Existence of axially symmetric solutions for a kind of planar Schrödinger-Poisson system

Qiongfen Zhang[a]*, Kai Chen[b], Shuqin Liu[a], Jinmei Fan[a]

[a] College of Science, Guilin University of Technology, Guilin, Guangxi 541004, PR China

[b] School of Science, Guilin University of Aerospace Technology, Guilin, Guangxi 541004, PR China

Abstract: In this paper, we study the following kind of Schrödinger-Poisson system in $\mathbb{R}^2$

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u), & x \in \mathbb{R}^2, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $V(x)$ and $K(x)$ are both axially symmetric functions. By constructing a new variational framework and using some new analytic techniques, we obtain an axially symmetric solution for the above planar system. our result improves and extends the existing works.

Keywords: Existence; Axially symmetric; Ground state solution; Logarithmic convolution potential; Planar Schrödinger-Poisson system

Mathematics Subject Classification. 35J20; 35J62; 35Q55

1. INTRODUCTION

We consider the following planar Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u), & x \in \mathbb{R}^2, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where K, V and f satisfy the following basic assumptions:

(V1) $V \in C(\mathbb{R}^2, (0, \infty))$, $V(x) = V(x_1, x_2) = V(|x_1|, |x_2|)$, $\forall\, x \in \mathbb{R}^2$ and $\liminf_{|x|\to\infty} V(x) > 0$;

(K1) $K \in C(\mathbb{R}^2, (0, \infty))$, $K(x) = V(x_1, x_2) = K(|x_1|, |x_2|)$, $\forall\, x \in \mathbb{R}^2$ and $\liminf_{|x|\to\infty} K(x) > 0$;

*Corresponding author: qfzhangcsu@163.com (Q. F. Zhang).

This work is supported by the National Natural Science Foundation of China (No. 11961014, No. 61563013) and Guangxi Natural Science Foundation (2018GXNSFAA281021, 2018GXNSFBA281019, 2019GXNSFAA245031).

(F1) $f(u) = o(|u|)$ as $u \to 0$;

(F2) $f \in C(\mathbb{R}, \mathbb{R})$, there exists $c_0 > 0$ and $p > 2$ such that $|f(u)| \leq c_0(1 + |u|^{p-2})$, $\forall\, u \in \mathbb{R}$.

It is pointed out that (V1) and (K1) imply that $V(x)$ and $K(x)$ are both axially symmetric functions. As shown in [1], axially symmetric functions are widely existing in real world, but axially symmetric functions are less used in the existing works because of the lack of compact embedding from the subspace of $H^1(\mathbb{R}^N)$ to $L^s(\mathbb{R}^N)$ for $N \geq 2$, where the elements of the subspace are axially symmetric functions. In recent years, the following nonlinear Schrödinger-Poisson equations have gained more attentions:

$$\begin{cases} -\Delta u + V(x)u + \mu\phi u = f(x,u), & x \in \mathbb{R}^N, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $\mu \in \mathbb{R}\backslash\{0\}$, $V \in (\mathbb{R}^N, (0,\infty))$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. It is easy to see that system (1.1) is a special form of system (1.2).

From [2], we know that system (1.2) comes from semiconductor theory and quantum mechanics theory. In the physical aspects, the solution ϕ of $-\Delta\phi = u^2$ in system (1.2) can be solved by $\phi = \Gamma_N * u^2$, where

$$\Gamma_N(x) = \begin{cases} \frac{\ln|x|}{2\pi}, & N = 2, \\ \frac{|x|^{N-2}}{N(2-N)\omega_N}, & N \geq 3, \end{cases}$$

is the fundamental solutions of the Laplacian, $*$ is the convolution in $\mathbb{R}^N$ and ω_N is the volume of the unit N-ball. With this formal inversion, an integro-differential equation is obtained as follows

$$-\Delta u + V(x)u + \mu(\Gamma_N * u^2)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

When $N = 2$ and $\mu \neq 0$, there are only a few works dealing with system (1.2) or (1.3). Chen, Chen and Tang [3] investigated system (1.2) in the periodic and asymptotically periodic cases using the non-Nehari manifold method derived from [4]. Bernini and Mugnai [5] rewritten a nonlinear planar Schrödinger-Poisson system as a nonlinear Hartree equation and obtained an existence result of radially symmetric solutions when $V(x)$ is a positive constant and $\mu = 1$. If $f(x,u) = f(u)$ and $\mu > 0$, the authors in [6, 7] dealt with periodic case and constructed a variational setting for (1.3). Recently, the author in [8] improved and extended the main results obtained in [7] with $V(x) = 1$ and more general nonlinearity $f(u)$. Very recently, Chen and Tang [9] dealt with axially symmetric potential instead of the periodic case and developed a natural constraint function space for system (1.2). More recently, Chen and Tang [1] considered the case that the nonlinearity is sub-cubic growth at infinity. As pointed out in [1] that this case is more difficult and the methods used in [9] is no longer available since it is not sure whether $\{u_n\}$ are bounded in $H^1(\mathbb{R}^N)$. Motivated by [1], Wen, Chen and Rădulescu [10] studied system (1.1) with $V(x) = 0$ and $K(x)$ is a axially symmetric function and obtained a main result.

