by $-u^-$ and integrating in Ω to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (u^-)^2 dx + \frac{\alpha}{2}\int_\Omega (\nabla u^-)^2 dx \leq C\int_\Omega (\nabla v)^2 (u^-)^2 dx + C\int_\Omega \theta (u^-)^2 dx \\ -a_1 \int_\Omega (u^-)^2 (1-u^- - v)(1-2u^- - v)dx - a_3 \int_\Omega (u^-)^2 v(v-u^-)dx. \end{aligned} \tag{4.1}$$

Using Gronwall' s lemma, we obtain that $(u^-)^2 = 0$ a.e. in Ω for all $t \in (0, T_e)$, and thus $u \geq 0$ a.e. in Q_{T_e}. We obtain in a similar way as above that $v \geq 0$, $w \geq 0$ a.e. in Q_{T_e}. Using a similar arguments to prove that $u(t,x) \leq 1, v(t,x) \leq 1, w(t,x) \leq 1$, define

$$u^+ = \begin{cases} u-1, & \text{if } u > 1, \\ 0, & \text{if } u \leq 1, \end{cases}$$

$$v^+ = \begin{cases} v-1, & \text{if } v > 1, \\ 0, & \text{if } v \leq 1, \end{cases}$$

$$w^+ = \begin{cases} w-1, & \text{if } w > 1, \\ 0, & \text{if } w \leq 1, \end{cases}$$

so that $u^+(t,x) \geq 0$ and satisfies boundary and initial conditions

$$\begin{aligned} u^+(t,x) = 0,\ v^+(t,x) = 0,\ w^+(t,x) = 0,\ t \geq 0,\ x \in \partial\Omega, \\ u^+(0,x) = 0,\ v^+(t,x) = 0,\ w^+(t,x) = 0,\ x \in \Omega. \end{aligned}$$

Multiplying (1.1), (1.2), (1.3) by u^+, v^+, w^+ and integrating in Ω with adding the result to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^+(1+u^+)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega v^+(1+v^+)dx + \frac{1}{2}\frac{d}{dt}\int_\Omega w^+(1+w^+)dx + \dots \\
&+\alpha\int_\Omega (\nabla u^+)^2dx + \gamma\int_\Omega(\nabla v^+)^2dx + \big(k_1u + k_2(1-u)\big)\int_\Omega(\nabla w^+)^2dx \\
&\le |\beta|\int_\Omega \Delta v u^+(1+u^+)dx + C\int_\Omega \Delta u v^+(1+v^+)dx + C\int_\Omega \Delta u w^+(1+w^+)dx \\
&-a_1\int_\Omega u^+(1+u^+)(-1-u^+-v^+)(-2-2u^+-v^+)dx + C\int_\Omega \theta u^+(1+u^+)dx \\
&-a_3\Big(\int_\Omega u^+(1+u^+)(1+v^+)(v^+-u^+)dx + C\int_\Omega \theta v^+(1+v^+)dx \\
&+\int_\Omega v^+(1+v^+)(1+u^+)(u^+-v^+)dx\Big) + C\int_\Omega w^+(1+w^+)\theta dx \\
&-a_2\int_\Omega v^+(1+v^+)(-1-u^+-v^+)(-2-2v^+-u^+)dx \\
&-a_1\int_\Omega w^+(1+w^+)(u^+-w^+)dx - a_2\int_\Omega w^+(1+w^+)(1+v^+)(v^+-w^+)dx
\end{aligned} \tag{4.2}$$

Using Gronwall's lemma, we obtain that $u^+ = 0$, $v^+ = 0$, $w^+ = 0$ a.e. in Ω for all $t \in (0, T_e)$, and thus $u \le 1$, $v \le 1$, $w \le 1$ a.e. in Q_{T_e}.

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