BLOW UP OF NONLINEAR MULTI-TIME FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the well-posedness of nonlinear multi-time fractional differential equations and show that the solutions of the system will blow up in finite time under certain assumptions. In particular, we apply the results to the nonlinear time fractional Burgers equations.

1. Introduction

Fractional calculus has attracted lots of attention in the recent years. Due to the growing applications in various fields such as fluid mechanics, biomathematics, finance and electrochemistry, etc, the properties of linear time-space fractional differential equations have been extensively studied. We refer the readers to [2] [3], [8], [14], [17], [20, 21, 22, 23] and references therein. Moreover, the nonlinear time fractional differential equations have also attracted some attention and properties such as the global existences, blow-up in finite time have been established under some conditions.

To be more precise, we recall some known results. In [4, 5], Fujita considered the nonlinear heat equation in $\mathbb{R}^n$

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u, \\ u(0,x) = \varphi(x). \end{cases}$$

He also proved that the solution will blow-up in finite time if $1 < p < 1 + \frac{2}{n}$. In [12], the authors generalized the results to the time fractional differential systems. In precious, they considered the following nonlinear nonlocal systems

$$\begin{cases} u_t + D^{\alpha}_{0^+}(u - u_0) = |v|^p, \\ v_t + D^{\alpha}_{0^+}(v - v_0) = |u|^p, \\ u(0) = u_0, v(0) = v_0, \end{cases}$$

where $D^{\alpha}_{0^+}$ is the left-handed Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$. Moreover, in [6, 7] the well-posedness of the nonlinear fractional differential equations with nonlinearities of form $\int_0^t (t-s)^{-\gamma}|u(s)|^{p-1}u(s)ds$ have been studied. For more results, see for example [1], [9]-[13].

Motivated by the interesting results in [10], [15], we concern the nonlinear multi-time fractional differential equations as follows

$$c_1 D^{\alpha_1}_{0^+,x_1}(|u|^{m_1} - \varphi_1^{m_1}) + \cdots + c_n D^{\alpha_n}_{0^+,x_n}(|u|^{m_n} - \varphi_n^{m_n}) = |u|^p \tag{1.1}$$

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with initial conditions

$$u(0,x_2,\cdots,x_n)=\varphi_1(x_2,\cdots,x_n),\cdots,u(x_1,\cdots,x_{n-1},0)=\varphi_n(x_1,\cdots,x_{n-1}).$$

Note that $D^{\alpha_1}_{0^+,x_1}$ is the left-handed Riemann-Liouville fractional derivatives with respect to variable x_i, $0<\alpha_n<\alpha_{n-1}<\cdots<\alpha_1\le 1, p>1, m_i>0$ and c_i are constants for $1\le i\le n$.

The main results are as follows

Theorem 1.1. *Assume that $\varphi_n>0, c_i>0$ for $1\le i\le n$, $p>c_1m_1+\cdots+c_nm_n$, and $n-1+\alpha_n-\alpha_j\frac{p}{p-m_j}<0$ for $1\le j\le n-1$, then any solution to* (1.1) *will blow up in finite time.*

Moreover, we also consider the system of multi-time fractional differential equations as follows

$$\begin{cases} c_1D^{\alpha_1}_{0^+,x_1}(|u|^{m_1}-\varphi_1^{m_1})+\cdots+c_nD^{\alpha_n}_{0^+,x_n}(|u|^{m_n}-\varphi_n^{m_n})=|v|^q,\\ b_1D^{\beta_1}_{0^+,x_1}(|v|^{k_1}-\psi_1^{k_1})+\cdots+b_nD^{\beta_n}_{0^+,x_n}(|v|^{k_n}-\psi_n^{k_n})=|u|^p, \end{cases} \tag{1.2}$$

with initial conditions

$$\begin{cases} u(0,x_2,\cdots,x_n)=\varphi_1(x_2,\cdots,x_n),\cdots,u(x_1,\cdots,x_{n-1},0)=\varphi_n(x_1,\cdots,x_{n-1}),\\ v(0,x_2,\cdots,x_n)=\psi_1(x_2,\cdots,x_n),\cdots,v(x_1,\cdots,x_{n-1},0)=\psi_n(x_1,\cdots,x_{n-1}). \end{cases}$$

Note that $0<\alpha_n<\alpha_{n-1}<\cdots<\alpha_1\le 1, 0<\beta_n<\beta_{n-1}<\cdots<\beta_1\le 1, p,q>1, m_i,k_i>0$ and b_i,c_i are constants for $1\le i\le n$.

