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Exact solution of the multi-component generalized six-vertex model

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A general q-component ($q \geqslant 2$) solution of the star–triangle equation under an "ice"-type restriction is considered. The Bethe ansatz equations are derived explicitly for it. The exact solution is presented for any q and reported in detail for $q = 3$. The free energy and the excitation energies are found as functions of the spectral and anisotropy parameters as well as the finite-size corrections yielding the central charges and conformal dimensions. A new QFT is associated to these models where the mass spectrum and the S-matrix are obtained through the light-cone approach. The new feature of this model is to present a mass spectrum dependent on the anisotropy parameter.

1. Introduction

The construction of exact solutions of 2D integrable statistical models has made impressive progress in recent years [1,2]. Eigenvalues and eigenvectors of these models have been constructed by means of the Bethe ansatz and its nested version. One of the first to be treated was the six-vertex model. This model was diagonalized for the q-component ($q \geqslant 2$) case [3] and for its generalization [4]. The solution of the Bethe ansatz equations is presented here for the last case.

The multi-component generalized six-vertex model, also called Perk–Schultz model, is defined by a multi-state "ice"-type condition associated to the vertex weights. These weights are a general solution of the Yang–Baxter equation which ensures integrability. Besides the dependence on the anisotropy (γ) and the spectral (θ) parameters, the weights present two new parameters: $G_{\sigma\rho}$ ($1 \leqslant \sigma \leqslant \rho \leqslant q$) and a discrete one $\epsilon_\rho = \pm 1$. Their influence in the solution of the Bethe ansatz equations is not the same. We will see that the solutions are dramatically affected by ϵ_ρ that defines the ferroelectric or antiferroelectric character of the weights. On the other hand, the $G_{\sigma\rho}$ parameter is equivalent to an external field.

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In this paper we sketch the algebraic Bethe ansatz construction of the eigenvalues and the eigenvectors. The solution of the Bethe ansatz equations is presented for q states per bound in the trigonometric regime, that is, when the weights are trigonometric functions of ϵ_ρ, γ and θ. The solution is derived in detail for $q=3$. We obtain the expressions for the free energy and excitation energies as functions of the spectral (θ) and the anisotropy (γ) parameters besides the discrete parameter ϵ_ρ. This lattice model yields a solvable quantum field theory and a conformal model in appropriated scaling limits within the lattice light-cone approach. One of the new features of the generalized model is to present a mass spectrum dependent on the anisotropy parameter. This fact is due to the presence of the discrete parameter ϵ_ρ.

A brief account of the present work has been reported in ref. [13].

2. The model

Let us consider a bi-dimensional lattice of order $M \times N$ with q possible colours on the lattice bonds. Four bonds come together at each vertex of the lattice, so if the number of colours is q there are q^4 distinct types of combinations of colours at a vertex. We ascribe to each allowed combination a positive number e_j ($j = 1, 2, \dots, q^4$), and then we associate a total energy, which is defined as the sum of the energies of the vertex:

$$E = \sum_{j=1}^{q^4} N_j e_j, \tag{1}$$

where N_j is the number of vertices with a combination of colours of type j in a given configuration. As a result we obtain a model of interacting colours situated along the edges of the lattice.

The partition function of the system writes

$$Z = \sum \exp(-\beta E), \tag{2}$$

where the summation is over all configurations of colours on the lattice, and E is the total energy of a configuration defined by (1). The parameter β is inversely proportional to the temperature, and $v_j = \exp(-\beta e_j)$ ($j = 1, 2, \dots, q^4$) are the Boltzmann weights.

The free energy is defined in the thermodynamic limit as

$$\beta f = -\lim_{M,N \to \infty} \frac{1}{MN} \log Z. \tag{3}$$

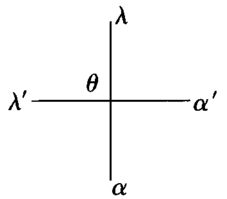

Fig. 1. $R^{\lambda'\lambda}_{\alpha\alpha'}(\theta)$.

It is useful to rewrite the Boltzmann weights in a more appropriate form, that is $R^{\lambda'\lambda}_{\alpha\alpha'}(\theta)$ for the combination of colours in fig. 1. Then, horizontal and vertical bonds can have a local colour belonging to the vector space $\mathscr{V}$ where $q = \dim \mathscr{V}$. We associate to a horizontal line of the lattice the monodromy operator $T_{ab}(\theta)$ defined by

$$T^{(N)}_{ab}(\theta) = \sum_{\lambda_1 \dots \lambda_{N-1}=1}^{q} t^{(1)}_{a\lambda_1}(\theta) \otimes t^{(2)}_{\lambda_1\lambda_2}(\theta) \otimes t^{(3)}_{\lambda_2\lambda_3}(\theta) \otimes \dots t^{(N)}_{\lambda_{N-1}b}(\theta), \tag{4}$$

where $[t_{\alpha'\lambda'}(\theta)]_{\alpha\lambda} = R^{\lambda'\lambda}_{\alpha\alpha'}(\theta)$ and $t^{(k)}_{\alpha'\lambda'}(\theta)$ acts in the q-dimensional vertical space $\mathscr{V}$ associated to the kth column of the lattice.

Considering periodic conditions in the horizontal direction we obtain the transfer matrix $\tau^{(N)}(\theta)$, defined as the trace of the monodromy matrix over the horizontal indices,

$$\tau(\theta) = \mathrm{Tr}_{\mathscr{V}} T^{[N]} = \sum_{a=1}^{q} T^{[N]}_{aa}(\theta), \tag{5}$$

and the expression (2) for Z becomes an outer summation over the vertical, and inner over the horizontal configurations, so

$$Z = \mathrm{Tr}_{\mathscr{V}}\left[\tau^{[N]}(\theta)^M\right], \tag{6}$$

where $\mathrm{Tr}_{\mathscr{V}}$ means the trace over the space $\mathscr{V}$.

With this transformation the problem of calculating the statistical sum (2) is reduced to a problem in quantum mechanics, namely the calculation of the eigenvalues of the transfer matrix $\tau^{(N)}(\theta)$.

The model considered here, that is, the generalized six-vertex model, is defined by a one-parameter family of vertex weights $R(\theta)$ with not q^4 but $q(2q-1)$ non-zero types of combinations of colours at a vertex. The only configurations

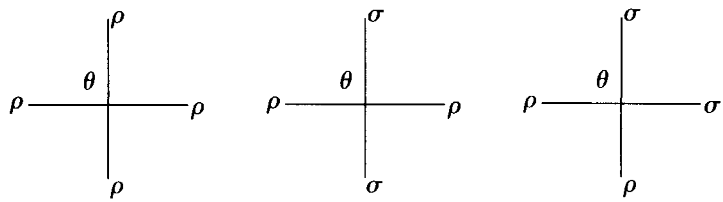

Fig. 2. Allowed configurations.

allowed to be nonzero are described in fig. 2. We have

$$\begin{aligned} R_{\rho\rho}^{\rho\rho}(\theta) &= \frac{\sin(\gamma+\epsilon_\rho\theta)}{\sin\gamma}, \\ R_{\sigma\rho}^{\rho\sigma}(\theta) &= G_{\rho\sigma}\frac{\sin(\theta)}{\sin\gamma} \quad \rho\neq\sigma, \\ R_{\rho\sigma}^{\rho\sigma}(\theta) &= \mathrm{e}^{i\theta\,\mathrm{sign}(\rho-\sigma)}, \quad \rho\neq\sigma; \end{aligned} \tag{7}$$

all other $R_{\alpha'\alpha}^{\lambda'\lambda}(\theta)=0$. Here $\epsilon_\rho=\pm1$, $G_{\rho\sigma}G_{\sigma\rho}^{-1}=1$ (no sum on ρ, σ) and $1\leqslant\rho$, $\sigma\leqslant q$.

We note that the weights with $\epsilon_\rho=1$ favour ferroelectric configurations, that is, all links with the same colour. Otherwise, for $\epsilon_\rho=-1$, the weights make more probable alternating coloured configurations. In this case we can obtain an antiferroelectric behavior if all colours are associated to $\epsilon_\rho=-1$ or a ferrielectric behavior if we have a mixed configuration of colours associated to $\epsilon_\rho=-1$ and $\epsilon_\rho=1$.