When $\mu = 0$, system (1.2) reduces to Schrödinger equations. Many researchers investigated Schrödinger equations and obtained many existence results of nontrivial solutions, see [11, 12, 13, 14, 15, 16, 17, 18] and references therein. However, most of the existing works of Schrödinger equations or Schrödinger-Poisson equations are dealt with one of the following two cases: i) $\inf_{x\in\mathbb{R}^N} V(x) > 0$; ii) $V(x)$ and $K(x)$ vanish at infinity. There is a question: what will happen if $\inf_{x\in\mathbb{R}^2} V(x) > 0$ and $\liminf_{|x|\to\infty} K(x) > 0$ in system (1.1). Moreover, the methods handling the case $N = 3$ are no longer available for $N = 2$ since the integral $\Gamma_2 = \frac{\ln|x|}{2\pi}$ is sign-changing and unbounded, which causes the functional associated with system (1.1) is not well-defined on $H^1(\mathbb{R}^2)$ even if $V \in L^\infty(\mathbb{R}^2)$ and $\inf_{x\in\mathbb{R}^2} V(x) > 0$. As far as we known, there seems no related works in the case of $\inf_{x\in\mathbb{R}^2} V(x) > 0$ and $\liminf_{|x|\to\infty} K(x) > 0$. In this paper, motivated by the aforementioned works, we will give a positive answer and obtain an axially symmetric solution for system (1.1) by establishing a new variational setting and using some new analytic tricks.

To present our result, the following assumptions are needed.

(V2) $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $t \mapsto t^2[2V(tx) - \nabla V(tx)(tx)]$ is nondecreasing on $(0, \infty)$ for all $x \in \mathbb{R}^2$;

(K2) $K \in C^1(\mathbb{R}^2, \mathbb{R})$, $\nabla K(x) \cdot x \leq 0$, $t \mapsto 4K(tx) - \nabla K(tx)(tx)$ is nonincreasing on $(0, \infty)$ for all $x \in \mathbb{R}^2$;

(F3) the function $\frac{f(u)u-F(u)}{u^3}$ is nondecreasing on both $(-\infty, 0)$ and $(0, \infty)$, where and in the sequel, $F(u) = \int_0^u f(s)ds$.

The main result is as follows.

Theorem 1.1. *Suppose that V, K and f satisfy* (V1), (V2), (K1), (K2) *and* (F1)-(F3). *Then (1.1) possesses an axially symmetric solution $\bar{u}$ satisfying*

$$\Phi(\bar{u}) = \inf_{u\in\mathscr{M}} \Phi(u) = \inf_{u\in E\setminus\{0\}} \max_{t>0} \Phi(t^2 u_t) \quad \text{with } \mathscr{M} := \{u \in E\setminus\{0\} : I(u) := 2\langle\Phi'(u), u\rangle - \mathcal{P}(u) = 0\},$$

where $u_t(x) = u(tx)$, the definitions of Φ, E and $\mathcal{P}$ will be given in the next section.

In the next section, we will construct a variational setting and give some preliminaries. In Section 3, we give the proof of Theorem 1.1. Throughout this paper, $\|\cdot\|_{H^1}$ and $\|\cdot\|_s$ denote the norms of $H^1(\mathbb{R}^2)$ and $L^s(\mathbb{R}^2)$ for $1 \leq s \leq \infty$, respectively. C_i are different positive constants in different places.