Theorem 1.2. *Assume that $\varphi_n,\psi_n>0, b_i,c_i>0$ for $1\le i\le n$, $p>c_1m_1+\cdots+c_nm_n, q>b_1k_1+\cdots+b_nk_n$ and $n-1+\alpha_n-\alpha_j\frac{p}{p-m_j}<0, n-1+\beta_n-\beta_j\frac{q}{q-k_j}<0$ for $1\le j\le n-1$, then any solution to* (1.2) *will blow up in finite time.*

The paper is organized as follows: In Section 2, we gather some basic facts and properties of the fractional calculus. The main results will be proved in Section 3. In Section 4, we will apply our results to some examples.

2. Preliminaries

In this section, we will recall some definitions and properties concerning the fractional calculus. For more details, we refer the readers to [17], [21].

The left-handed and right-handed Riemann-Liouville fractional derivatives for continuous function f, $0<\alpha<1$ are defined as

$$D^{\alpha}_{0^+}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{f(s)}{(t-s)^{\alpha}}ds,$$

and

$$D^{\alpha}_{T^-}f(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T\frac{f(s)}{(t-s)^{\alpha}}ds,$$

respectively. Then the integration by parts reads (see[21], p.46)

$$\int_0^T D^{\alpha}_{0^+}f(t)g(t)dt=\int_0^T f(t)D^{\alpha}_{T^-}g(t)dt$$

where $f,g\in C([0,T])$.

The test function ϕ is defined as

$$\phi(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^\lambda, & 0 \le t \le T \\ 0, & t > T. \end{cases}$$

The following lemmas about the test function are proved in [9]. For the completeness, we will give the proof here.

Lemma 2.1. *Let ϕ be defined as above and the following hold:*

$$\int_0^T D_{T^-}^\alpha \phi(t) dt = C_{\alpha,\lambda} T^{1-\alpha}, \tag{2.1}$$

$$\int_0^T \phi^{1-p}(t) |\phi'(t)|^p dt = C_{p,\lambda} T^{1-p} \qquad \text{if} \quad p < \lambda + 1 \tag{2.2}$$

$$\int_0^T \phi^{1-p}(t) |D_{T^-}^\alpha \phi(t)|^p dt = C_{p,\alpha,\lambda} T^{1-\alpha p} \qquad \text{if} \quad \lambda > \alpha p - 1 \tag{2.3}$$

where $C_{\alpha,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)}, C_{p,\lambda} = \frac{\lambda^p}{1+\lambda-p}, C_{p,\alpha,\lambda} = \frac{\lambda^p}{1+\lambda-\alpha p} \left\{ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} \right\}^p.$

Proof. Note first that

$$\begin{aligned} D_{T^-}^\alpha \phi(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \left(1 - \frac{t}{T}\right)^\lambda (t-s)^{-\alpha} ds \\ &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[T^{-\lambda} (T-t)^{1+\lambda-\alpha} \int_0^1 (1-y)^\lambda y^{-\alpha} dy \right] \\ &= \frac{B(1-\alpha, \lambda+1)}{\Gamma(1-\alpha)} (1+\lambda-\alpha) T^{-\lambda} (T-t)^{\lambda-\alpha} \\ &= C_{\alpha,\lambda} (1+\lambda-\alpha) T^{-\lambda} (T-t)^{\lambda-\alpha}, \end{aligned}$$

where we have used change of variable $y = \frac{s-t}{T-t}$ in the second equality and $C_{\alpha,\lambda} = \frac{B(1-\alpha,\lambda+1)}{\Gamma(1-\alpha)} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)}$. It is direct to check the Lemma holds by the above equality. $\square$