Notice that reversing signs of all $\epsilon_\rho(1\leqslant\rho\leqslant q)$ is equivalent to changing γ into $\pi-\gamma$.

The generalized model is a solution of the Yang–Baxter equation [5]

$$R(\theta-\theta')[T(\theta)\otimes T(\theta')]=[T(\theta')\otimes T(\theta)]R(\theta-\theta'). \tag{8}$$

This condition is sufficient to ensure the integrability of the model.

If $q=2$, then eq. (7) gives the weights of a six-vertex model with three possible regimes depending on the values of ϵ_1 and ϵ_2. For $\epsilon_1=\epsilon_2=-1$ or $+1$ we obtain respectively the ordinary six-vertex model in the antiferroelectric and ferroelectric regime. For $\epsilon_1\neq\epsilon_2$ we obtain a six-vertex model with a ferrielectric character. If $q>2$ then we have a multi-component generalization of the six-vertex model with the three possible regimes. In fact, the weights of (7) are found to be a general solution of the Yang–Baxter equation (8) under the restriction of a generalized "ice" rule, which state that, of the $q^4R_{\alpha'\alpha}^{\lambda'\lambda}$ only $R_{\rho\rho}^{\sigma\sigma}$, $R_{\rho\rho}^{\rho\rho}$ and $R_{\rho\sigma}^{\sigma\rho}$ are different from zero. This restriction requires the vertices to obey a conservation law, just as in the case of the original ice rule. That is, if λ' and λ have the values ρ and σ than α' and α must also be ρ and σ or σ and ρ.

A more general solution is obtained if we consider vertical field factors. In this case the R-matrix reads

$$R^{\lambda'\lambda}_{\alpha\alpha'}(\theta) \to b_\alpha b'_\alpha R^{\lambda'\lambda}_{\alpha\alpha'}(\theta). \tag{9}$$

So, the transfer matrix becomes

$$T_{\alpha'\alpha}(\theta) \to b_1^{2N_1} b_2^{2N_2} \dots b_q^{2N_q} T_{\alpha'\alpha}(\theta). \tag{10}$$

As we will see later $T(\theta)$ is block diagonal, where each block is specified by a set of integers $N_1, N_2, \dots, N_q$, and hence, the transfer matrix with vertical fields can be diagonalized simply by adding the field factors to the expression (23) for the eigenvalues, according to (10). In this way we will not consider this parameter in the solution of the model treated in the following sections.

3. Algebraic Bethe ansatz

Let us now sketch how to construct the exact eigenvectors and eigenvalues of the transfer matrix $\tau^{[N]}(\theta)$ using the nested Bethe ansatz.

Taking the ferroelectric state

$$\|1\rangle = \otimes_{s=1}^{N} |1\rangle^{(s)}, \tag{11}$$

where the q-component vectors $|1\rangle$ have all components zero except for the first that equals one, as the reference state, we observe that applying the monodromy operators $T^{[N]}_{ab}(\theta, \tilde{\alpha})$ to this state we get

$$\begin{aligned}
&T^{[N]}_{11}(\theta, \tilde{\alpha}) \|1\rangle = \prod_{s=1}^{N} \frac{\sin[\gamma + \epsilon_1(\theta - \alpha_s)]}{\sin\gamma} \|1\rangle, \\
&T^{[N]}_{kk}(\theta, \tilde{\alpha}) \|1\rangle = \prod_{s=1}^{N} G_{1k} \frac{\sin(\theta - \alpha_s)}{\sin\gamma} \|1\rangle, \quad 2 \leqslant k \leqslant q, \\
&T^{[N]}_{ki}(\theta, \tilde{\alpha}) \|1\rangle = 0, \quad k \neq i, \quad 2 \leqslant i \leqslant q, \\
&T^{[N]}_{1i}(\theta, \tilde{\alpha}) \|1\rangle \neq 0,
\end{aligned} \tag{12}$$

where $\tilde{\alpha} = (\alpha_1, \dots, \alpha_N)$ is an inhomogeneity that varies from site to site.

The last operator is the only one that gives new states when applied to the reference state $\|1\rangle$. Then, to obtain all the possible physical states we apply the operator $T^{[N]}_{1i}(\theta, \tilde{\alpha})$ on the reference state many times. In this way we obtain states of antiferroelectric or ferrielectric character from $\|1\rangle$.

The properties of the reference state $\|1\rangle$ suggest to decompose $T_{ab}^{[N]}$ and R in blocks of the following form:

$$T^{[N]}(\theta, \alpha) = \begin{pmatrix} A(\theta, \tilde{\alpha}) & B_j(\theta, \tilde{\alpha}) \\ C_i(\theta, \tilde{\alpha}) & D_{ij}(\theta, \tilde{\alpha}) \end{pmatrix}, \tag{13}$$

where

$$A(\theta, \tilde{\alpha}) = T_{11}^{[N]}(\theta, \tilde{\alpha}), \quad B_j(\theta, \tilde{\alpha}) = T_{1j}^{[N]}(\theta, \tilde{\alpha}), \quad C_i(\theta, \tilde{\alpha}) = T_{i1}^{[N]}(\theta, \tilde{\alpha})$$

$$D_{ij}(\theta, \tilde{\alpha}) = T_{ij}^{[N]}(\theta, \tilde{\alpha}), \quad 2 \leqslant i, j \leqslant q.$$

and

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_{ij} b_1^- & \delta_{ij} c_1^+ & 0 \\ 0 & \delta_{ij} c_1^- & \delta_{ij} b_1^+ & 0 \\ 0 & 0 & 0 & R_1^{(2)} \end{pmatrix}, \tag{14}$$

where

$$b_1^{\pm} = \frac{\sin \gamma}{\sin[\gamma + \epsilon_1(\theta)]} e^{\pm\theta}, \qquad R_1^{(2)} = \frac{\sin \gamma}{\sin[\gamma + \epsilon_1(\theta)]} R_{ab}^{ij(2)}, \quad 2 \leqslant i, j, a, b \leqslant q,$$

$$c_1^{\pm} = \frac{\sin \theta}{\sin[\gamma + \epsilon_1(\theta)]} G_{1j}^{\pm\theta}, \qquad G_{1j}^{-} = G_{j1}.$$

The bilinear algebra for the operators $A(\theta)$, $B_j(\theta)$, $C_i(\theta)$ and $D_{ij}(\theta)$ follows by inserting eqs. (13) and (14) in eq. (8). One finds

$$[A(\theta), A(\theta')] = 0, \tag{15}$$

$$B(\theta) \otimes B(\theta') = [B(\theta') \otimes B(\theta)] R^{(2)}(\theta - \theta') \frac{\sin \gamma}{\sin[\gamma + \epsilon_1(\theta - \theta')]}, \tag{16}$$

$$A(\theta) B(\theta') = g_1^+(\theta' - \theta) B(\theta') A(\theta) - h_1^+(\theta' - \theta) B(\theta) A(\theta'), \tag{17}$$

$$D(\theta) \otimes B(\theta') = g_1^+(\theta - \theta') [B(\theta') \otimes D(\theta)] R^{(2)}(\theta - \theta') \frac{\sin \gamma}{\sin[\gamma + \epsilon_1(\theta - \theta')]}$$

$$- h_1^+(\theta - \theta') [B(\theta) \otimes D(\theta')], \tag{18}$$

where

$$g_1^+(\theta) = \frac{1}{c_1^+} = G_{j1}\frac{\sin[\gamma + \epsilon_1(\theta)]}{\sin\theta},$$

$$h_1^\pm(\theta) = \frac{b_1^\pm}{c_1^\pm} = G_{j1}\,\mathrm{e}^{\pm\theta}\frac{\sin\gamma}{\sin\theta}.$$

We are interested in the eigenstate of $\tau^{[N]}(\theta, \tilde{\alpha})$ having the following structure:

$$\begin{aligned}\Psi\left(\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_{p_1}^{(1)}\right) &= \sum_{i_1,\dots,i_{p_1}=2}^{q} X^1_{i_1,\dots,i_{p_1}} B_{i_1}\left(\lambda_1^{(1)}\right) \otimes \dots \otimes B_{i_{p_1}}\left(\lambda_{p_1}^{(1)}\right) \|1\rangle \\ &= X_1^{\mathrm{t}} B\left(\lambda_1^{(1)}\right) \otimes \dots \otimes B\left(\lambda_{p_1}^{(1)}\right) \|1\rangle. \end{aligned} \tag{19}$$

X is a vector in the tensor product of p_1 horizontal spaces of dimension $(q-1)$. The numbers $\lambda_1^{(1)}, \dots, \lambda_{p_1}^{(1)}$ and $X_{i_1 \cdots i_{p_1}}$ will be determined by the eigenvalue equation

$$\tau^{[N]}(\theta, \tilde{\alpha})\Psi(\{\lambda^{(1)}\}) = \Lambda(\theta, \tilde{\alpha})\Psi(\{\lambda^{(1)}\}). \tag{20}$$

Notice that Ψ is assumed to be independent of θ. This is reasonable because the family commuting $\tau^{[N]}(\theta, \tilde{\alpha})$ may have common eigenvectors Ψ for all θ.

