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RESEARCH ARTICLE

A new approach for constructing mock-Chebyshev grids

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Summary

Polynomial interpolation with equidistant nodes is notoriously unreliable due to the Runge phenomenon, and is also numerically ill-conditioned. By taking advantage of the optimality of the interpolation processes on Chebyshev nodes, one of the best strategies to defeat the Runge phenomenon is to use the mock-Chebyshev points, which are selected from a satisfactory uniform grid, for polynomial interpolation. Yet, little literature exists on the computation of these points. In this study, we investigate the properties of the mock-Chebyshev nodes and propose a subsetting method for constructing mock-Chebyshev grids. Moreover, we provide a precise formula for the cardinality of a satisfactory uniform grid. Some numerical experiments using the points obtained by the method are given to show the effectiveness of the proposed method and numerical results are also provided.

KEYWORDS:
Polynomial interpolation, Runge phenomenon, mock-Chebyshev nodes

1 | INTRODUCTION

In polynomial interpolation, the quality of the polynomial interpolant depends heavily on the distribution of the interpolation points. Moreover, some sets of interpolation points deliver near-best polynomial approximants, while others lead to divergence of the interpolation scheme. It is well-known that the best choice is to use point sets that are clustered at the endpoints of the interval with an asymptotic density proportional to $\frac{1}{\sqrt{1-x^2}}$ as $n \to \infty$. The family of Chebyshev points have such density properties. Moreover, if a smooth function f defined on the interval $[-1, 1]$ is interpolated by polynomials in Chebyshev points or in any other system of points with the asymptotic density $(1-x^2)^{-1/2}$, the rate of convergence of the interpolants to f as $n \to \infty$ is remarkably fast[1]. In applications, however, this is not always the case. For instance, experimental data may be available only at equally spaced intervals. Thus, we are led to consider the problem of interpolation on equispaced points. In this case, however, polynomial interpolation has the well-known drawback that the polynomial interpolants on equally spaced points might not converge, even if the considered function is analytic anywhere on the interval. For this, a classical example is given by using equidistant points to interpolate the Runge function $f(x) = \frac{1}{1+25x^2}$ over the interval $[-1, 1]$. Here, the error made by interpolating f with polynomials has wild oscillations near the the endpoints of the interval[2] - an effect known as *Runge Phenomenon*. There are many strategies to defeat this phenomenon[3,4]. By taking advantage of the optimality of the interpolation processes on Chebyshev nodes, one of the best strategies to defeat the Runge phenomenon is to use the mock-Chebyshev points for polynomial interpolation, which asymptotically follow the Chebyshev distribution and they are selected from a larger set of equispaced nodes. Yet, little literature exists on the computation of these points. In his work, Boyd[5] first introduced a nice strategy for defeating the Runge phenomenon by interpolation on a "mock-Chebyshev" grid, choosing the nodes from a uniform satisfactory grid. Since the computational cost is about $\mathcal{O}(n^3)$ for this strategy, the computation of these

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

nodes is challenging especially when the number of nodes is large. In a recent paper[6], we have presented a fast algorithm for computing the mock-Chebyshev nodes for a given set of $(n+1)$ Chebyshev-Lobatto points. In this study, we investigate the properties of the mock-Chebyshev nodes and propose a subsetting method for constructing mock-Chebyshev grids. Moreover, we provide a precise formula for the smallest value of the cardinality of a satisfactory uniform grid.

The rest of this paper is organized as follows. Section 2 introduces Lebesgue constants in polynomial interpolation. In Section 3, we prove the precise formula for the minimum value of the cardinality of a satisfactory uniform grid, and discuss some details on the mock-Chebyshev interpolation. In Section 4, we investigate the properties of the mock-Chebyshev nodes and propose a subsetting method for constructing mock-Chebyshev grids. Section 5 is devoted to some numerical results. The concluding remarks are given in Section 6.