2. VARIATIONAL SETTING AND PRELIMINARIES

The following bilinear forms are given as

$$(u, v) \mapsto A_1(u, v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(2 + |x - y|)u(x)v(y)dxdy,$$

$$(u, v) \mapsto A_2(u, v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{2}{|x - y|}\right)u(x)v(y)dxdy,$$

and

$$(u, v) \mapsto A_0(u, v) = A_1(u, v) - A_2(u, v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x - y|)u(x)v(y)dxdy,$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are measurable functions. Since u, v are measurable functions, $A_1(u, v)$, $A_2(u, v)$ and $A_0(u, v)$ are well defined in Lebesgue sense. From the Hardy-Littlewood-Sobolev inequality [19] and $0 \leq \ln(1 + t) \leq t$ for $t \geq 0$, we have

$$|A_2(u, v)| \leq \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x - y|}|u(x)v(y)|dxdy \leq C_1\|u\|_{4/3}\|v\|_{4/3}, \tag{2.1}$$

where C_1 is a positive constant. In order to obtain the existence of ground state solution for system (1.1), we develop a new variational framework for system (1.1). The working function space is

$$E := X \cap H^1_{as} = \left\{ u \in H^1_{as}(\mathbb{R}^2) : \int_{\mathbb{R}^2} [V(x) + \ln(2 + |x|)] u^2(x) dx < \infty \right\},$$

where

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} [V(x) + \ln(2 + |x|)] u^2(x) dx < \infty \right\},$$

and

$$H^1_{as} = \{ u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|), \ \forall\, x \in \mathbb{R}^2 \}.$$

Under (V1) and (K1), it is easy to see that the space E is a suitable constraint to study system (1.1). The norms of E is given by

$$\|u\|_E := (\|u\|^2 + \|u\|_*^2)^{\frac{1}{2}}, \tag{2.2}$$

where

$$\|u\|^2 = \int_{\mathbb{R}^2} [|\nabla V(x)|^2 + V(x) u^2(x)] dx, \quad \forall\, u \in X,$$

$$\|u\|_*^2 = \int_{\mathbb{R}^2} \ln(2 + |x|) u^2(x) dx, \quad \forall\, u \in X.$$

The energy functional of system (1.1) on E is given by

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^2} [|\nabla V(x)|^2 + V(x) u^2(x)] dx + \frac{1}{4} A_0(u^2, u^2) - \int_{\mathbb{R}^2} K(x) F(u) dx. \tag{2.3}$$

From (F1), (F2) and [1, (2.9)], we have $\Phi \in C^1(X, \mathbb{R})$ and the embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $s \in [2, \infty)$, moreover,

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x) uv) dx + A_0(u^2, uv) - \int_{\mathbb{R}^2} K(x) f(u) v dx.$$

Now, the Pohozaev functional associated to (1.1) is defined as follows:

$$\begin{aligned} \mathcal{P}(u) \quad := \quad & \frac{1}{2} \int_{\mathbb{R}^2} [\nabla V(x) x + 2V(x)] u^2(x) dx - \int_{\mathbb{R}^2} F(u) \nabla K(x) \cdot x dx \\ & -2 \int_{\mathbb{R}^2} K(x) F(u) dx + A_0(u^2, u^2) + \frac{1}{8\pi} \|u\|_2^4. \end{aligned}$$

Similar to [1], any solution u of (1.1) satisfies $\mathcal{P}(u) = 0$. The following constraint is defined as:

$$\mathscr{M} = \{ u \in E \backslash \{0\} : I(u) = 2\langle \Phi'(u), u \rangle - \mathcal{P}(u) = 0 \},$$

where

$$\begin{aligned} I(u) \quad &= \quad 2\langle \Phi'(u), u \rangle - \mathcal{P}(u) \\ &= \quad 2\|\nabla u\|_2^2 - \frac{1}{8\pi} \|u\|_2^4 - 2 \int_{\mathbb{R}^2} [f(u) u - F(u)] K(x) dx - \int_{\mathbb{R}^2} F(u) \nabla K(x) \cdot x dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^2} [2V(x) - \nabla V(x) \cdot x] u^2(x) dx + A_0(u^2, u^2), \quad \forall\, u \in E. \end{aligned} \tag{2.4}$$

Similar to [1, 10], the following lemmas are obtained.

Lemma 2.1. *Assume that* (V1), (K1), (F1) *and* (F2) *hold. If u is a critical point of Φ restricted to E, then u is a critical point of Φ on X.*

Lemma 2.2. *Assume that* (V1) *and* (V2) *hold. Then*

$$A_1(u^2, v^2) \geq \frac{1}{8\pi}\|u\|_2^2\|v\|_*^2, \quad \forall\ u, v \in E,$$

and there exists a constant $\gamma > 0$ such that

$$\gamma\|u\|_{H^1}^2 \leq 2\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2}[2V(x) - \nabla V(x) \cdot x]u^2 dx + A_1(u^2, u^2), \quad \forall\ u, v \in E.$$