We have here

$$\tau^{[N]}(\theta, \tilde{\alpha}) = A(\theta) + \mathrm{tr}^{(2)} D(\theta), \tag{21}$$

where

$$\mathrm{tr}^{(2)} D(\theta) = \sum_{a=2}^{q} D_{aa}(\theta).$$

That is $\mathrm{tr}^{(2)}$ is the trace in the $(q-1)$-dimensional horizontal space.

The application of $A(\theta)$ and $D_{ab}(\theta)$ on Ψ, that depends on $B(\lambda)$, implies passing these operators through the $B(\lambda)$ till we reach the reference state $\|1\rangle$, whose eigenvalues we know by eq. (12). This is done using the commutation relations (15)–(18).

This procedure is the general strategy of the algebraic Bethe ansatz and generates a lot of terms. We are interested in those keeping the product

$$B\left(\lambda_1^{(1)}\right) \otimes B\left(\lambda_2^{(1)}\right) \otimes \dots \otimes B\left(\lambda_{p_1}^{(1)}\right) \tag{22}$$

unchanged. All the others, where some $B(\lambda_j^{(1)})$ is replaced by $B(\theta)$, are imposed to have a vanishing sum. In this way we get only states proportional to Ψ providing an eigenvector and its eigenvalue.

Using the same procedure as the one applied to the multi-component six-vertex model in ref. [2], we find the following expression for the eigenvalues:

$$\begin{aligned} &\Lambda\left(\theta;\, \lambda_1^{(1)} \dots \lambda_{p_1}^{(1)};\, \lambda_1^{(2)} \dots \lambda_{p_2}^{(2)};\, \dots\, ;\, \lambda_1^{(q-1)} \dots \lambda_{p_{q-1}}^{(q-1)}\right) \\ &\quad = \Lambda^{(1)}(\theta) + \Lambda^{(2)}(\theta) + \dots + \Lambda^{(q)}(\theta), \end{aligned} \tag{23}$$

where

$$\Lambda^{(1)}(\theta) = \prod_{j=2}^{q} \left[G_{j,1}\right]^{p_{j-1}-p_j} \left[\frac{\sin(\gamma + \epsilon_1 \theta)}{\sin\gamma}\right]^N \prod_{\alpha=1}^{p_1} \epsilon_1 \frac{\sinh\left(\lambda_\alpha^{(1)} + i\theta - i\gamma\epsilon_1/2\right)}{\sinh\left(\lambda_\alpha^{(1)} + i\theta + i\gamma\epsilon_1/2\right)},$$

$$\begin{aligned} \Lambda^{(\sigma)}(\theta) = &\prod_{j=1, j\neq\sigma}^{q} \left[G_{j,\sigma}\right]^{p_{j-1}-p_j} \left[\frac{\sin\theta}{\sin\gamma}\right]^N \\ &\times \prod_{\alpha=1}^{p_{\sigma-1}} \epsilon_\sigma \frac{\sinh\left(\lambda_\alpha^{(\sigma-1)} + i\theta + i\left(\sum_{s=1}^{\sigma} \epsilon_s + \epsilon_\sigma\right)\gamma/2\right)}{\sinh\left(\lambda_\alpha^{(\sigma-1)} + i\theta + i\left(\sum_{s=1}^{\sigma} \epsilon_s - \epsilon_\sigma\right)\gamma/2\right)} \\ &\times \prod_{\alpha=1}^{p_\sigma} \epsilon_\sigma \frac{\sinh\left(\lambda_\alpha^{(\sigma)} + i\theta + i\left(\sum_{s=1}^{\sigma-1} \epsilon_s - \epsilon_\sigma\right)\gamma/2\right)}{\sinh\left(\lambda_\alpha^{(\sigma)} + i\theta + i\left(\sum_{s=1}^{\sigma-1} \epsilon_s + \epsilon_\sigma\right)\gamma/2\right)}, \end{aligned}$$

$$\Lambda^{(q)}(\theta) = \prod_{j=1}^{q-1} \left[G_{j,q}\right]^{p_{j-1}-p_j} \left[\frac{\sin\theta}{\sin\gamma}\right]^N \prod_{\alpha=1}^{p_{q-1}} \epsilon_q \frac{\sinh\left(\lambda_\alpha^{(q-1)} + i\theta + i\left(\sum_{s=1}^{q} \epsilon_s + \epsilon_q\right)\gamma/2\right)}{\sinh\left(\lambda_a^{(q-1)} + i\theta + i\left(\sum_{s=1}^{q} \epsilon_s - \epsilon_q\right)\gamma/2\right)}.$$

The set of numbers $\lambda_j^{(k)}(1 \leqslant k \leqslant q,\ 1 \leqslant j \leqslant p_k)$ are determined as functions of γ and ϵ_k by coupled algebraic equations, that is, the Bethe ansatz equations (BAE):

$$\begin{aligned} &\prod_{j=1}^{q} \left[G_{ij} G_{j,i+1}\right]^{N_j} \left[\epsilon_{i+1}\right]^{p_{i+1}} \left[\epsilon_i\right]^{p_{i-1}} \\ &\quad = \prod_{l=1}^{p_j} \frac{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j)} - i\gamma\epsilon_{j+1}\right]}{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j)} + i\gamma\epsilon_j\right]} \times \prod_{l=1}^{p_{j+1}} \frac{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j+1)} + i\gamma\epsilon_{j+1}/2\right]}{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j+1)} - i\gamma\epsilon_{j+1}/2\right]} \\ &\qquad \times \prod_{l=1}^{p_{j-1}} \frac{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j-1)} + i\gamma\epsilon_j/2\right]}{\sin\left[\lambda_k^{(j)} - \lambda_l^{(j-1)} - i\gamma\epsilon_j/2\right]}, \end{aligned} \tag{24}$$

where

$$1 \leqslant k \leqslant p_k, \quad 1 \leqslant \sigma \leqslant (q-1), \quad p_q \equiv 0, \quad \lambda^{(0)} = 0, \quad p_0 \equiv N, \quad N_j \equiv p_{j-1} - p_j. \tag{25}$$

The expression (24) is written in the trigonometric regime, that is, when the weights are trigonometric functions. The same expression follows from the hyperbolic regime (that is, when the weights are hyperbolic functions) with the hyperbolic functions replaced by the trigonometric ones.

The expression for the eigenvalues in the hyperbolic regime was obtained in a different way in ref. [4]. This expression can be fit in our results setting in this paper:

$$\lambda^{(\sigma)} \to i\lambda^{(\sigma)} - (\epsilon_1 + \epsilon_2 + \dots + \epsilon_\sigma)\gamma/2. \tag{26}$$

Eq. (24) shows that the role of the parameters G_{ij} and ϵ_a is not the same. The first one appears as multiplicative factors and has the meaning of external fields. It is equivalent to gauge transformation on the eigenvalues or twist on the boundary conditions [2,6] when $|G_{\sigma\rho}| = 1$.

The second one, on the contrary, takes a prominent part in the solutions of the Bethe ansatz equations. When $\epsilon_j = -\epsilon_{j+1}$ (for a given j), we see in eq. (24) that the phase describing the interaction of pseudoparticles in the jth step between them ($\lambda_l^{(j)}$, $1 \leqslant l \leqslant p_j$), vanishes. The interaction between pseudoparticles in different steps is always present. The attractive or repulsive character of the interactions can be changed at will by choosing the ϵ_a appropriately.

Notice that putting $\epsilon_1 = \epsilon_2 = \dots = \epsilon_q = -1$ and $G_{ij} = 1$ in eq. (24) we get the usual Bethe ansatz equations for the original multi-component six-vertex model.