2 | LEBESGUE CONSTANTS IN POLYNOMIAL INTERPOLATION

Let $X_n = \left\{x_j : 0 \le j \le n\right\}$ be a set of $(n+1)$ distinct interpolation points (or nodes) on the real interval $\Omega = [-1, 1]$, increasingly ordered from $x_0 = -1$ to $x_0 = 1$. Let $f \in C(\Omega)$ and denote by $\mathcal{P}_n$, the set of all polynomials of degree at most n. Then there is a unique polynomial interpolant $p_n \in \mathcal{P}_n$ of the function f at the points of X_n, which satisfies

$$p(x_j) = f(x_j), \qquad j = 0, \dots, n. \tag{1}$$

Suppose $f \in C^{(n+1)}(\Omega)$. Then, it is well known that the interpolation error satisfies[7, pp. 56-57]

$$||f - p_n||_\infty \le \max_{x\in\Omega} \left(\frac{|f^{(n+1)}(x)|}{(n+1)!} \right) \max_{x\in\Omega} \prod_{j=0}^{n} |x - x_j|. \tag{2}$$

Since f is given, we cannot control the factor $|f^{(n+1)}(x)|$, but we can try to minimise the error by choosing an appropriate distribution of the points x_j. It is well-known that $||(x-x_0)\cdots(x-x_n)||_\infty$ is minimal on Ω if the points x_j are the zeroes of the $(n+1)$th degree Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos x)$. In fact, $\max_{x\in\Omega} \prod_{j=0}^{n} |x - x_j|$ is bounded by $\frac{1}{2^n}$ and $\frac{1}{2^{n-1}}$ for the zeros of $T_{n+1}(x)$ and the extrema of $T_n(x)$ (Chebyshev points of the first kind and Chebyshev-Lobatto points), respectively[7].

The operator that associates $f \in C(\Omega)$ to its interpolant is linear and given by

$$\begin{aligned} P_n : C(\Omega) &\to \mathcal{P}_n : \\ f(x) &\to p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x) \end{aligned} \tag{3}$$

where the i-th basic Lagrange polynomial is defined by

$$\ell_i(x) = \prod_{j=0,\, i\neq j}^{n} \frac{x - x_j}{x_i - x_j} \tag{4}$$

the unique polynomial of degree n which equals 0 at all points except x_i, at which it equals 1. So another bound for the interpolation error is given by[8]

$$||f - p_n||_\infty \le \left(1 + ||P_n||\right) ||f - p_n^*||_\infty, \qquad ||P_n|| = \max_{x\in[-1,1]} \sum_{i=0}^{n} \left|\ell_i(x)\right|. \tag{5}$$

Here, $\Lambda_n := \Lambda_n(x_0, \dots, x_n) = ||P_n||$ is called the Lebesgue constant, and p_n^* is the best polynomial approximation to f of degree n. It is clear that Λ_n depends on the location of the interpolation points x_j but not on the function values $f(x_i)$. Since the computation of the best approximation p_n^* is more complicated than that of the interpolant p_n, there is an interest in using interpolating polynomials that are near-best approximants. It follows from (5) that if Λ_n is small, then p_n is a near-best polynomial approximation to f.

The Lebesgue constant also expresses the conditioning of the polynomial interpolation problem in the Lagrange basis. Let $\tilde{p}_n(x)$ denote the polynomial interpolant of degree n for the perturbed function $\tilde{f}$ in the same interpolation points:

$$\tilde{p}_n(x) = \sum_{i=0}^{n} \tilde{f}(x_i)\ell_i(x).$$

Since $||p_n||_\infty \geq \max_{0\leq i\leq n} |f(x_i)|$ we have

$$\begin{aligned}\frac{||p_n - \tilde{p}_n||_\infty}{||p_n||_\infty} &\leq \frac{\max_{x\in\Omega} \sum_{i=0}^{n} |f(x_i) - \tilde{f}(x_i)||\ell_i(x)|}{\max_{0\leq i\leq n} |f(x_i)|} \\ &\leq \Lambda_n(x_0, \dots, x_n) \frac{\max_{0\leq i\leq n} |f(x_i) - \tilde{f}(x_i)|}{\max_{0\leq i\leq n} |f(x_i)|}.\end{aligned} \tag{6}$$