3. Main results

3.1. **Proof of Theorem 1.1.**

Proof. Our method is by contradiction. Set $Q = [0,T]^n$ and

$$\int_Q f = \int_0^T \cdots \int_0^T f(x_1, \cdots, x_n) dx_1 \cdots dx_n.$$

Denote by $\Phi(x_1, \cdots, x_n) = \phi(x_1) \cdots \phi(x_n)$. Multiply Φ to both side of (1.1) and integration by parts gives:

$$c_1 \int_Q (|u|^{m_1} - \varphi_1^{m_1}) D_{T^-,x_1}^{\alpha_1} \Phi + \cdots + c_n \int_Q (|u|^{m_n} - \varphi_n^{m_n}) D_{T^-,x_n}^{\alpha_n} \Phi = \int_Q |u(t)|^p \Phi,$$

where $D_{T^-,x_i}^{\alpha_i}$ is the right-handed Riemann-Liouville fractional derivatives with respect to variable x_i for $1 \le i \le n$.
In turn, it follows that

$$\sum_{i=1}^n c_i \int_Q |u|^{m_i} D_{T^-,x_i}^{\alpha_i} \Phi = \int_Q |u|^p \Phi + \sum_{i=1}^n c_i \int_Q \varphi_i^{m_i} D_{T^-,x_i}^{\alpha_i} \Phi.$$

Note that by Hölder inequality, we have

$$\begin{aligned}
&\sum_{i=1}^{n} c_i \int_Q |u|^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi \\
\leq &\sum_{i=1}^{n} c_i \left(\int_Q |u|^p\Phi\right)^{\frac{m_i}{p}} \left(\int_Q |\Phi|^{-\frac{m_i}{p-m_i}} |D^{\alpha_i}_{T^-,x_i}\Phi|^{\frac{p}{p-m_i}}\right)^{\frac{p-m_i}{p}} \\
\leq &\sum_{i=1}^{n} c_i \left(\frac{m_i}{p}\int_Q |u|^p\Phi + \frac{p-m_i}{p}\int_Q |\Phi|^{-\frac{m_i}{p-m_i}} |D^{\alpha_i}_{T^-,x_i}\Phi|^{\frac{p}{p-m_i}}\right).
\end{aligned}$$

Moreover, by Lemma 2.1, we obtain

$$\begin{aligned}
&\int_Q |\Phi|^{-\frac{m_i}{p-m_i}} |D^{\alpha_i}_{T^-,x_i}\Phi|^{\frac{p}{p-m_i}} \\
= &\int_0^T |\phi(x_i)|^{-\frac{m_i}{p-m_i}} |D^{\alpha_i}_{T^-,x_i}\phi(x_i)|^{\frac{p}{p-m_i}}\, dx_i \prod_{j\neq i}\int_0^T |\phi(x_j)|^{-\frac{m_j}{p-m_j}} |\phi(x_j)|^{\frac{p}{p-m_j}}\, dx_j \\
= &C_1 T^{n-\alpha_i\frac{p}{p-m_i}}.
\end{aligned}$$

And hence

$$\begin{aligned}
&\sum_{i=1}^{n} c_i \int_Q |u|^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi \\
\leq &\frac{\sum_{i=1}^{n} c_i m_i}{p}\int_Q |u|^p\Phi + \sum_{i=1}^{n} c_i' T^{n-\alpha_i\frac{p}{p-m_i}},
\end{aligned}$$

where $c_i' = C_1 c_i \frac{p-m_i}{p}$.
Thus it follows that

$$\frac{p-\sum_{i=1}^{n} c_i m_i}{p}\int_Q |u|^p\Phi + \sum_{i=1}^{n} c_i \int_Q \varphi_i^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi \leq \sum_{i=1}^{n} c_i' T^{n-\alpha_i\frac{p}{p-m_i}}.$$