4. Exact solution of the $q = 3$ critical model

We will concentrate in this section on the particular case where each bond can have three different colours and the variable $G_{\sigma\rho}$ is always equal one.

The variables $\epsilon_s (1 \leqslant s \leqslant 3)$ may assume the values ± 1, and thus we have eight possible situations. The first four are:

$$\begin{array}{llll} \text{(i)} & \epsilon_1 = -1, & \epsilon_2 = -1, & \epsilon_3 = +1, \\ \text{(ii)} & \epsilon_1 = -1, & \epsilon_2 = +1, & \epsilon_3 = -1, \\ \text{(iii)} & \epsilon_1 = -1, & \epsilon_2 = +1, & \epsilon_3 = +1, \\ \text{(iv)} & \epsilon_1 = -1, & \epsilon_2 = -1, & \epsilon_3 = -1, \end{array} \tag{27}$$

and the other possibilities corresponding to reversing all signs are described by the same solutions upon the change $\gamma \to (\pi - \gamma)$.

The expression for the eigenvalues is

$$\Lambda(\theta) = \Lambda^{(1)}(\theta) + \Lambda^{(2)}(\theta) + \Lambda^{(3)}(\theta), \tag{28}$$

where $\Lambda^{(i)}(\theta)$ is given by eq. (23). We notice that for large N and fixed p_1 and p_2 the expression (28) is dominated by the first term $\Lambda^{(1)}(\theta)$ when $0 \leqslant \theta \leqslant \gamma/2$, that is

$$\Lambda^{(1)}(\theta, \tilde{\lambda}) = \left(\frac{\sin(\gamma + \epsilon_1 \theta)}{\sin \gamma} \right)^N \prod_{\alpha=1}^{p_1} \epsilon_1 \frac{\sinh\left(\lambda_\alpha^{(1)} + i\theta - i\gamma\epsilon_1/2\right)}{\sinh\left(\lambda_\alpha^{(1)} + i\theta + i\gamma\epsilon_1/2\right)}. \tag{29}$$

A closed expression for the eigenvalue $\Lambda(\theta)$ can be derived in the thermodynamic limit $N \to \infty$ where eq. (24) becomes easily solvable with the help of the following transformations:

$$\lambda^{(1)} \to \lambda^{(1)}, \qquad \lambda^{(2)} \to \lambda^{(2)} + i(1 + \epsilon_2)\frac{\pi}{4}. \tag{30}$$

Substituting (30) in (24) we get two coupled equations:

$$\left(\frac{\sinh\left(\lambda_k^{(1)} + i\gamma\epsilon_1/2\right)}{\sinh\left(\lambda_k^{(1)} - i\gamma\epsilon_1/2\right)} \right)^N = \prod_{j=1}^{p_2} \frac{\sinh\left[\lambda_k^{(1)} - \lambda_j^{(2)} - i\gamma\epsilon_2/2 + i(1+\epsilon_2)\pi/4\right]}{\sinh\left[\lambda_k^{(1)} - \lambda_j^{(2)} + i\gamma\epsilon_2/2 + i(1+\epsilon_2)\pi/4\right]}$$
$$\times \prod_{j=1}^{p_1} \frac{\sinh\left(\lambda_k^{(1)} - \lambda_j^{(1)} + i\gamma\epsilon_1/2\right)}{\sinh\left(\lambda_k^{(1)} - \lambda_j^{(1)} - i\gamma\epsilon_2/2\right)}, \tag{31}$$

$$\prod_{j=1}^{p_1} \frac{\sinh\left[\lambda_l^{(2)} - \lambda_j^{(1)} + i\gamma\epsilon_2/2 + i(1+\epsilon_2)\pi/4\right]}{\sinh\left[\lambda_l^{(2)} - \lambda_j^{(1)} - i\gamma\epsilon_2/2 + i(1+\epsilon_2)\pi/4\right]}$$
$$= \prod_{j=1}^{p_2} \frac{\sinh\left[\lambda_l^{(2)} - \lambda_j^{(2)} + i\gamma\epsilon_2 + i(1+\epsilon_2)\pi/4\right]}{\sinh\left[\lambda_l^{(2)} - \lambda_j^{(2)} - i\gamma\epsilon_3 + i(1+\epsilon_2)\pi/4\right]}, \tag{32}$$

where $1 \leqslant k \leqslant p_1$ and $1 \leqslant l \leqslant p_2$.

To solve this system of equations we first take the logarithm of eqs. (31) and (32), to get

$$\delta_{\epsilon_j, \epsilon_{j+1}} \epsilon_j \sum_{l=1}^{p_j} \Phi\left(\lambda_k^{(j)} - \lambda_l^{(j)}, \gamma\right) - \epsilon_j \sum_{l=1}^{p_{j-1}} \Phi\left(\lambda_k^{(j)} - \lambda_l^{(j-1)} + i\frac{\pi}{4}(\epsilon_j - \epsilon_{j-1}), \gamma/2\right)$$
$$- \epsilon_{j+1} \sum_{l=1}^{p_{j+1}} \Phi\left(\lambda_k^{(j)} - \lambda_l^{(j+1)} + i\frac{\pi}{4}(\epsilon_j - \epsilon_{j+1}), \gamma/2\right) + \delta_{l1} N \Phi\left(\lambda_k^{(1)}, \gamma/2\right) = 2\pi I_k^{(j)}, \tag{33}$$

where $j = 1, 2$ corresponds respectively to the first and second equation, $1 \leqslant k \leqslant p_j$, and we call

$$\Phi(\lambda, z) \equiv i \log\frac{\sinh(\lambda + iz)}{\sinh(\lambda - iz)}, \qquad \overline{\Phi}(\lambda, z) \equiv -\Phi(\lambda + i\pi/2, z). \tag{34}$$

The numbers $I_k^{(j)}$ are half-integers. For large N the number of roots $(\lambda^{(j)})$ is very large but they become closer and closer and in the real axis so one can define a continuous density in the $N = \infty$ limit:

$$\rho_\infty\big(\lambda_k^{(j)}\big) = \lim_{N\to\infty} \frac{1}{N\big(\lambda_{k+1}^{(j)} - \lambda_k^{(j)}\big)}. \tag{35}$$

For the ground state the half-integers $I_k^{(j)}$ form a monotonic sequence

$$I_{k+1}^{(j)} - I_k^{(j)} = 1 \tag{36}$$

and, for excited states $I_k^{(j)}$ exhibits jumps for some values of k.

Taking the difference between eq. (33) for $k = j + 1$ and $k = j$, we obtain

$$\sigma_\infty^{(j)}(\lambda) - \sum_{k=1}^{2} \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} K_{jk}(\lambda - \mu)\sigma_\infty^{(k)}(\mu) = \frac{\delta_{j1}}{2\pi}\Phi'(\lambda, \gamma/2)$$
$$- \frac{1}{N}\sum_{k=1}^{2}\left\{\sum_{k=1}^{N_h^{(j)}} K_{jk}\big(\lambda - \theta_h^{(j)}\big) - \sum_{k=1}^{N_c^{(j)}}\left[K_{jk}\big(\lambda - \xi_c^{(j)}\big) + \text{c.c.}\right]\right\}, \tag{37}$$

where $\xi_c^{(j)}$ denote the complex root ($\operatorname{Im} \xi_c^{(j)} > 0$) and

$$\begin{aligned} K_{jk}(\mu) = &-\epsilon_j\delta_{j,k+1}\left[\delta_{\epsilon_j,\epsilon_{j-1}}\Phi'(\mu, \gamma/2) - \delta_{\epsilon_j,-\epsilon_{j-1}}\overline{\Phi}'(\mu, \gamma/2)\right] \\ &-\epsilon_j\delta_{j,k-1}\left[\delta_{\epsilon_j,\epsilon_{j+1}}\Phi'(\mu, \gamma/2) - \delta_{\epsilon_j,-\epsilon_{j+1}}\overline{\Phi}'(\mu, \gamma/2)\right] \\ &+\epsilon_j\delta_{jk}\delta_{\epsilon_j,\epsilon_{j+1}}\Phi'(\mu, \gamma). \end{aligned} \tag{38}$$

Here we used

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{p_j} f\big(\lambda_k^{(j)}\big) = \int_{-\infty}^{\infty} d\lambda\ \rho_\infty^{(j)}(\lambda) f(\lambda), \tag{39}$$

$$\Phi'(\lambda, \alpha) = \frac{d\Phi(\lambda, \alpha)}{d\lambda}, \qquad \overline{\Phi}'(\lambda, \alpha) = \frac{d\overline{\Phi}(\lambda, \alpha)}{d\lambda}, \tag{40}$$

$$\lim_{N\to\infty}\frac{I_{k+1}^{(j)} - I_k^{(j)}}{N\big(\lambda_{k+1}^{(j)} - \lambda_k^{(j)}\big)} = \sigma_\infty^{(j)}(\lambda) \equiv \rho_\infty^{(j)}(\lambda) + \frac{1}{N}\sum_{h=1}^{N_h^{(j)}} \delta\big(\lambda - \theta_h^{(j)}\big). \tag{41}$$

From the above relation one sees that each jump in the sequence $I_k^{(j)}$ corresponds to remove a root at the position $\lambda_k^{(j)} = \theta_h^{(j)}$. One says that there is a hole at the position $\theta_h^{(j)}$ at the lth level ($h = 1, \dots, N_{\mathrm{h}}^{(j)}$).