By virtue of (5) and (6), it is of interest to find interpolation points that lead to a small Lebesgue constant. An explicit formula for the points x_j that minimize the Lebesgue constant is not yet known, and if no further constraints are imposed on the interpolation points, then the solution is not even unique. Interpolation sets with the constraints that include the endpoints of the interval as interpolation points are called canonical interpolation sets. If one imposes these constraints on the interpolation points, then such a nodal set that minimizes the Lebesgue constant is unique[9,10]. Although no explicit formulation of this canonical nodal set is known, it is proved that the minimal growth of the Lebesgue constant satisfies the following estimate[11,12, pp. 110-121]:

$$\min\Lambda_n = \frac{2}{\pi}\log(n+1) + 0.52125\ldots + \mathcal{O}\left(\left(\frac{\log\log(n+1)}{\log(n+1)}\right)^2\right) \tag{7}$$

where $\gamma = 0.5772156\ldots$ is the Euler constant and $n+1$ is the number of interpolation points.

Several sets of interpolation points come close to realizing this minimal growth, among which the Chebyshev-Lobatto points are. Recall that these points are the extrema of the Chebyshev polynomial of the first kind of degree n on Ω, i.e.,

$$x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0, \dots, n. \tag{8}$$

Including the endpoints of the interval ± 1, this is a canonical interpolation set. The Lebesgue constant of the Chebyshev-Lobatto nodes satisfies the following estimate[13,14]:

$$\Lambda_n(x_0, \dots, x_n) = \frac{2}{\pi}\log(n) + 0.9625\ldots + \mathcal{O}(\frac{1}{n^2}), \tag{9}$$

which is very close to the minimal growth. In contrast, for equispaced nodes,

$$x_j = -1 + \frac{2j}{n}, \quad j = 0, \dots, n,$$

the Lebesgue constant grows exponentially with the asymptotic estimate[15,16]:

$$\Lambda_n(x_0, \dots, x_n) = \frac{2^{n+1}}{e\, n(\log(n) + \gamma)} + o(1).$$

3 | MOCK-CHEBYSHEV GRIDS

Polynomial interpolation on equidistant nodes is notoriously unreliable due to the Runge phenomenon, and also the interpolation process is exponentially ill-conditioned with Lebesgue constants of size about 2^n. This ill-conditioning means that, for instance, polynomial interpolants in equispaced points on $[-1, 1]$ will diverge on a computer as $n \to \infty$, even for a function like $f(x) = \exp(x)$ for which in principle they should converge. Moreover, in the paper[17], it has been shown that no stable method for approximating functions from equally spaced samples can converge exponentially for analytic functions.

Polynomial interpolation on Chebyshev-Lobatto points, by contrast, converge geometrically and stably for analytic functions. Moreover, as we have discussed in the previous section, for any sets of interpolation points, the Lebesgue constant grows at least logarithmically. The Chebyshev-Lobatto points are one system of points that has this optimal (logarithmically growing) Lebesgue constant. In order to make use of this result in real-world applications, where experimental data is available only at equally spaced intervals, interpolation at the Chebyshev-Lobatto points is mimicked, for instance, by selecting a proper subset of interpolation points $\tilde{x}_j$ from a sufficiently dense uniform grid with $\tilde{x}_j \approx x_j$ from 8. In other words, a mock-Chebyshev grid is a subset of $(n+1)$ points from a satisfactory uniform grid with $\mathcal{O}(n^2)$ points chosen to mimic the non-uniform $(n+1)$-point Chebyshev-Lobatto grid. Hence, we can say that a mock-Chebyshev grid is considered to satisfy the following properties:

(i) The points are distinct.

