The Eigenspace Spectral Regularization Method for solving Discrete Ill-Posed Systems

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Abstract

In this paper, it is shown that discrete equations with Hilbert matrix operator, circulant matrix operator, conference matrix operator, banded matrix operator and sparse matrix operator are ill-posed in the sense of Hadamard. These ill-posed problems cannot be regularized by Gauss Least Square Method (GLSM), QR Factorization Method (QRFM), Cholesky Decomposition Method (CDM) and Singular Value Decomposition (SVDM). To overcome the limitations of these methods of regularization, an Eigenspace Spectral Regularization Method (ESRM) is introduced which solves ill-posed discrete equations with Hilbert matrix operator, circulant matrix operator, conference matrix operator, banded matrix operator and sparse matrix operator. Unlike GLSM, QRFM, CDM, and SVDM, the ESRM regularizes such a system. In addition, the ESRM has a unique property, the norm of the eigenspace spectral matrix operator $\kappa(K) = ||K^{-1}K|| = 1$. Thus, the condition number of ESRM is bounded by unity unlike the other regularization methods such as SVDM, GLSM, CDM, and QRFM.

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1 Introduction

Many mathematical problems are classified to be ill-posed problems, for instance, the discretization of linear ill-posed problems like the Fredholm integral equations of the first kind with a smooth kernel is ill-posed, hence referred to as a linear discrete ill-posed problem [1]. Moreover, image processing problems are mostly ill-posed, that is, when a blurred text or image is produced. In such cases, the image need to be reconstructed to produce a clear image. This study is concerned with regularization of such linear discrete systems.

Given a system of linear equations $A : X \longrightarrow Y$ defined by

$$Ax = b, \tag{1}$$

where A is a matrix operator $R^m \times R^n$ from a domain set X into a codomain set Y, $x \subseteq X$ is a column vector $R^m \times 1$ and $b \in Y$ is also a column vector $R^m \times 1$ is well-posed in the sense of Hadamard if

1) A is an surjective operator. Thus, the solution to the discrete equation (1) exists.

2) A is injective operator. Thus, there is a unique solution to the discrete equation (1).

3) The inverse, $A^{-1} : Y \longrightarrow X$ is continuous. Thus, the solution to the equation (1) is stable. [2]

Although Hadamard provided the criteria for detecting ill-posed problem but he did not provide any method of solving such a problem. The question of restoration of well-posedness of an equation was problem at heart of Mathematicians in the beginning of the 20th century. Lavrentiev observed well-posed discrete equations as a system whose solution is sought on a compact subspace of a topological space which maps to another compact topological space. This type of well-posedness is generally termed as conditionally well-posed discrete equations [14].

Based on the varying definitions of well-posedness of discrete equations, Tikhonov defines the solution space M as a compact set which is a subset of the Euclidean space but there is no restriction on the data function, the vector that appears on the right hand side of equation (1). The Tikhonov and Lavrentiev observations of well-posedness are generics definition of Hadamard definitions. On the other hand, if one or more of the conditions stated above is violated by a discrete equations then the equation is said to be ill-posed [15]. Equation (1) may have either no solution, or infinitely many solutions, or a unique solution.

In 1944, Tikhonov first constructed an iterative method which involves a regularization parameter for regularizing ill-posed discrete equations. There is no unique choice of value of this parameter of even the same set of problems. In addition, Tikhonov regularization method cannot restore well-posedness of a discrete equation which does not have a solution. Ill-posed discrete equations can be regularized either by a direct approach or an iterative approach. Some direct regularization methods like Gauss Least Square Method (GLSM) [8], QR factorization Method (QRFM) [13], Singular Value Decomposition Method (SVDM) [16] and Cholesky Decomposition method (CDM) [10].

Theorem 1.1 (Gauss Least Square Method) *If A is an $m \times n$ matrix with $rank A = n$, then $A^T A$ is nonsingular and the discrete equation (1) has a unique least squares solution given by*

$$\widehat{x} = (A^T A)^{-1} A^T b$$

. *[3]*

The GLSM fails to solve the problem of stability and also performs poorly when solving underdetermined discrete equations. The GLSM though regularise the system but the operator maintain to be unstable which fails to satisfy the stability condition for a well-posed system. In the work of [17, 19] the GLSM has been extended to meet the lipchitz condition to satisfy the boundedness condition.