Lemma 2.3. *Assume that* (V1), (V2), (K1), (K2), (F1)-(F3) *hold. Then for all $t > 0$, $u \in \mathbb{R}$ and $x \in \mathbb{R}^2$,*

$$\begin{aligned} g(t, x, u) \quad := \quad & \frac{1}{t^2}F(t^2u)K(t^{-1}x) + \frac{1-t^4}{2}[f(u)u - F(u)]K(x) \\ & -\frac{1-t^4}{2}F(u)\nabla K(x) \cdot x - F(u)K(x) \geq 0. \end{aligned} \tag{2.5}$$

Lemma 2.4. *Assume that* (V2) *holds. Then*

$$\alpha(t, x) := (1 + t^4)V(x) + \frac{1-t^4}{2}\nabla V(x) \cdot x - 2t^2V(t^{-1}x) \geq 0, \quad \forall\ x \in \mathbb{R}^2,\ t > 0. \tag{2.6}$$

3. PROOF OF THEOREM 1.1

In this section, we first establish an energy estimate inequality related to $\Phi(u)$, $\Phi(t^2u_t)$ and $I(u)$, where

$$\begin{aligned} \Phi(t^2u_t) \quad = \quad & \frac{t^4}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} t^2V(t^{-1}x)u^2dx + \frac{t^4}{4}A_0(u^2, u^2) \\ & -\frac{t^4 \ln t}{8\pi}\|u\|_2^4 - \frac{1}{t^2}F(t^2u)K(t^{-1}x)dx, \quad \forall\ u \in E,\ t > 0. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then*

$$\Phi(u) \geq \Phi(t^2u_t) + \frac{1-t^4}{4}I(u) + \frac{1 - t^4 + 4\ln t}{32}\|u\|_2^4, \quad \forall\ t > 0,\ u \in E, \tag{3.2}$$

and

$$\Phi(u) \geq \frac{1}{4}I(u) + \frac{1}{32\pi}\|u\|_4^2, \quad \forall\ u \in E. \tag{3.3}$$

Proof. From (2.3), (2.5), (2.6) and (3.1), we have

$$\begin{aligned}
\Phi(u) - \Phi(t^2 u_t) &= \frac{1-t^4}{2}\|\nabla u\|_2^2 + \frac{1-t^4}{4}A_0(u^2,u^2) + \frac{1}{2}\int_{\mathbb{R}^2}[V(x) - t^2 V(t^{-1}x)]u^2 dx \\
&\quad + \frac{t^4 \ln t}{8\pi}\|u\|_2^4 + \int_{\mathbb{R}^2}\frac{1}{t^2}F(t^2 u)K(t^{-1}x)dx - \int_{\mathbb{R}^2}F(u)K(x)dx \\
&= \frac{1-t^4}{4}I(u) + \frac{1}{4}\int_{\mathbb{R}^2}\left[(1+t^4)V(x) + \frac{1-t^4}{2}\nabla V(x)\cdot x - 2t^2 V(t^{-1}x)\right]u^2 dx \\
&\quad + \frac{1-t^4+4\ln t}{32}\|u\|_2^4 + \int_{\mathbb{R}^2}\left\{\frac{1}{t^2}F(t^2u)K(t^{-1}x) + \frac{1-t^4}{2}[f(u)u - F(u)]K(x)\right. \\
&\quad \left. - \frac{1-t^4}{4}F(u)\nabla K(x)\cdot x - F(u)K(x)\right\} dx \\
&\geq \frac{1-t^4}{4}I(u) + \frac{1-t^4+4\ln t}{32}\|u\|_2^4, \quad \forall\, t>0,\ u\in E. \qquad (3.4)
\end{aligned}$$

From (F1)-(F3), we have

$$\beta(t,u) := \frac{1-t^4}{2}f(u)u + \frac{t^4-3}{2}F(u) + \frac{1}{t^2}F(t^2u) \geq 0, \quad \forall\, t>0,\ u\in\mathbb{R}. \qquad (3.5)$$

By (3.5), we get

$$\lim_{t\to 0}\beta(t,u) = \frac{1}{2}[f(u)u - 3F(u)] \geq 0. \qquad (3.6)$$

From (V2) and (2.6), we have

$$(1+t^4)V(x) + \frac{1-t^4}{2}\nabla V(x)\cdot x \geq 2t^2 V(t^{-1}x) \geq 0, \quad \forall\, t>0,\ x\in\mathbb{R}^2. \qquad (3.7)$$

Let $t\to 0$ in (3.7), we obtain

$$2V(x) + \nabla V(x)\cdot x \geq 0, \quad \forall\, x\in\mathbb{R}^2. \qquad (3.8)$$