Since

$$\int_Q \varphi_i^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi = \int_0^T D^{\alpha_i}_{T^-,x_i}\phi(x_i)dx_i \int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx, = C_2 T^{1-\alpha_i}\int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx,$$

where dx is the Lebesgue measure on $\mathbb{R}^{n-1}$ and $\Phi_i = \prod_{j\neq i}\phi(x_j)$.
By the assumption $p > c_1 m_1 + \cdots + c_n m_n$, we obtain

$$C_2 \sum_{i=1}^{n} c_i T^{1-\alpha_i}\int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx \leq \sum_{i=1}^{n} c_i' T^{n-\alpha_i\frac{p}{p-m_i}}.$$

Multiply T^{α_n-1} to each side of the inequality and by the facts $\varphi_n > 0$, we obtain for T large enough

$$C_2 \sum_{i=1}^{n-1} c_i T^{\alpha_n-\alpha_i}\int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx + C_3\int_{B(0,1)} \varphi_n dx \leq \sum_{i=1}^{n} c_i' T^{n-1+\alpha_n-\alpha_i\frac{p}{p-m_i}}.$$

By the assumption $0 < \alpha_n < \cdots < \alpha_1 \leq 1$ and $n-1+\alpha_n-\alpha_i\frac{p}{p-m_i} < 0$ for $1 \leq i \leq n-1$, the above inequality leads to contradiction as $T \to \infty$ since we have $C_3 > 0$. $\square$

3.2. **Proof of Theorem 1.2.**

Proof. We still use the method of contradiction. Multiply each term in (1.2) by $\phi(t)$ and integration by parts leads

$$\sum_{i=1}^{n} c_i \int_Q |u|^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi = \int_Q |v|^q \Phi + \sum_{i=1}^{n} c_i \int_Q \varphi_i^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi$$

and

$$\sum_{i=1}^{n} b_i \int_Q |v|^{k_i} D^{\beta_i}_{T^-,x_i}\Phi = \int_Q |u|^p \Phi + \sum_{i=1}^{n} b_i \int_Q \psi_i^{k_i} D^{\beta_i}_{T^-,x_i}\Phi.$$

By Hölder inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} b_i \int_Q |v|^{k_i} D^{\beta_i}_{T^-,x_i}\Phi \\
\le&\sum_{i=1}^{n} b_i \left(\int_Q |v|^q \Phi\right)^{\frac{k_i}{q}} \left(\int_Q |\Phi|^{-\frac{k_i}{q-k_i}} |D^{\beta_i}_{T^-,x_i}\Phi|^{\frac{q}{q-k_i}}\right)^{\frac{q-k_i}{q}} \\
\le&\sum_{i=1}^{n} b_i \left(\frac{k_i}{q}\int_Q |v|^q \Phi + \frac{q-k_i}{q}\int_Q |\Phi|^{-\frac{k_i}{q-k_i}} |D^{\beta_i}_{T^-,x_i}\Phi|^{\frac{q}{q-k_i}}\right).
\end{aligned}$$

and

$$\int_Q |\Phi|^{-\frac{k_i}{q-k_i}} |D^{\beta_i}_{T^-,x_i}\Phi|^{\frac{q}{q-k_i}} = B_1 T^{n-\beta_i \frac{q}{q-k_i}}.$$

Then we have the estimates

$$\int_Q |v|^q\Phi + \sum_{i=1}^{n} c_i \int_Q \varphi_i^{m_i} D^{\alpha_i}_{T^-,x_i}\Phi \le \frac{\sum_{i=1}^{n} c_i m_i}{p}\int_Q |u|^p\Phi + \sum_{i=1}^{n} c_i' T^{n-\alpha_i\frac{p}{p-m_i}}$$

as well as

$$\int_Q |u|^p\Phi + \sum_{i=1}^{n} b_i \int_Q \psi_i^{m_i} D^{\beta_i}_{T^-,x_i}\Phi \le \frac{\sum_{i=1}^{n} b_i k_i}{p}\int_Q |v|^q\Phi + \sum_{i=1}^{n} + c_i' T^{n-\beta_i\frac{q}{q-k_i}}.$$