Theorem 1.2 (QR Factorization Method) *Let A be a $m \times n$ matrix with entries in F and linearly independent columns. Then there exist a $n \times n$ matrix R and $m \times n$ matrix Q, both having entries in F, such that*

1) $A = QR$
2) $Q^H Q = I_n$
3) *R is a nonsingular upper triangular matrix with $r_{kk} > 0$*
4) *The columns of Q are an orthonormal basis for $R(A)$.*
6) *Q and R are unique.* *[7]*

In the work of [21], the QR decomposition has been merged with the Cholesky decomposition method to provide a more faster algorithm in computations. The matrix operator R, the upper triangular matrix, is closed in R^m. We can see that theorem (1.2) dwells on the existence of the solution to a discrete equation. QRFM restores only the existence and the uniqueness of the discrete equation. Also, the QRFM requires the matrix operator of the discrete equation given to be a linearly independent matrix operator [12]

Theorem 1.3 (Singular Value Decomposition Method) *Let A be an $m \times n$ matrix. There exixts an integer $r, \sigma_1 \geq sigma_2 \geq \cdots \geq sigma_r > 0$, an $m \times m$ unitary matrix U, an $n \times n$ unitary matrix V, and an $m \times n$ matrix S, all of whose entries are 0 except $S_{ii} = \sigma_i, i = 1, 2, \cdots, r$ such that*

$$A = USV^H \qquad [4]$$

In the work of [20], the SVDM has been applied in a correspondence analysis using R. On the other hand, the SVDM do not produce a stable results dealing with discrete equations with $Rank(A) < n$ or infinitely many solution or no no solution. The SVDM mostly have problems with the loss of small singular values thereby giving undesirable solutions in a functional space [9, 13].

Theorem 1.4 (Cholesky Factorization Method) *Given a symmetric positive definite matrix A, there exists a lower triangular matrix L such that $A = LL^T$, where L is the lower triangular matrix (Cholesky factor) and L^T is the Cholesky factorization of A. [11]*

the Cholesky decomposition do not have a unique Cholesky decomposition for matrices that are not hermitian and positive definite in nature. Also, the CDM requires the matrix operator to be a symmetric positive definite matrix operator [10, 18]. This therefore suffices that the Cholesky method is unable to regularize some matrix operators and hence may still be ill-posed in the sense of Hadamard.

All these regularization methods in restoring the well-posedness of the discrete equation (1), fail to restore the well-posedness of a discrete equation [3]. Even a discrete equation with a sparse matrix operator, these existing methods of regularization fail to restore the stability of the solution of the equation. In numerical analysis, matrix operators with large condition number are prone to large numerical errors and also very sensitive to variations in either the right hand side or even the matrix operator of equation (1) [5].

In addition, the stability of the discrete equation with Hilbert matrix, circulant matrix, conferences matrix, banded matrix could not be resolved by GLSM, QRM, SVDM, and CDM on the grounds that, after the applications of these methods to the discrete equation, the regularized matrix operator is nearly singular and generally attributed to large condition numbers. This makes these existing methods not reliable for approximating solution of discrete equations.

In the area of application, the GLSM, QRFM, SVDM, and CDM have not been much consistently used to regularize a discrete equation with perturbed right hand side b in equation (1) [15]. Nothwithstanding, the populous matrix operators like Hilbert matrix operator, Conference matrix operator, Circulant matrix operator, and Banded matrix operator occurring in a discrete equation have not been looked at and then even talk about their regularization process visa-vis regularization method. The Sparse matrix operator occurring in a discrete equation which is commonly studied many authors accross the globe, when solved with SVDM, GLSM, CDM, and QRFM, do not yield the desired solution in functional space.

In order to overcome the myraids of ill-posed discrete equations raised in the method of regularization which solves these problems is paramount. A method devoid of computational errors or little computational errors is interesting and must be focus point of regularizing discrte equations. This paper seeks to Introduce a new regularization method, "the Eigenspace Spectral Regularization Method (ESRM)" for solving ill-posed discrete equations and check how efficient the ESRM is compared to the existing regularization methods.