From (K1), (K2), (2.3), (2.4), (3.6) and (3.8), we get

$$\begin{aligned}
\Phi(u) - \frac{1}{4}I(u) &= \frac{1}{32\pi}\|u\|_4^2 + \frac{1}{8}\int_{\mathbb{R}^2}[2V(x) + \nabla V(x)\cdot x]u^2 dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}[f(u)u - 3F(u)]K(x)dx - \frac{1}{4}\int_{\mathbb{R}^2}F(u)\nabla K(x)\cdot x dx \\
&\geq \frac{1}{32\pi}\|u\|_4^2, \quad \forall\, u\in E. \qquad (3.9)
\end{aligned}$$

It follows from (3.4) and (3.9) that Lemma 3.1 holds. $\square$

From (3.2) and the fact that $1 + 4t^4\ln t - t^4 \geq 0$ for $t>0$, the following corollary is obtained.

Corollary 3.2. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then* $\Phi(u) = \max_{t>0}\Phi(t^2u_t)$ *for all* $u\in\mathscr{M}$.

Lemma 3.3. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then for any* $u\in E\backslash\{0\}$, *there exists a constant* $t_u>0$ *such that* $t_u^2 u_{t_u}\in\mathscr{M}$.

Proof. Fix $u \in E\backslash\{0\}$ and define $\xi(t) := \Phi(t^2u_t)$ on $(0,\infty)$. From (2.4) and (3.1), one has

$$\begin{aligned}\xi'(t) = 0 \quad \Leftrightarrow \quad & 2t^2\|\nabla u\|_2^2 + t^3A_0(u^2,u^2) + \frac{t}{2}\int_{\mathbb{R}^2}[2V(t^{-1}x) - \nabla V(t^{-1}x)\cdot t^{-1}x]u^2dx \\ & -\frac{4t^3\ln t + t^3}{8\pi}\|u\|_2^4 + \frac{1}{t^3}\int_{\mathbb{R}^2}F(t^2u)\nabla K(t^{-1}x)\cdot(t^{-1}x)dx \\ & -\frac{2}{t^3}\int_{\mathbb{R}^2}[f(t^2u)t^2u - F(t^2u)]K(t^{-1}x)dx = 0 \\ \Leftrightarrow \quad & I(t^2u_t) = 0 \Leftrightarrow t^2u_t \in \mathscr{M}, \quad \forall\, t > 0.\end{aligned}$$

From (3.6), we have

$$\frac{F(u)}{u^3} \text{ is nondecreasing on } (-\infty,0)\cup(0,+\infty). \tag{3.10}$$

By (V2), we have

$$2V(x) - \nabla V(x)x \geq 0, \quad \forall\, x \in \mathbb{R}^2, \tag{3.11}$$

and

$$t^{-2}[2V(t^{-1}x) - \nabla V(t^{-1}x)t^{-1}x] \leq 2V(x) - \nabla V(x)x, \quad \forall\, t > 1,\ x \in \mathbb{R}^2. \tag{3.12}$$

By (K2), we obtain

$$-2K(x) \leq \nabla K(x)x \leq 2K(x), \quad \forall\, x \in \mathbb{R}^2. \tag{3.13}$$

It follows from (F3), (3.10), (3.11), (3.12) and (3.13) that

$$\frac{\xi'(t)}{t^3} \geq 2\|\nabla u\|_2^2 + A_0(u^2,u^2) - \frac{4\ln t + 1}{8\pi}\|u\|_2^4 - 2K_\infty\int_{\mathbb{R}^2}f(u)udx, \quad \forall\, 0 < t < 1 \tag{3.14}$$

and

$$\begin{aligned}\frac{\xi'(t)}{t^3} \quad \leq \quad & 2\|\nabla u\|_2^2 + A_0(u^2,u^2) + \frac{1}{2}\int_{\mathbb{R}^2}[2V(x) - \nabla V(x)x]u^2dx \\ & -\frac{4\ln t + 1}{8\pi}\|u\|_2^4 + 4K_\infty\int_{\mathbb{R}^2}F(u)dx, \quad \forall\, t > 1.\end{aligned} \tag{3.15}$$

Then, from (3.14) and (3.15), for $t \in (0,1)$ small enough, one has $\xi'(t) > 0$ and for $t > 1$ large enough, $\xi'(t) < 0$. Hence, there exists $t_u > 0$ such that $\xi'(t_u) = 0$ and $t_u^2u_{t_u} \in \mathscr{M}$. □

From Corollary 3.2 and Lemma 3.3, we get the following lemma.