By the assumption $p > c_1 m_1 + \cdots + c_n m_n$ and $q > b_1 k_1 + \cdots + b_n k_n$ the above two inequalities lead to

$$\begin{aligned}
&C_2 \sum_{i=1}^{n} c_i T^{1-\alpha_i} \int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx + B_2 \sum_{i=1}^{n} b_i T^{1-\beta_i}\int_{\mathbb{R}^{n-1}} \psi_i^{k_i}\Phi_i dx \\
\le&\sum_{i=1}^{n} c_i' T^{n-\alpha_i\frac{p}{p-m_i}} + b_i' T^{n-\beta_i\frac{q}{q-k_i}}
\end{aligned}$$

Without loss of generality, we can assume that $\alpha_n \le \beta_n$ and multiplying $T^{\alpha_n - 1}$ to each side of the above inequality gives

$$\begin{aligned}
&C_3 \int_{B(0,1)} \varphi_n dx + C_2 \sum_{i=1}^{n-1} c_i T^{\alpha_n-\alpha_i}\int_{\mathbb{R}^{n-1}} \varphi_i^{m_i}\Phi_i dx + B_2\sum_{i=1}^{n} b_i T^{\alpha_n-\beta_i}\int_{\mathbb{R}^{n-1}}\psi_i^{k_i}\Phi_i dx \\
\le&\sum_{i=1}^{n} c_i' T^{n-1-\alpha_n-\alpha_i\frac{p}{p-m_i}} + b_i' T^{n-1-\alpha_n-\beta_i\frac{q}{q-k_i}}
\end{aligned}$$

By the assumption $0 < \alpha_n < \cdots < \alpha_1 \leq 1$, $0 < \beta_n < \cdots < \beta_1 \leq 1$, $n - 1 + \alpha_n - \alpha_i \frac{p}{p-m_i} < 0$, $n - 1 + \beta_n - \beta_i \frac{p}{p-m_i} < 0$ for $1 \leq i \leq n-1$, the above inequality leads to contradiction as $T \to \infty$ since we have $C_3 > 0$. $\square$

4. Applications

In this section, we will consider several examples and give the corresponding blow up results.

Example 4.1. (*Nonlinear time fractional Burgers Equation*)
Now consider the equation for $0 < \alpha < 1$

$$D^{\alpha}_{0^+,t}(u(x,t) - \varphi(x)) + \frac{1}{2} D_x(u^2(x,t)) = |u|^p,$$

with initial data $u(0,x) = \varphi(x) > 0$.
According to Theorem 1.1, we have for p satisfying

$$p > 2, 1 + \alpha - \frac{p}{p-2}, 1 + \alpha - \frac{2p}{p-1},$$

which is $2 < p < \frac{2(1+\alpha)}{\alpha}$, the solution will blow up in finite time.

Example 4.2. (*Nonlinear coupled time fractional Burgers Equations*)
Now consider the following systems

$$\begin{cases} D^{\alpha}_{0^+,t}(u - \varphi) + \frac{1}{2} D_x(|u|^2) = |v|^q, \\ D^{\beta}_{0^+,t}(v - \psi) + \frac{1}{2} D_x(|v|^2) = |u|^p, \end{cases}$$

with initial condition $u(0,x) = \varphi(x) > 0, v(0,x) = \psi(x) > 0$ where $0 < \alpha, \beta < 1$.
Thus by Theorem 1.2 for

$$2 < p < \frac{2(1+\alpha)}{\alpha}, 2 < q < \frac{2(1+\beta)}{\beta},$$

the solution will blow up in finite time.

5. Conclusions

In this paper, the testing function methods are used to study the nonlinear multi-time fractional differential equations. As a result, we prove the systems will blow up in finite time provided the nonlinear term satisfying some condition which is determined by the structure of the equations. Then we apply our results to the time fractional Burgers equations and give the explicit range which will lead to a blow up of the systems.

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