The paper is organized in this order. Section 1 contains a background to ill-posed problems. Also, in section 2 we present the drawbacks of GLSM, QRFM, SVDM, and CDM when they are applied in regularizing discrete equations. Section 3 introduces the Eigenspace Spectral Regularization Method (ESRM) and apply it to regularize discrete equations in which the GLSM, QRFM, SVDM, and CDM can not solve. In addition, we also compare ESRM with GLSM, QRFM, SVDM, and CDM. Finally, section 4 contains the conclusion of this paper.

2 The Ill-Posed Linear Discrete Equations and Drawbacks of Some Existing Methods

In this section, the highlights definitions that are relevant to establishing ill-posed discrete equation and regularization of such ill-posed discrete equation. In the next section, the rigorous proofs for ill-posed discrete equation are provided.

2.1 Ill-Posed Discrete Equations

In this subsection a number of discrete equations are discussed. We in this case prove the type of ill-posedness of the discrete equations in the Hadamard sense.

Example 2.1 *Consider the discrete equations below*

$$\begin{aligned} x+y &= 1 \\ x-y &= 3 \\ -x+2y &= -2 \end{aligned} \tag{2}$$

To determine there is a solution to equation (2), the The augumented matrix of equation (2) is

$$[A|b] = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right)$$

and its corresponding reduced row-echelon form is

$$[A|b] = \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

By the Rouche-Capelli theorem, the discrete equation (2) does not have a solution in R^3 since the rank of the matrix $A = 2$, which is less than the rank of the extended matrix, $\kappa = 3$. Hence the discrete equation (2) is ill-posed in the sense of Hadamard.

Example 2.2 *Also, consider the discrete equattion*

$$\begin{aligned} x+3y+2z &= 1 \\ 2x+8y+6z &= 6 \\ x+2y+z &= 1 \end{aligned} \tag{3}$$

Similarly, the discrete equation in (3) does not have a solution in R^3, on the grounds that, the rank of the matrix is $r = 2$, which is less than the extended matrix. Hence the discrete equation (3) is ill-posed in the sense of Hadamard.

Example 2.3 *consider the discrete equation below*

$$\begin{aligned} x+2y+z+w &= 8 \\ x+2y+2z-w &= 12 \\ 2x+4y+6w &= 4 \end{aligned} \tag{4}$$

Similarly, the discrete equation in (4) does not have a solution in R^3. Hence the discrete equation is ill-posed equation in the sense of Hadamard.

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Example 2.4 *consider the discrete equation below*

$$\begin{aligned} -3x - 5y + 36z &= 10 \\ -x + 7z &= 5 \\ x + y - 10z &= -4 \end{aligned} \tag{5}$$

The reduced row-echelon of the discrete equation in (5) is

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here, the z is arbitrarily. This implies that the discrete equation in (5) has a solution but not unique (infinitely many solutions). Hence, the discrete equation in (5) is ill-posed in the sense of Hadamard.

Example 2.5 *consider the discrete equation*

$$\begin{aligned} x_1 + x_4 &= 4 \\ 2x_3 + x_6 &= 6 \\ 2x_4 &= 2 \end{aligned} \tag{6}$$

Imilarly, the discrete equation in (6) has no unique solution in R^6. Hence, the discrete equation is ill-posed in the sense of Hadamard.

Example 2.6 *Let us again consider the discrete equation with Hilbert matrix operator.*

$$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 3 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z &= 2 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z &= 5 \end{aligned} \tag{7}$$

The discrete equation (7) has a unique solution but not stable because it has $cond(A) = 504$. Hence, the discrete equation is ill-posed in the sense of Hadamard.

Example 2.7 *Considering also the discrete equation with banded matrix operator*

$$\begin{aligned} x - 8y + 0z + 0w &= 3 \\ 2x - 2y - 7z + 0w &= -5 \\ 0x + 7y + 3z - 6w &= 1 \\ 0x + 0y + 8z - 7w &= 6 \end{aligned} \tag{8}$$

The discrete equation (7) has a unique solution but not stable since it has $cond(A) = 12.6139$. Hence, the discrete equation is ill-posed in the sense of Hadamard.