The linear integral equation (37) can be solved by Fourier integrals. In order to do that one needs the following Fourier representations:

$$\phi(\lambda, \alpha) = \pi + \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k} \sin(k\lambda) \frac{\sinh k(\pi/2 - \alpha)}{\sinh k\pi/2}, \tag{42}$$

$$\overline{\phi}(\lambda, \alpha) = -\pi + \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{k} \sin(k\lambda) \frac{\sinh k\alpha}{\sinh k\pi/2}, \tag{43}$$

$$\sigma^{(j)}(\lambda) = \int_{-\infty}^{\infty} \hat{\sigma}^{(j)}(k)\, \mathrm{e}^{i\lambda k}\, \mathrm{d}k, \tag{44}$$

where $\lambda \in \mathrm{R}$. The Fourier tranforms of $\phi(\lambda - \xi_i^{(j)}, \alpha)$ and $\overline{\phi}(\lambda - \xi_i^{(j)}, \alpha)$, where $\lambda - \xi_i^{(j)} = \lambda - \tau^{(j)} + i\eta^{(j)} \in \mathscr{C}$ are defined in appendix A. The complex roots $\xi_i^{(j)}$ are classified as: wide, middle and close roots depending on the region where $\operatorname{Im} \xi_i^{(j)}$ is defined.

The solution of eq. (37) reads

$$\sigma_\infty^{(j)}(\lambda) = \sigma_{\text{vacuum}}^{(j)}(\lambda) + \frac{1}{N}\left[\sigma_{\text{holes}}^{(j)} + \sigma_{\text{complex}}^{(j)}\right]. \tag{45}$$

The Fourier transforms of the ground-state root density is given by

$$\hat{\sigma}_{\text{vacuum}}^{(j)}(k) = \frac{\sinh\left\{\frac{1}{2}k\left[(3-j)\pi/2 + (\pi/2 - \gamma)\sum_{k=j+1}^{3} \epsilon_k\right]\right\}}{\sinh\left\{\frac{1}{2}k\left[3\pi/2 + (\pi/2 - \gamma)\sum_{k=1}^{3} \epsilon_k\right]\right\}}. \tag{46}$$

We obtain the following expression for the ground-state spin:

$$S_j = 2p_j - p_{j-1} - p_{j+1} = \frac{\epsilon_{j+1} - \epsilon_j}{3(1 - 2\gamma/\pi)^{-1} + \sum_{l=1}^{3} \epsilon_l}. \tag{47}$$

This equation shows that $|S_l| < 1$ for $1 < \gamma < \pi$. Therefore, unless all ϵ_j are equal we find a ferrielectric behavior. That is, the $|S_j|$ values are larger than in the antiferroelectric case ($S_j = 0$) and smaller than in the ferroelectric case ($S_j = \pm 1$). More precisely, S_j behaves ferrielectrically provided $\epsilon_j \neq \epsilon_{j+1}$. Otherwise, when $\epsilon_j = \epsilon_{j+1}$, S_j exhibits an antiferroelectric character.

The free energy follows from eqs. (28), (29) and (46),

$$f(\theta, \gamma) = -\lim_{N\to\infty} \frac{1}{N} \log \Lambda(\theta, \lambda) = i\int_{-\infty}^{\infty} \mathrm{d}\lambda\ \sigma^{(1)}_{\text{vacuum}} \phi(\lambda + i\theta, \gamma/2)$$

$$= 2\int_0^{\infty} \frac{\mathrm{d}x}{x} \frac{\sinh(2x\theta)}{\sinh \pi x} \frac{\sinh x\left[\pi + (\pi/2-\gamma)\sum_{k=2}^{3} \epsilon_k\right]}{\sinh x\left[3\pi/2 + (\pi/2-\gamma)\sum_{k=1}^{3} \epsilon_k\right]}$$

$$\times \sinh x\left[\pi/2 - \epsilon_1(\pi/2-\gamma)\right]. \tag{48}$$

The solution of eq. (37) can be written in general in terms of the resolvent $R = (1-K)^{-1}$, that is

$$R_{lk}(\lambda) - \sum_{j=1}^{2} \int_{-\infty}^{\infty} \mathrm{d}\mu\ K_{lj}(\lambda - \mu) R_{jk}(\mu) = \delta_{lk}\delta(\lambda). \tag{49}$$

One finds upon Fourier transformation,

$$R_{lk}(\lambda) = \int_{-\infty}^{\infty} \hat{R}_{lk}(k)\ \mathrm{e}^{i\lambda k}\,\mathrm{d}k, \tag{50}$$

and using eqs. (38), (42) and (43) an explicit solution for eq. (49):

$$\hat{R}_{ll'}(2x) = \sinh(\pi x) \frac{\sinh x\left[l_< \pi/2 + (\pi/2-\gamma)\sum_{k=1}^{l_<} \epsilon_k\right]}{\sinh[x(\pi-\gamma)]\ \sinh(x\gamma)}$$

$$\times \frac{\sinh x\left[(3-l_>)\pi/2 + (\pi/2-\gamma)\sum_{k=l_>+1}^{3} \epsilon_k\right]}{\sinh x\left[3\pi/2 + (\pi/2-\gamma)\sum_{k=1}^{3} \epsilon_k\right]}. \tag{51}$$

Now, we can also express the hole contributions to the roots density $\sigma^{(l)}_{\infty}(\lambda)$ as

$$\sigma^{(j)}_{\text{holes}}(\lambda) = \sum_{k=1}^{2} \sum_{h=1}^{N_{\text{h}}^{(j)}} \left[\delta_{jk}\delta\left(\lambda - \theta_h^{(j)}\right) - R_{jk}\left(\lambda - \theta_h^{(j)}\right)\right], \tag{52}$$

where we used eqs. (38) and (49).

The matrix transfer eigenvalues for a state with a hole at $\theta_h^{(j)}$ in the jth branch behaves for large N as

$$\Lambda_{\text{exc}}\left(\theta,\, \theta_h^{(j)}\right) \underset{N\to\infty}{=} \exp\left[-Nf(\theta,\,\gamma) - ig_j\left(\theta + i\theta_h^{(j)},\, \gamma\right)\right], \tag{53}$$

where $\Lambda_0(\theta) = \exp[-Nf(\theta,\,\gamma)]$ is the ground-state eigenvalue and g_j follows from eqs. (45), (48) and (52),

$$g_j\left(\theta_h^{(j)} + i\theta, \gamma\right) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\ R_{j1}\left(\lambda - \theta_h^{(j)}\right)\Phi(\lambda + i\theta,\, \gamma/2). \tag{54}$$

So, we get

$$g^{(j)}(\phi,\,\gamma) = \Phi\left(\kappa\phi,\, \frac{\pi}{4}\kappa\left[\left(j + \left(1 - 2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]\right]\right) - \frac{\pi}{2}\kappa\left[\left(j + \left(1 - 2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]. \tag{55}$$

Here $\kappa \equiv [\frac{3}{2} + (\frac{1}{2} - \gamma/\pi)\Sigma_{k=1}^{3}\epsilon_k]^{-1}$ and $\phi = \theta + i\theta_h^{(j)}$. By these expressions we see that $|\mathrm{e}^{-ig(\theta)}| < 1$, so, this confirms our identification of the ground state since any deviation from it increases the eigenvalue of $\tau(\theta)$.