Lemma 3.4. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then*

$$\inf_{u\in\mathscr{M}}\Phi(u) := c = \inf_{u\in E\backslash\{0\}}\max_{t>0}\Phi(t^2u_t). \tag{3.16}$$

Lemma 3.5. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then* $c = \inf_{u\in\mathscr{M}}\Phi(u) > 0$.

Proof. By a standard argument, by (F1), (F2) and $I(u) = 0$ for $u \in \mathscr{M}$, one can easily show that there exists $\sigma > 0$ such that $\|u\|_{H^1} \geq \sigma$, $\forall\, u \in \mathscr{M}$. Let $\{u_n\} \subset \mathscr{M}$ be such that $\Phi(u_n) \to c$. We consider two cases:

Case 1). $\inf_{n\in\mathbb{N}} \|u\|_2 := \sigma_1 > 0$. From (3.3), we get

$$c + o(1) = \Phi(u_n) \geq \frac{1}{32\pi}\|u_n\|_2^4 \geq \frac{1}{32\pi}\sigma_1^4.$$

Case 2). $\inf_{n\in\mathbb{N}} \|u\|_2 := 0$. Since $\|u\|_{H^1} \geq \sigma$ for all $u \in \mathscr{M}$, passing to a subsequence, one obtains

$$\|u_n\|_2 \to 0, \quad \|\nabla u_n\|_2 \geq \frac{\sigma}{2}. \tag{3.17}$$

From (2.1) and the Gagliardo-Nirenberg inequality, we get

$$0 \leq A_2(u_n^2, u_n^2) \leq C_1\|u_n\|_{8/3}^4 \leq C_2\|u_n\|_2^3\|\nabla u_n\|_2, \quad \|u_n\|_p^p \leq C_3\|u_n\|_2^2\|\nabla u_n\|_2^{p-2}. \tag{3.18}$$

From (3.17), we have

$$\frac{|\ln(\|\nabla u_n\|_2)|}{\|\nabla u_n\|_2^2} \leq C_4. \tag{3.19}$$

Let $t_n = \|\nabla u_n\|_2^{-1/2}$. Since $I(u_n) = 0$, from (F1), (F2), (3.1), (3.17), (3.18), (3.19) and Corollary 3.2, we have

$$\begin{aligned}
c + o(1) &= \Phi(u_n) \geq \Phi(t_n^2(u_n)_{t_n}) \\
&= \frac{t_n^4}{2}\|\nabla u_n\|_2^2 + \frac{t_n^4}{4}[A_1(u_n^2, u_n^2) - A_2(u_n^2, u_n^2)] - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V(t_n^{-1}x)u_n^2 dx - \frac{1}{t_n^2}\int_{\mathbb{R}^2} K(t_n^{-1}x)F(t_n^2 u_n)dx \\
&\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}A_2(u_n^2, u_n^2) - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 - \frac{\|K\|_\infty}{t_n^2}\int_{\mathbb{R}^2}[|t_n^2 u_n|^2 + C_5|t_n^2 u_n|^p]dx \\
&\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{C_2 t_n^4}{4}\|u_n\|_2^3\|\nabla u_n\|_2 - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad - t_n^2\|K\|_\infty\|u_n\|_2^2 - C_6\|K\|_\infty t_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} \\
&= \frac{1}{2} - \frac{C_2\|u_n\|_2^3}{4\|\nabla u_n\|_2} + \frac{\ln(\|\nabla u_n\|_2)}{16\pi\|\nabla u_n\|_2^2}\|u_n\|_2^4 - \frac{\|K\|_\infty\|u_n\|_2^2}{\|\nabla u_n\|_2} - \frac{C_6\|K\|_\infty\|u_n\|_2^2}{\|\nabla u_n\|_2} \\
&= \frac{1}{2} + o(1).
\end{aligned}$$

It follows from the above two cases that $c = \inf_{u\in\mathscr{M}} \Phi(u) > 0$. □

Lemma 3.6. *Assume that* (V1), (V2), (K1), (K2) *and* (F1)-(F3) *hold. Then c is achieved. Moreover, if $\bar{u} \in \mathscr{M}$ and $\Phi(\bar{u}) = c$, then $\bar{u}$ is a critical point of Φ in E.*

Proof. Let $\{u_n\} \subset \mathscr{M}$ be such that $\Phi(u_n) \to c$. From (3.9) and $I(u_n) = 0$, it yields

$$c + o(1) = \Phi(u_n) - \frac{1}{4}I(u_n) \geq \frac{1}{32\pi}\|u_n\|_2^4. \tag{3.20}$$