Example 2.8 *A discrete equation with conference matrix operator*

$$\begin{aligned} 0x + y + z + w &= 4 \\ x + 0y + z - w &= 5 \\ x + y + 0z + w &= 3 \\ x - y + z + 0w &= 2 \end{aligned} \tag{9}$$

Similarly, we can see that although the discrete equation in (9) has unique solution but not stable as $cond(A) = 2.2361$. Hence it is ill-posed in the sense of Hadamard.

Example 2.9 *let us consider the discrete equation with a sparse matrix operator*

$$\begin{aligned} 1.1x_1 + 0.5x_7 &= 3 \\ 1.9x_2 + 0.5x_7 &= 2 \\ 2.6x_3 + 0.5x_7 &= 5 \\ 7.8x_3 + 0.6x_4 &= -1 \\ 1.5x_4 + 2.7x_5 &= 6 \\ 1.6x_1 + 0.4x_5 &= -4 \\ 0.9x_6 + 1.7x_7 &= 9 \end{aligned} \tag{10}$$

Similarly, we realize that the discrete equation in (10) has unique solution but not stable as $cond(A) = 44.4164$. Hence it is an ill-posed discrete equation in the sense of Hadamard.

Example 2.10 *Finally we also consider the discrete equation with circulant matrix operator*

$$\begin{aligned} x + 2y + 3z + 4w &= -1 \\ 2x + y + 4z + 3w &= 1 \\ 3x + 4y + 2z + w &= 3 \\ 4x + 3y + z + 2w &= 5 \end{aligned} \tag{11}$$

Similarly, the discrete equation in (11) also has unique solution but not stable as $cond(A) = 16.4375$. Hence it is ill-posed in the sense of Hadamard.

2.2 Drawbacks of Existing Methods of Regularizing Discrete Equations

In this subsection the shortcomings of GLSM, QRFM, SVDM and CDM are showed. Firstly, the shortcomings of GLSM is discussed in detail.

2.2.1 Gauss Least Square Method

Applying the GLSM to the discrete equation (2), we observed that the GLSM operator is

$$A^T A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$$

The GLSM operator $A^T A$ yields a unique solution, that is $x = 1.7143\, y = -0.4286$, on the grounds that $|A^T A| = 14$, which is nonsingular. On the other hand, $cond(A^T A) = 3.5$, which is far from one. This implies that the solution of the discrete equation in (2) is not stable when solved, using the GLSM. Hence, the GLSM does not restore the well-posedness of the discrete equation in (2).

Also in (3), the existence and uniqueness of solution to this equation is met but

$$cond(A^T A) = 227037379566481184$$

and also

$$|A^T A| = -0.00000000000000116573417585655503671$$

We can see that not only the condition number is very large (far from unity) but also the determinant of GLSM operator is nearly singular. This implies instability of the solution of discrete equation in (3), when solved by GLSM. Hence, the GLSM does not yield well-posed equation when applied to the discrete equation in (3).

Similarly the discrete equation in (4), the

$$cond(A^T A) = 76509541981751072$$

Also it has

$$|A^T A| = 0$$

thus, could not restore the uniqueness condition of well-posedness, and again, the condition number is large and fails to meet the third requirement of well-posedness. Hence the GLSM fails to regularize the discrete equation in (4).

Moreso, given the discrete equation in (5), it has

$$cond(A^T A) = 1822.2651$$

which is large and fails to meet the third requirement of well-posedness. Hence the GLSM fails to regularize the discrete equation in (5).

Similarly given the discrete equation in (6), the

$$cond(A^T A) = \infty$$

Also it has

$$|A^T A| = 0$$

thus, fails to restore the uniqueness condition of well-posedness, and again, the condition number is very large and hence fails to meet the third requirement of well-posedness. Hence the GLSM fails to regularize the discrete equation in (6).

Also, given the discrete equation in (7), it has

$$cond(A^T A) = 274635.5061$$

thus, a large condition number which fails to meet the third condition of well-posedness. Hence the GLSM fails to regularize the discrete equation in (7).

Again, in equation (8), the

$$cond(A^T A) = 93955.72$$

which is large and fails to meet the third condition of well-posedness. Hence the GLSM fails to regularize the discrete equation in (8).

Considering also, equation (9), we realize that

$$cond(A^T A) = 5$$

which is large and fails to meet the third condition of well-posedness. Hence the GLSM fails to regularize the discrete equation in (9).