The solutions for complex roots densities are written down explicitly just for case (ii):
Case (ii), close: $0 \leqslant \eta_c \leqslant (\pi - \gamma)/2$:

$$\hat{\sigma}_{\mathrm{c}}^{(1)} = A\left\{\sum_{c=1}^{N_{\mathrm{c}}^{(1)}} \mathrm{e}^{-ik\tau_c^{(1)}} \cosh\, k\eta_c^{(1)} \sinh\, \tfrac{1}{2}k\gamma + \sum_{c=1}^{N_{\mathrm{c}}^{(2)}} \mathrm{e}^{-ik\tau_c^{(2)}} \cosh\, k\eta_c^{(2)} \sinh\, \tfrac{1}{2}k\pi\right\}, \tag{56}$$

$$\hat{\sigma}_{\mathrm{c}}^{(2)} = A\left\{\sum_{c=1}^{N_{\mathrm{c}}^{(1)}} \mathrm{e}^{-ik\tau_c^{(1)}} \cosh\, k\eta_c^{(1)} \sinh\, \tfrac{1}{2}k\pi + \sum_{c=1}^{N_{\mathrm{c}}^{(2)}} \mathrm{e}^{-ik\tau_c^{(2)}} \cosh\, k\eta_c^{(2)} \sinh\, \tfrac{1}{2}k\gamma\right\}. \tag{57}$$

Case (ii), wide: $(\pi - \gamma)/2 \leqslant \eta_w \leqslant \pi/2$

$$\hat{\sigma}_{\mathrm{w}}^{(1)} = B\left\{\sum_{w=1}^{N_{\mathrm{w}}^{(1)}} \mathrm{e}^{-ik\tau_w^{(1)}} \cosh\, k\left(\eta_w^{(1)} + \pi/2\right) \sinh\, \tfrac{1}{2}k\gamma\right.$$
$$\left. + \sum_{w=1}^{N_{\mathrm{w}}^{(2)}} \mathrm{e}^{-ik\tau_w^{(2)}} \cosh\, k\left(\eta_w^{(2)} + \pi/2\right) \sinh\, \tfrac{1}{2}k\pi\right\}. \tag{58}$$

$$\hat{\sigma}_{\mathrm{w}}^{(2)} = B\left\{ \sum_{w=1}^{N_{\mathrm{w}}^{(1)}} \mathrm{e}^{-ik\tau_w^{(1)}} \cosh\, k\left(\eta_w^{(1)} + \pi/2\right) \sinh\, \tfrac{1}{2}k\pi \right.$$

$$\left. + \sum_{w=1}^{N_{\mathrm{w}}^{(2)}} \mathrm{e}^{-ik\tau_w^{(2)}} \cosh\, k\left(\eta_w^{(2)} + \pi/2\right) \sinh\, \tfrac{1}{2}k\gamma \right\}, \tag{59}$$

where $A = 2 \sinh \frac{1}{2}k\gamma(\sinh \frac{1}{2}k(\pi+\gamma) \sinh \frac{1}{2}k(\pi-\gamma))^{-1}$ and $B = -2(\sinh \frac{1}{2}k(\pi + \gamma))^{-1}$.

The number of roots in each step is given by

$$p_1 = -\frac{\pi}{\pi+\gamma}\left[2N_{\mathrm{w}}^{(2)} + \frac{\gamma}{\pi}\left(2N_{\mathrm{w}}^{(2)} + 2N_{\mathrm{c}}^{(1)} + N_{\mathrm{h}}^{(1)}\right)\right.$$

$$\left. - \frac{\gamma}{(\pi-\gamma)}\left(2N_{\mathrm{c}}^{(2)} + 2N_{\mathrm{c}}^{(1)} + N_{\mathrm{h}}^{(1)} + N_{\mathrm{h}}^{(2)}\right)\right] + \frac{N\pi}{\pi+\gamma}, \tag{60}$$

$$p_2 = -\frac{\pi}{\pi+\gamma}\left[2N_{\mathrm{w}}^{(1)} + \frac{\gamma}{\pi}\left(2N_{\mathrm{w}}^{(1)} + 2N_{\mathrm{c}}^{(2)} + N_{\mathrm{h}}^{(2)}\right)\right.$$

$$\left. - \frac{\gamma}{(\pi-\gamma)}\left(2N_{\mathrm{c}}^{(1)} + 2N_{\mathrm{c}}^{(2)} + N_{\mathrm{h}}^{(1)} + N_{\mathrm{h}}^{(2)}\right)\right] + \frac{N\gamma}{\pi+\gamma}. \tag{61}$$

Since p_1 and p_2 must be integers, the relation γ/π must be a rational number.

We observe that expression (55) is gapless since $g^{(j)}(\theta, -\infty) = 0$. From the light-cone approach (see ref. [2]) the gapless models describe a massive quantum field theory where the energy and momentum eigenvalues are given by

$$E \pm P = \lim_{a \to 0, i\theta \to \infty} \frac{g^{(j)}\left(\pm\theta + i\theta_h^{(j)}, \gamma\right)}{a} \tag{62}$$

where a is the lattice spacing. We let $i\theta \to \infty$ and the lattice spacing $a \to 0$ such that

$$\mu = \frac{1}{a}\exp(-i\theta\kappa). \tag{63}$$

μ is a fixed mass unit. The energy-momentum dispersion law results in

$$E_j = m_j \cosh\left(\kappa\theta_h^{(j)}\right), \qquad P_j = m_j \sinh\left(\kappa\theta_h^{(j)}\right), \tag{64}$$

where

$$m_j = \mu \sin\left\{\frac{\pi}{2}\kappa\left[j + \left(1 - 2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j} \epsilon_k\right]\right\}. \tag{65}$$

We observe here the interesting effect of the ϵ_p parameters in the generalized model. In fact, their presence produces a dependence of the mass spectrum on the anisotropy parameter (γ). That is the case for all models where we have not all the ϵ_p equal among them. This is in contrast with all mass spectra find up to now for integrable models which are γ-independent [1,2,10,11]. At the quantum field theory level γ stands for a coupling constant (or a function of it).

The S-matrix between a hole at branch l and another one at l' follows from eq. (51) by applying the method of ref. [9]. (That is the S-matrix between a particle m_l and a particle $m_{l'}$.) It reads $S_{ll'}(\phi) = \exp[i\delta_{ll'}(\phi)]$, where $\phi = \kappa\theta_h^{(j)}$ is the relativistic rapidity and

$$\delta_{ll'}(\phi) = 2\pi\int_0^{\phi/\kappa} \sigma_{ll'}(\lambda)\, \mathrm{d}\lambda. \tag{66}$$

A look at eqs. (51) and (60) shows that this S-matrix can be expressed as an infinite product of Γ-functions.

Besides the scaling limit yielding an integrable massive QFT we can take the trivial continuous limit ($a \to 0$) leading to conformal invariant models.

A systematic procedure for computing finite-size corrections for integrable theories was proposed in ref. [7]. We will just sketch here the derivation of the finite-size corrections to the free energy, that is

$$\begin{aligned} L_N(\theta, \gamma) &= f_N(\theta, \gamma) - f(\theta, \gamma) \\ &= -\sum_{l=1}^{2}\left(\int_{-\infty}^{-\Lambda_l^-} + \int_{\Lambda_l^+}^{\infty}\right) \mathrm{d}\lambda_l\, f_l(\lambda_l)\sigma_N^{(l)}(\lambda_l) \\ &\quad + \frac{1}{2N}\sum_{l=1}^{2}\left[f_l(\Lambda_l^-) + f_l(\Lambda_l^+)\right] + \frac{1}{12N^2}\sum_{l=1}^{2}\left[\frac{f_l'(\Lambda_l^+)}{\sigma_N^{(l)}(\Lambda_l^+)} - \frac{f_l'(\Lambda_l^-)}{\sigma_N^{(l)}(\Lambda_l^-)}\right] \\ &\quad + \text{higher-order terms}, \end{aligned} \tag{67}$$

where $\pm\Lambda_l^{\pm}$ are the largest positive and negative roots of the Bethe ansatz equations (31) and (32) in the lth branch,