From (3.20), we know that $\{\|u_n\|_2\}$ is bounded. It is needed to prove that $\{\|\nabla u_n\|_2\}$ is bounded too. Arguing by indirectly, assume that $\|\nabla u_n\|_2 \to \infty$ as $n \to \infty$. Let $t_n = (2\sqrt{c}/\|\nabla u_n\|_2)^{1/2}$, then $t_n \to 0$ as

$n \to \infty$. Hence, $t_n^4 \ln t_n \to 0$ as $n \to \infty$. From (K1), (F1), (F2), (2.1), (3.1), (3.18) and Corollary 3.2, we have

$$\begin{aligned} c + o(1) &= \Phi(u_n) \geq \Phi(t_n^2 (u_n)_{t_n}) \\ &= \frac{t_n^4}{2}\|\nabla u_n\|_2^2 + \frac{t_n^4}{4}[A_1(u_n^2, u_n^2) - A_2(u_n^2, u_n^2)] - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 \\ &\quad + \frac{t_n^2}{2}\int_{\mathbb{R}^2} V(t_n^{-1}x)u_n^2 dx - \frac{1}{t_n^2}\int_{\mathbb{R}^2} K(t_n^{-1}x)F(t_n^2 u_n)dx \\ &\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}A_2(u_n^2, u_n^2) - \frac{\|K\|_\infty}{t_n^2}\int_{\mathbb{R}^2} F(t_n^2 u_n)dx + o(1) \\ &\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}A_2(u_n^2, u_n^2) - \frac{\|K\|_\infty}{t_n^2}\int_{\mathbb{R}^2} [|t_n^2 u_n|^2 + C_5|t_n^2 u_n|^p]dx + o(1) \\ &\geq \frac{t^4}{2}\|\nabla u_n\|_2^2 - \frac{C_2 t_n^4}{4}\|u_n\|_2^3\|\nabla u_n\|_2 - t_n^2\|K\|_\infty\|u_n\|_2^2 \\ &\quad - C_6\|K\|_\infty t_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} + o(1) \\ &= 2c - \frac{C_2 c\|u_n\|_2^3}{\|\nabla u_n\|_2} - \frac{C_3(2\sqrt{c})^{p-1}\|K\|_\infty\|u_n\|_2^2}{\|\nabla u_n\|_2} + o(1) \\ &= 2c + o(1), \end{aligned} \tag{3.21}$$

a contradiction, hence, we have that $\{\|\nabla u_n\|_2\}$ is bounded too, so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. Thanks to (K1), (F1), (F2), (2.1) and (2.2), we know that $\{\|u_n\|\}$ and $A_1(u_n^2, u_n^2)$ are both bounded. From [1, Lemma 3.5], one has

$$\limsup_{n\to\infty} \|u_n\|_2 > 0, \tag{3.22}$$

which together with Lemma 2.2 shows that $\{\|u_n\|_*\}$ is bounded. Then $\{u_n\}$ is bounded in E. Passing to a subsequence, one may assume that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \in [2, \infty)$, $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. By a standard argument, we have

$$I(\bar{u}) \leq \liminf_{n\to\infty} I(u_n) = 0. \tag{3.23}$$

It follows from (3.22) and (3.23) that $\bar{u} \neq 0$. From Lemma 3.3, there exists $\bar{t} > 0$ such that $\bar{t}^2\bar{u}_{\bar{t}} \in \mathscr{M}$ and $\Phi(\bar{t}^2\bar{u}_{\bar{t}}) \geq c$. Hence, by (2.2), (2.4), (3.22), Fatou's Lemma, Lebesgue's dominated convergence theorem and the fact $1+4t^4\ln t - t^4 \geq 0$ for $t>0$, we obtain

$$\begin{aligned}
c &= \lim_{n\to\infty}\left[\Phi(u_n) - \frac{1}{4}I(u_n)\right] \\
&= \lim_{n\to\infty}\left\{\frac{1}{32\pi}\|u_n\|_4^2 + \frac{1}{8}\int_{\mathbb{R}^2}[2V(x)+\nabla V(x)\cdot x]u_n^2dx \right.\\
&\quad \left. + \frac{1}{2}\int_{\mathbb{R}^2}\{[f(u_n)u_n - 3F(u_n)]K(x) - F(u_n)\nabla K(x)\cdot x\}dx\right\} \\
&\geq \frac{1}{32\pi}\|\bar{u}\|_4^2 + \frac{1}{8}\int_{\mathbb{R}^2}[2V(x)+\nabla V(x)\cdot x]\bar{u}^2dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}[f(\bar{u})\bar{u} - 3F(\bar{u})]K(x)dx - \frac{1}{4}\int_{\mathbb{R}^2}F(\bar{u})\nabla K(x)\cdot xdx \\
&= \Phi(\bar{u}) - \frac{1}{4}I(\bar{u}) \geq \Phi(\bar{t}^2\bar{u}_{\bar{t}}) - \frac{\bar{t}^4}{4}I(\bar{u}) \geq c - \frac{\bar{t}^4}{4}I(\bar{u}) \geq c.
\end{aligned} \tag{3.24}$$

From (3.24), we have $I(\bar{u})=0$ and $\Phi(\bar{u})=c$. Similar to [1, Lemma 4.1], we can obtain that $\bar{u}$ is a critical point of Φ in E.