Again, in equation (10), the

$$cond(A^T A) = 1972.8151$$

which is also large and fails to satisfy the third condition of well-posedness. Thus, the GLSM fails to regularize the discrete equation in (10).

Finally, given the discrete equation in (11), the

$$cond(A^T A) = 50$$

which is large and fails to satisfy the third condition of well-posedness. Thus, the GLSM fails to regularize the discrete equation in (11).

2.2.2 QR Factorization Method

In this subsection the concentration is on the regularization of discrete equations discussed in previous section using the QRFM.

Using the QRFM, we observed that

$$R = \begin{pmatrix} -1.7321 & 1.1547 \\ 0 & 2.1602 \\ 0 & 0 \end{pmatrix}$$

,

which is not feasible to proceed with this method. Hence, the QRFM fails to regularize the discrete equation in (2). Similar observations were made when applying the QRFM to the discrete equations in (3), (4), and (6).

Applying the QRFM to the discrete equation in (5), the computed

$$R = \begin{pmatrix} 3.3166 & 4.8242 & -37.6889 \\ 0 & -1.6514 & 4.9543 \\ 0 & 0 & -0.0000 \end{pmatrix}$$

and the unique solution $x = -4,\ y = 2$ and $z = 0$. However, the stability of this solution of equation (5) is determined by

$$cond(R) = 231004359338148256$$

which is very large and far from unity. Moreover, the determinant of the operator in the QRFM is

$$|R| = 0.000000000000007008$$

which is nearly singular. All these results imply instability of the solution of equation (5) yield by the QRFM. Hence, QRFM cannot regularize the discrete equation in (5) in the sense of Hadamard.

Similar trends are observed on the stabilities of solutions of discrete equations in (7), (8), (9), (10) and (11) when solved by the QRFM. From the foregoing analysis, the QRFM cannot regularize the discrete equations in (2) to (11).

2.2.3 Cholesky Decomposition method

We apply CDM on the discrete equations in (2) - (11) in this subsection of the paper. Applying this method to discrete equation in (2) the operator R could not be computed since matrix A is not a positive definite matrix operator. Similar challenge was encountered when applying the CDM to the discrete equations (3), (4), (5), (6), (8), (9), (10), and (11).

Now Applying the CDM to the discrete equation in (7) gives the matrix operator

$$R = \begin{pmatrix} 1.0000 & 0.5000 & 0.3333 \\ 0 & 0.2887 & 0.2887 \\ 0 & 0 & 0.0745 \end{pmatrix}$$

it was observed that

$$cond(R) = 22.8923$$

,

This implies that the CDM fails to restore the stability condition. Therefore the discrete equation in (7) remains an ill-posed discrete equation in the sense of Hadamard.

2.2.4 Singular Value Decomposition

The SVDM presents similar results, thus, it is able to resolve the existence and uniqueness conditions of the discrete equations (2) - (11), but fails to satisfy the stability condition of well-posedness in the sense of Hadamard.

All these preliminary results signifies that the GLSM, SVD, QR factorisation and Cholesky decomposition all fails to restore the stability of the Hilbert and banded matrix operator. They also fail to regularize discrete equations with a conference matrix operator and also circulant matrix operator. Thus in the sense of Hadamard all of these regularization methods produces an ill-posed discrete equation.

3 Main Results

This section introduces the Eigenspace Spectral Regularization Method (ESRM) and also proved. Then the ESRM is applied to regularize the discrete equations in (2) - (11). We discuss in detail the three conditions of well-posedness are restored using the ESRM.

Theorem 3.1 (The Eigenspace Spectral Regularization Method) *Suppose a discrete equation (1) where A is the matrix operator, X is an $m \times 1$ vector and b is also an $m \times 1$ vector. The eigenspace spectral matrix operator, $K =$ eigenspace of A_1, where $A_1 = A^*A$. Then a discrete equations in (1) has a stable unique solution given by*

$$x = (K^*K)^{-1}K^*\wedge, \tag{12}$$

where $\wedge$ is the spectrum of A_1

Proof: Given a discrete equation (1), setting eigenspace of $K =$ **eigenspace of** A_1 and $\wedge$ the corresponding spectrumof A_1, where $A_1 = A^*A$. Replacing A and b in equation (1) by K and $\wedge$ respectively, we have:

$$Kx = \wedge$$

Multiplying both sides of the above equationby its adjoint operator of K^* yields

$$K^*Kx = K^*\wedge$$

Since the columns of K are linearly independent, thus, $||K|| \neq 0$, it implies that K^{-1} exists. This suffices that

$$(K^*K)^{-1} K^TKx = (K^*K)^{-1} K^*B$$

implies

$$X = (K^*K)^{-1} K^*B$$

This is a stable unique solution. In matrix theory $A^* = A^T$, where A^T is the transpose of the matrix operator A.