$$\sigma_N^{(l)}(\lambda) = \frac{\mathrm{d}Z_N^{(l)}}{\mathrm{d}\lambda}(\lambda), \tag{68}$$

where

$$2\pi N Z_N^{(l)} = \epsilon_{l+1}\delta_{\epsilon_l,-\epsilon_{l+1}} \sum_{j=1}^{p_{l+1}} \overline{\phi}\left(\lambda_k^{(l)} - \lambda_j^{(l+1)}, \gamma/2\right) - \epsilon_{l+1}\delta_{\epsilon_l,\epsilon_{l+1}} \sum_{j=1}^{p_{l+1}} \phi\left(\lambda_k^{(l)} - \lambda_j^{(l+1)}, \gamma/2\right)$$
$$+ \epsilon_l \delta_{\epsilon_l,-\epsilon_{l-1}} \sum_{j=1}^{p_{l-1}} \overline{\phi}\left(\lambda_k^{(1)} - \lambda_j^{(l-1)}, \gamma/2\right) - \epsilon_l \delta_{\epsilon_l,\epsilon_{l-1}} \sum_{j=1}^{p_{l-1}} \phi\left(\lambda_k^{(l)} - \lambda_j^{(l-1)}, \gamma/2\right)$$
$$+ \epsilon_l \delta_{\epsilon_l,\epsilon_{l+1}} \sum_{j=1}^{p_l} \phi\left(\lambda_k^{(l)} - \lambda_j^{(l)}, \gamma\right) \tag{69}$$

where $\epsilon_0 = \epsilon_1$.

We define the Fourier transforms

$$X_l^{\pm}(w) = \int_{-\infty}^{\infty} \exp(iwt)\ \theta(\pm t) \sigma_N^{(l)}(\Lambda_l^+ + t)\ \mathrm{d}t \tag{70}$$

which are analytic functions in $\pm \operatorname{Im} w > 0$. We get the following matrix Riemann–Hilbert problem from the Bethe ansatz equations (24) approximated as (67):

$$X_k^-(w) + \sum_{l=1}^{2} \hat{R}_{lk}(w) X_l^+(w)$$
$$= \mathrm{e}^{-iw\Lambda_k^+}\ \hat{\sigma}_k(w) + \frac{1}{2N}\left[-1 + \sum_{l=1}^{2} \hat{R}_{lk}(w)\right] - \frac{iw}{12N^2} \sum_{l=1}^{2} (\delta_{kl} - 1) \frac{\hat{R}_{lk}(w)}{\sigma_N^{(l)}(\Lambda^+)}, \tag{71}$$

where $R_{kl}(w) = \delta_{kl} - \sigma_{kl}(w)$. The resolution of this problem is analogous to ref. [2]. The finite-size corrections $L_N(\theta)$ for an excited state with weights $S_l(1 \leqslant l \leqslant 2)$ [eq. (4.7)] and $h_{\pm}^{(l)}$ holes beyond $\pm\Lambda_l^{\pm}$, that is a generic low-energy excitation, reads

$$L_N(\theta, \gamma) = -\frac{\pi}{3N^2}\sin(\kappa\theta) - \frac{2\pi i}{N^2}\left[\Delta\ \mathrm{e}^{-i\kappa\theta} - \overline{\Delta}\ \mathrm{e}^{i\kappa\theta}\right], \tag{72}$$

where

$$\Delta = \sum_{l',l=1}^{2} \left(2h_l^+ + \frac{\gamma}{2\pi}(\epsilon_{l+1} + \epsilon_l) S^l\right) \hat{R}_{ll'}(0) \left(2h_{l'}^+ + \frac{\gamma}{2\pi}(\epsilon_{l'+1} + \epsilon_{l'}) S^{l'}\right) \tag{73}$$

with the weight of the Bethe ansatz state given by

$$S^l = 2p_l - p_{l+1} - p_{l-1},$$

and the $R_{lk}^{-1}(0) = [1-K]$ matrix follows from eq. (38), that is

$$\hat{R}_{lj}^{-1}(0) = -2\delta_{lj}\left[\frac{\epsilon_{l+1}+\epsilon_l-2}{4} - \frac{(\epsilon_{l+1}+\epsilon_l)\gamma}{2\pi}\right] + \delta_{l,j+1}\left[\frac{\epsilon_{l-1}+\epsilon_l}{2} - \frac{(\epsilon_l)\gamma}{\pi}\right]$$
$$+\delta_{l,j-1}\left[\frac{\epsilon_{l+1}+\epsilon_l}{2} - \frac{(\epsilon_{l+1})\gamma}{\pi}\right]. \tag{74}$$

$\bar{\Delta}$ follows from Δ by exchanging $h_+^l \leftrightarrow h_-^l$. Since the speed of sound here is $v = \sin\kappa\theta$, eq. (72) tells us that the central charge $c = 2$. We recall that the parameter κ gives the finite renormalization of the rapidity [see eq. (64)].

A quantum hamiltonian for a chain of SU(3) spins can be obtained through the well-known relation [2]

$$H = -\sin\gamma \frac{\mathrm{d}}{\mathrm{d}\theta} \log \tau(\theta)\,|_{\theta=0}. \tag{75}$$

For the $q = 3$ generalized model we obtain [12]

$$H = -\sum_{j=1}^{N} h_{j,j+1} \tag{76}$$

where

$$h_{j,j+1} = \sigma_j + (\sigma_j)^2 + \tfrac{1}{2}\left[\tfrac{1}{2}(\epsilon_1+\epsilon_3)\cos\gamma - 1\right]\sigma_j^z - 2(\epsilon_2\cos\gamma - 1)\left(S_j^z\right)^2$$
$$+\tfrac{1}{2}i\sin\gamma\sigma_j^z\left(S_j^z - S_{j+1}^z\right) + \tfrac{1}{4}(\epsilon_1-\epsilon_3)\cos\gamma\;\sigma_j^z\left(S_j^z + S_{j+1}^z\right)$$
$$+\tfrac{1}{2}\left[\tfrac{1}{2}(\epsilon_1+4\epsilon_2+\epsilon_3)\cos\gamma - 3\right]\left[\cos\left(\tfrac{1}{2}\gamma\right) - 1\right]\left(\sigma_j^z\right)^2$$

and

$$\sigma_j^z \equiv S_j^z S_{j+1}^z, \qquad \sigma_j \equiv \sum_{a=1}^{3} S_{j+1}^a S_j^a, \qquad \sigma_j^{\perp} \equiv \sigma_j - \sigma_j^z. \tag{77}$$

Here $S_j^a (a = x, y, z)$ are the spin-1 matrices,

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{78}$$

j is the site index $(1 \leqslant j \leqslant N)$ and we choose periodic boundary conditions: $S_{N+1}^a \equiv S_1^a$. The hamiltonian H describes nearest neighbors interactions invariant under rotations around the z-axis, it includes biquadratic couplings (σ_j^z, σ_j) besides the usual Heisenberg exchange terms $S_{j+1}^a S_j^a$.

5. Multi-component generalized model

We present in this section the exact solution of the multi-component generalized six-vertex model where each bond can assume q different colours and where all the variables $G_{\sigma\rho}$ equal 1 [13].