From Lemmas 2.1, 3.4 and 3.6, it is easy to get Theorem 1.1. The proof is complete.

□

ACKNOWLEDGEMENTS

The authors would like to thank the editors and referees for their useful suggestions and comments.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Qiongfen Zhang https://orcid.org/0000-0002-7037-1961

References

[1] Chen ST, Tang XH. On the planar Schrödinger-Poisson system with the axially symmetric potential. J. Differential Equations 2020; 268: 945-976.

[2] Benguria R, Brezis H, Lieb E. The Thomas-Fermi-von Weizsäcker theory of atoms and molecules. Comm. Math. Phys. 1981; 79: 167-180.

[3] Chen J, Chen ST, Tang XH. Ground state solutions for asymptotically periodic Schrödinger-Poisson systems in $\mathbb{R}^2$. Electronic J. Differential Equations 2018; 192: 1-18.

[4] Tang XH. Non-Nehari manifold method for asymptotically periodic Schrödinger equation. Sci. China Math. 2015; 58: 715-728.

[5] Bernini F, Mugnai D. On a logarithmic Hartree equation. Adv. Nonlinear Anal. 2020; 9: 850-865.

[6] Cingolani S, Weth T. On the planar Schrödinger-Poisson system. Ann. Inst. Henri Poincaré, Anal. Non Linéaire 2016; 33: 169-197.

[7] Du M, Weth T. Ground states and high energy solutions of the planar Schrödinger-Poisson system. Nonlinearity 2017; 30: 3492-3515.

[8] Chen ST, Shi JP, Tang XH. Ground state solutions of Nehari-Pohozaev type for the planar Schrödinger-Poisson system with general nonlinearity. Discrete Contin. Dyn. Syst., Ser. A 2019; 39: 5867-5889.

[9] Chen ST, Tang XH. Existence of ground state solutions for the planar axially symmetric Schrödinger-Poisson system. Discrete Contin. Dyn. Syst., Ser. B 2019; 24: 4685-4702.

[10] Wen LX, Chen ST, Rădulescu VD. Axially symmetric solutions of Schrödinger-Poisson system with zero mass potential in $\mathbb{R}^2$. Appl. Math. Lett. 2020; 104: 106244.

[11] Chen ZJ, Zou WM. Ground states for a system of Schrödinger equations with critical exponent. J. Func. Anal. 2012; 262: 3091-3107.

[12] Chen ST, Tang XH. Berestycki-Lions conditions on ground state solutions for a nonlinear Schrödinger equation with variable potentials. Adv. Nonlinear Anal. 2020; 9: 496-515.

[13] Peng JY, Chen ST, Tang XH. Semiclassical solutions for linearly coupled Schrödinger equations without compactness. Complex Variables and Elliptic Equations 2019; 64: 548-556.

[14] Tang XH. New super-quadratic conditions for asymptotically periodic Schrödinger equations. Canadian Mathematical Bulletin-Bulletin Canadien DE Mathematiques 2017; 60: 422-435.

[15] Tang XH, Lin XY, Yu JS. Nontrivial solutions for Schrödinger equation with local super-quadratic conditions. J. Dyn. Diff. Equat. 2019; 31: 369-383.

[16] Che GF, Chen HB. Existence of multiple nontrivial solutions for a class of quasilinear Schrödinger equations on $\mathbb{R}^N$. Bull. of the Belg. Math. Soc. Simon Stevin 2018; 25: 39-53.

[17] Tang XH. Non-Nehari manifold method for asymptotically linear Schrödinger equation. J. Aust. Math. Soc. 2015; 98: 104-116.

[18] Sirakov B. Standing wave solutions of the nonlinear Schrödinger equations in $\mathbb{R}^N$. Ann Mat Pura Appl. 2002; 183: 73-83.

[19] Lieb E, Loss M. Analysis, 2nd ed., Graduate Studies in Mathematics, vol.14, American Mathematical Society, Providence, RI, 2001.