Applying the ESRM to a disceret equation (2), we observed that

$$A_1 = A^T A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \tag{13}$$

and

$$K = \begin{pmatrix} -0.8944 & -0.4472 \\ -0.4472 & 0.8944 \end{pmatrix}, \tag{14}$$

and

$$\wedge = \begin{pmatrix} 2.0000 \\ 7.0000 \end{pmatrix} \tag{15}$$

We can see that ESRM matrix operator is

$$(K^T K)^{-1} = \begin{pmatrix} 1.0000 & 0 \\ 0 & 1.0000 \end{pmatrix}$$

and

$$K^T \wedge = \begin{pmatrix} -4.9193 \\ 5.3666 \end{pmatrix}$$

The solution is

$$x = \begin{pmatrix} -4.9193 \\ 5.3666 \end{pmatrix}$$

this implies that using the ESRM $x = -4.9193$ and $y = 5.3666$. There is a solution to the discrete equation in equation (2). Since the determinant of the operatot in ESRM is nonzero, it implies that the above solution is unique.

Also, the condition number is

$$cond(K) = 1$$

This implies that the above solution is stable. Hence, the ESRM regularizes a discrete equation in (2). Thus, all the three conditions of well-posedness are met. Hence, solution of a discrete equation in (2) is well-posed using ESRM.

Also applying the ESRM to a discrete equation (3), we observed that yields

$$K = \begin{pmatrix} -0.5774 & -0.7878 & 0.2145 \\ 0.5774 & -0.2081 & 0.7895 \\ -0.5774 & 0.5797 & 0.5750 \end{pmatrix}$$

and

$$\wedge = \begin{pmatrix} -0.0000 \\ 0.5102 \\ 123.4898 \end{pmatrix}$$

therefore, the solution to the discrete equation (3) will be computed similarly as;

$$x = \begin{pmatrix} -71.0023 \\ 71.4800 \\ 71.4092 \end{pmatrix}$$

this implies that using the ESRM $x = -71.0023$. and $y = 71.4800$ and $z = 71.4092$. Thus there is a solution to the discrete equation in equation (3). Since $|K| = 1$, it implies that the above solution is unique.

Also, the condition number is

$$cond(K) = 1$$

This implies that the above solution is stable. Hence, the ESRM regularizes a discrete equation in (3). Thus, all the three conditions of well-posedness are met. Hence, solution of a discrete equation in (3) is well-posed using ESRM.

Applying the ESRM to the discrte equation in (4)

$$K = \begin{pmatrix} 0.2301 & 0.8944 & -0.2466 & 0.2937 \\ 0.4602 & -0.4472 & -0.4932 & 0.5874 \\ -0.7670 & -0.0000 & -0.6384 & 0.0648 \\ -0.3835 & 0 & 0.5370 & 0.7514 \end{pmatrix}$$

and

$$\wedge = \begin{pmatrix} -0.0000 \\ -0.0000 \\ 11.6354 \\ 61.3646 \end{pmatrix}$$

the solution to the discrete equations (4) then becomes

$$x = \begin{pmatrix} -32.4562 \\ -0.0000 \\ 25.5248 \\ 46.8624 \end{pmatrix}$$

So by the ESRM, the discrete equation (4) satisfies the existence condition of well-posedness in the sense of Hadamard.

By the ESRM, it was observed that $|K| = 1$. Thus, the solution of the discrete equation in (4) is unique.

Also, by the ESRM

$$cond(K) = 1$$

,

Thus, the discrete equation is stable. Hence, the discrete equation (4) is a well-posed system in the sense of Hadamard by the ESRM.