The Bethe ansatz equations (24) are now solved using the following generalized transformation:

$$\lambda_\alpha^{(j)} = \mu_\alpha^{(j)} + i\frac{\pi}{4}(1+\epsilon_j). \tag{79}$$

This results for the expression (24)

$$\begin{aligned}[\epsilon_{j+1}]^{p_{j+1}}[\epsilon_j]^{p_{j-1}} = &\prod_{\beta=1}^{p_j} \frac{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j)} + i\gamma\epsilon_j\right]}{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j)} - i\gamma\epsilon_{j+1}\right]} \\ &\times \prod_{\beta=1}^{p_{j+1}} \frac{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j+1)} - i\gamma\epsilon_{j+1}/2 + i(\epsilon_j - \epsilon_{j+1})\pi/4\right]}{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j+1)} + i\gamma\epsilon_{j+1}/2 + i(\epsilon_j - \epsilon_{j+1})\pi/4\right]} \\ &\times \prod_{\beta=1}^{p_{j-1}} \frac{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j-1)} - i\gamma\epsilon_j/2 + i(\epsilon_j - \epsilon_{j-1})\pi/4\right]}{\sin\left[\lambda_\alpha^{(j)} - \lambda_\beta^{(j-1)} + i\gamma\epsilon_j/2 + i(\epsilon_j - \epsilon_{j-1})\pi/4\right]}. \end{aligned} \tag{80}$$

Using the same procedure as in sect. 4, we obtain for the ground-state density the Fourier transform:

$$\hat{\sigma}_{\text{vacuum}}^{(j)}(k) = \frac{\sinh\left\{\frac{1}{2}k\left[(q-j)\pi/2 + (\pi/2-\gamma)\sum_{l=j+1}^{q}\epsilon_l\right]\right\}}{\sinh\left\{\frac{1}{2}k\left[q\pi/2 + (\pi/2-\gamma)\sum_{l=1}^{q}\epsilon_l\right]\right\}}, \tag{81}$$

and for the holes density, expressed in terms of the resolvent $R = (1-K)^{-1}$,

$$\begin{aligned}\hat{R}_{ll'}(2x) = \sinh(\pi x) &\frac{\sinh x\left[l_<\pi/2 + (\pi/2-\gamma)\sum_{k=1}^{l_<}\epsilon_k\right]}{\sinh[x(\pi-\gamma)]\ \sinh(x\gamma)} \\ &\times \frac{\sinh x\left[(q-l_>)\pi/2 + (\pi/2-\gamma)\sum_{k=l_>+1}^{q}\epsilon_k\right]}{\sinh x\left[q\pi/2 + (\pi/2-\gamma)\sum_{k=1}^{q}\epsilon_k\right]}. \end{aligned} \tag{82}$$

The free energy follows from eqs. (48) and (81). It reads

$$f(q,\theta,\gamma) = -\lim_{N\to\infty}\frac{1}{N}\log \Lambda(\theta,\lambda)$$

$$= 2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\sinh(2x\theta)}{\sinh \pi x}\,\frac{\sinh x\left[(q-1)\pi/2+(\pi/2-\gamma)\sum_{k=2}^{q}\epsilon_k\right]}{\sinh x\left[q\pi/2+(\pi/2-\gamma)\sum_{k=1}^{q}\epsilon_k\right]}$$

$$\times \sinh x\left[\pi/2-\epsilon_1(\pi/2-\gamma)\right]. \tag{83}$$

The eigenvalues of the excited states writes as in the $q=3$ case by the expression (53) where

$$g_j(\phi,\ \gamma) = \Phi\left(\kappa_q\phi,\ \frac{\pi}{4}\kappa_q\left[j+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]\right) - \frac{\pi}{2}\kappa_q\left[j+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]. \tag{84}$$

Here $\kappa_q \equiv [q/2+(\frac{1}{2}-\gamma/\pi)\Sigma_{k=1}^{q}\epsilon_k]^{-1}$. Since these models are gapless, $g_j(\theta,\ -\infty,\ \gamma,\ \epsilon)=0$, we can apply the light-cone approach and we get

$$E \pm P = \lim_{a\to 0, i\theta\to\infty} \frac{g^{(j)}(\pm\theta + i\theta_h^{(j)},\gamma)}{a}. \tag{85}$$

We let $i\theta\to\infty$ and the lattice spacing $a\to 0$ such that

$$\mu = \frac{1}{a}\exp(-i\theta\kappa_q). \tag{86}$$

μ is a fixed mass unit. The energy–momentum dispersion law results in

$$E_l = m_l\cosh\left(\kappa_q\theta_h^{(l)}\right), \qquad P_l = m_l\sinh\left(\kappa_q\theta_h^{(l)}\right),$$

where

$$m_l = \mu\sin\left\{\frac{\pi}{2}\kappa_q\left[l+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{l}\epsilon_k\right]\right\}. \tag{87}$$

The decisive influence of the parameters ϵ_p in all results obtained just now is evident. The same conclusion for the mass spectrum holds here, that is the dependence on the anisotropy parameter γ is linked to the values that ϵ_p assume for each different colour. We recover the results for the ordinary multi-component six-vertex model, that is, a γ-independent mass spectrum putting all $\epsilon_s = -1$ $(1 \leqslant s \leqslant q)$.

The systematic procedure for computing the finite-size corrections can also be applied here. We find

$$L_N(q, \theta, \gamma) = -\frac{\pi(q-1)}{6N^2}\sin(\kappa_q\theta) - \frac{2\pi i}{N^2}\left[\Delta\, \mathrm{e}^{-i\kappa_q\theta} - \bar{\Delta}\, \mathrm{e}^{i\kappa_q\theta}\right] \tag{88}$$

where

$$\Delta = \sum_{l',l=1}^{q-1}\left(2h_l^+ + \frac{\gamma}{2\pi}(\epsilon_{l+1}+\epsilon_l)S^l\right)\hat{R}_{ll'}(0)\left(2h_{l'}^+ + \frac{\gamma}{2\pi}(\epsilon_{l'+1}+\epsilon_{l'})S^{l'}\right) \tag{89}$$

with $\hat{R}_{ll'}(0)$ given by

$$\hat{R}_{lj}^{-1}(0) = -2\delta_{lj}\left[\frac{\epsilon_{l+1}+\epsilon_l-2}{4} - \frac{(\epsilon_{l+1}+\epsilon_l)\gamma}{2\pi}\right] + \delta_{l,j+1}\left[\frac{\epsilon_{l-1}+\epsilon_l}{2} - \frac{(\epsilon_l)\gamma}{\pi}\right]$$
$$+\delta_{l,j-1}\left[\frac{\epsilon_{l+1}+\epsilon_l}{2} - \frac{(\epsilon_{l+1})\gamma}{\pi}\right]. \tag{90}$$

$\bar{\Delta}$ can be obtained from Δ by exchanging $h_+^{(l)} \leftrightarrow h_-^{(l)}$.

Since the speed of sound here is $v = \sin\kappa_q\theta$, eq. (88) tell us that the central charge c equals $(q-1)$.

6. Conclusion

We have considered a multi-component generalization of the six-vertex model whose weights are a general solution of the star–triangle equations under a multi-colour "ice"-type condition which, by requiring certain vertex weights to be zero, ensures that the model has some useful conservation properties. The eigenvalues and eigenvectors of the transfer matrix are obtained by the nested Bethe ansatz. From this we see the different role played by the parameters $G_{\rho\sigma}$ and ϵ_ρ. The first one has the meaning of an external field and the second one defines the ferroelectric or antiferroelectric or still the ferrielectric character of the model. In the thermodynamic limit we solve the Bethe ansatz equations for the case where $|G_{\sigma\rho}| = 1$. We note the presence of the parameters ϵ_ρ in all results. In particular for the mass spectrum where the ϵ_ρ presence makes it dependent on the anisotropy parameter (γ).

We would like to point out that this model is connected with the deformed Lie algebra $A_n(q)$. γ is related with the quantum group variable q through $q = \epsilon^{i\gamma}$. Models associated to other Lie algebras can be generalized like the model treated here with the addition of the discrete parameters ϵ_ρ [14].

Notice that in the special case $q = 3$ and $\epsilon_1 = -\epsilon_2 = -\epsilon_3 = -1$ the isotropic limit ($\gamma \to 0$) of our solution yields the integrable case (a) of the models solved in ref. [15]. The roots density for the second step, $\hat{\sigma}^{(2)}(k)$ identically vanishes in this limit and $\hat{\sigma}^{(1)}(k/\gamma) = \exp - |k|/2$ for $\gamma \to 0$, in agreement with ref. [15].

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Appendix A

$$\Phi(\lambda, \alpha) \equiv i \log \frac{\sinh(\lambda + i\alpha)}{\sinh(\lambda - i\alpha)}.$$

(i) $0 < \alpha < \pi$:

$$|\mathrm{Im}\ \lambda| < \min(\alpha, \pi - \alpha),$$

$$\Phi(\lambda, \alpha) = \pi + \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{ik} \exp(ik\lambda) \frac{\sinh k(\pi/2 - \alpha)}{\sinh k\pi/2}. \tag{A.1}$$

(ii) $0 < \alpha < \pi/2$:

$$|\mathrm{Im}\ \lambda| > \alpha,$$

$$\Phi(\lambda, \alpha) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{ik} \exp[ik\lambda + \pi k(\mathrm{Im}\ \lambda)/2] \frac{\sinh k\alpha}{\sinh k\pi/2}. \tag{A.2}$$

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