Similarly, applying the ESRM to the discrete Equation (5) with the ESRM matrix operator

$$K = \begin{pmatrix} -0.9113 & -0.4027 & -0.0854 \\ -0.3906 & 0.9114 & -0.1300 \\ -0.1302 & 0.0851 & 0.9878 \end{pmatrix}$$

yields a nondegenrate system, on the grounds that $|K| = -1$.

Also

$$cond(K) = 1$$

,

that is, a stable solution. Therefore by the ESRM the discrete equation (5) is homeomorphic system of linear equation in the Hadamard sense.

We also realised that example 2.20 whose matrix operator is a sparse matrix failed to satisfy the second and third conditions of well-posedness in the sense of Hadamard. All existing regularization methods also failed to restore the well-posedness of example 2.20. We observed that the ESRM whose matrix operator is given by

Moreover, applying the ESRM to the discrete equation (6) with the ESRM matrix operator

$$K = \begin{pmatrix} 0 & 0 & 0 & 0.9732 & 0 & 0.2298 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4472 & 0 & 0.8944 & 0 \\ 0 & 0 & 0 & -0.2298 & 0 & 0.9732 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8944 & 0 & 0.4472 & 0 \end{pmatrix}$$

The ESRM matrix operator has $|K| = -1$, thus the discrete equation has a unique solution

Also,

$$cond(K) = 1$$

.

Thus, the ESRM has a stable solution. Hence, the discrete equation (6) is a homeamorphic system of linear equation in the sense of Hadamard.

Applying the ESRM to the discrete equation (7), we obtain the ESRM matrix operator as

$$K = \begin{pmatrix} -0.1277 & 0.5474 & 0.8270 \\ 0.7137 & -0.5283 & 0.4599 \\ -0.6887 & -0.6490 & 0.3233 \end{pmatrix}$$

we observed that the ESRM matrix operator has $|K| = -1$, thus, the discrete equation (7) has a unique solution.

Also,

$$cond(K) = 1$$

,

thus, the discrte equations (7) has a stable solution. Hence the discrete equation (7) is a homeomorphic discrete equation in the Hadamard sense.

Again, applying the ESRM to the discrete equation (8), the ESRM matrix operator is given as

$$K = \begin{pmatrix} -0.9305 & 0.3564 & -0.0401 & 0.0739 \\ -0.1198 & -0.1134 & 0.8504 & -0.4996 \\ -0.2295 & -0.5170 & -0.4907 & -0.6628 \\ -0.2590 & -0.7699 & 0.1857 & 0.5528 \end{pmatrix}$$

Using the ESRM, we observed that $|K| = -1$, thus the discrete equation (8) has a unique solution.

Also,

$$cond(K) = 1$$

,

thus the discrete equation (8) has a stable solution. Therefore, the discrete equation (8) is a well-posed discrete equuation in the Hadamard sense.

In addition, applying the ESRM to the discrete equation (9), we obtain the ESRM matrix operator as

$$K = \begin{pmatrix} -0.7071 & 0 & 0 & -0.7071 \\ 0 & 0.7071 & 0.7071 & 0 \\ 0.70710 & 0 & -0.7071 & \\ 0 & -0.7071 & 0.7071 & 0 \end{pmatrix}$$

we observed K has $|K| = -1$ implies that the discrete equation (9) has a unique solution.

Also,

$$cond(K) = 1$$

,

Thus the discrete equation has a stable solution. Hence the discrete equation (9) is homeomorphic in the Hadamard sense.

Moreover, applying the ESRM to the discrete equation (10), we obtain the ESRM matrix operator as

$$K = \begin{pmatrix} 0.0582 & -0.0866 & 0.3457 & 0.8704 & 0.3210 & 0.0943 & 0.0002 \\ -0.0366 & -0.1091 & 0.7128 & -0.4766 & 0.5016 & 0.0001 & 0.0003 \\ -0.0601 & 0.0103 & 0.0125 & 0.0085 & -0.0124 & -0.0383 & 0.9972 \\ 0.8299 & -0.2611 & -0.0353 & -0.1027 & -0.0437 & 0.4737 & 0.0717 \\ -0.4582 & 0.1510 & -0.0174 & -0.0378 & -0.0119 & 0.8748 & 0.0048 \\ -0.2720 & -0.8517 & -0.3541 & -0.0266 & 0.2729 & 0.0001 & 0.0005 \\ 0.1379 & 0.4051 & -0.4953 & -0.0504 & 0.7541 & 0.0004 & 0.0202 \end{pmatrix}$$

It is again observed that $|K| = 1$ implies that the discrete equation (10) has a unique solution.

Also,

$$cond(K) = 1$$

.

Thus the discrete equation also satisfies the third condition of well-posedness. Hence the discrete equation (10) is homeomorphic in the Hadamard sense.

Last but not the least, applying the ESRM to the discrete equation (11), we obtain the ESRM matrix operator

$$K = \begin{pmatrix} -0.5478 & -0.4471 & -0.5000 & 0.5000 \\ 0.5478 & 0.4471 & -0.5000 & 0.5000 \\ 0.4471 & -0.5478 & 0.5000 & 0.5000 \\ -0.4471 & 0.5478 & 0.5000 & 0.5000 \end{pmatrix}$$

We observed that $|K| = 1$, which implies that the discrete equation (11) has a unique solution.

Also,

$$cond(K) = 1$$

,

thus an inversely bounded solution. Thus the discrete eqution (11) is a homeomorphic system of linear equation in the Hadamard sense.

3.1 Main Findings in this Paper

We therefore represent the performance of the ESRM against the GLSM, QRFM, SVDM, and CDM in regularizing the above discussed discrete equations in the tables below.

4 Conclusion

We observed that only ESRM is able to regularize discrete equations (2), (3), (4), (5), (6), (7), (8), (9), (10), and (11). Thus, all three conditions of well-posedness are restored in the sense of Hadamard. However, the existing methods of regularization: GLSM, QRFM, SVDM, and CDM failed to regularize discrete equations (2 - 11). In addition, we observed that ESRM has a unique property, the norm of $\kappa(K) = ||K^{-1}K|| = 1$. Thus, the condition number of ESRM is bounded by unity unlike the other regularization methods such as SVDM, GLSM, CDM, and QRFM.

Table 1: Shows comparison of ESRM and GLSM, QRFM, SVDM, and CDM for regularizing ill-posed discrete equations.

Matrix	ESRM	GLSM	QRFM	SVDM	CDM
$\begin{aligned} x+y &= 1 \\ x-y &= 3 \\ -x+2y &= -2 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} x+3y+2z &= 1 \\ 2x+8y+6z &= 6 \\ x+2y+z &= 1 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} x+2y+z+w &= 8 \\ x+2y+2z-w &= 12 \\ 2x+4y+6w &= 4 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} -3x-5y+36z &= 10 \\ -x+7z &= 5 \\ x+y-10z &= -4 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize

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Table 2: Continuation of Table 1

Matrix	ESRM	GLSM	QRFM	SVDM	CDM
$\begin{aligned} x_1 + x_4 &= 4 \\ 2x_3 + x_6 &= 6 \\ 2x_4 &= 2 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} x + \frac{1}{2}y + \frac{1}{3}z &= 3 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z &= 2 \\ \frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z &= 5 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} x - 8y + 0z + 0w &= 3 \\ 2x - 2y - 7z + 0w &= -5 \\ 0x + 7y + 3z - 6w &= 1 \\ 0x + 0y + 8z - 7w &= 6 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} 0x + y + z + w &= 4 \\ x + 0y + z - w &= 5 \\ x + y + 0z + w &= 3 \\ x - y + z + 0w &= 2 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize

Table 3: Continuation of Table 1

Matrix	ESRM	GLSM	QRFM	SVDM	CDM
$\begin{aligned} 1.1x_1 + 0.5x_7 &= 3 \\ 1.9x_2 + 0.5x_7 &= 2 \\ 2.6x_3 + 0.5x_7 &= 5 \\ 7.8x_3 + 0.6x_4 &= -1 \\ 1.5x_4 + 2.7x_5 &= 6 \\ 1.6x_1 + 0.4x_5 &= -4 \\ 0.9x_6 + 1.7x_7 &= 9 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize
$\begin{aligned} x + 2y + 3z + 4w &= -1 \\ 2x + y + 4z + 3w &= 1 \\ 3x + 4y + 2z + w &= 3 \\ 4x + 3y + z + 2w &= 5 \end{aligned}$	It Regularizes	Fail to regularize	Fail to regularize	Fail to regularize	Fail to regularize

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