(ii) The points are distributed like Chebyshev points in the interval.

When computing a mock-Chebyshev grid, we should consider the above properties.

3.1 | Satisfactory Uniform Grid

In the mock-Chebyshev interpolation, a sufficiently dense uniform grid is considered to be a uniform satisfactory grid when the spacing between the points ensures that a grid point nearest to the endpoints of the interval $[-1, 1]$ is never repeated. In order to explain and prove the upper bound of the spacing for a sufficiently dense uniform grid, let us first introduce some notations as follows:

$$\begin{aligned}
X_n^{CL} &: \text{the set of (n + 1) Chebyshev-Lobatto points,} \\
& X_n^{CL} = \{x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, \dots, n\}, && (10) \\
\bar{X}_m &: \text{the set of (m + 1) equispaced points, with } \; m \geq \mathcal{O}(n^2) \\
& \bar{X}_m = \{\bar{x}_k = -1 + \frac{2k}{m}, \quad k = 0, 1, \dots, m\}, && (11) \\
\tilde{X}_n &: \text{the set of (n + 1) mock-Chebyshev points,} \\
& \tilde{X}_n = \{\tilde{x}_j : \tilde{x}_j \in \bar{X}_m, \text{ with } \tilde{x}_j \approx x_j, \quad j = 0, 1, \dots, n\}. && (12)
\end{aligned}$$

Let us now take a closer look at the distance between each of the two consecutive Chebyshev-Lobatto points. Since the Chebyshev-Lobatto points are more dense near the endpoints of the interval $[-1, 1]$ and less near its center, by using the symmetry we can write

$$x_j - x_{j-1} < x_{j+1} - x_j, \qquad j = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{13}$$

From this, the minimum distance between any two points of the set of Chebyshev-Lobatto points X_n^{CL} is $x_1 - x_0$ (or $x_n - x_{n-1}$). If the grid spacing of the set of $(m + 1)$ equispaced points $\bar{X}_m$, which is $\frac{2}{m}$, is smaller than $x_1 - x_0$, this ensures that a grid point nearest to a Chebyshev-Lobatto point x_j is never repeated. We now prove how to determine the minimum value of m for a uniform satisfactory grid. To do this, let us expand in Taylor series $x_1 - x_0$

$$x_1 - x_0 = \frac{\pi^2}{2n^2} - \frac{1}{24}\frac{\pi^4}{n^4} + \mathcal{O}(\frac{\pi^6}{n^6}).$$

Now, by employing this, we can write

$$\frac{2}{m} < \frac{\pi^2}{2n^2} - \frac{1}{24}\frac{\pi^4}{n^4}.$$

Indeed, it is not necessary to mimic the Chebyshev-Lobatto grid this closely. Using Taylor expansion of the cosine function for $x_2 - x_1$

$$x_2 - x_1 = \frac{3\pi^2}{2n^2} - \frac{5}{8}\frac{\pi^4}{n^4} + \mathcal{O}(\frac{\pi^6}{n^6}),$$

it is easy to verify that we have a satisfactory grid with

$$\frac{2}{m} < \frac{\pi^2}{n^2} - \frac{1}{12}\frac{\pi^4}{n^4}, \tag{14}$$

because this satisfies that the difference $\bar{x}_1 - x_1$ is less than both $x_2 - \bar{x}_1$ and $x_1 - x_0$ for $\bar{x}_1 > x_1$. Since

$$\frac{2n^2}{\pi^2} + 1 > \frac{2}{\frac{\pi^2}{n^2} - \frac{1}{12}\frac{\pi^4}{n^4}},$$

it is easy to conclude that the cardinality of a satisfactory uniform grid satisfies

$$m \geq \left\lceil \frac{2n^2}{\pi^2} \right\rceil + 1, \tag{15}$$

where $\lceil \cdot \rceil$ denotes the ceil function. If the positive integer m is less than this lower bound, then two grid points $\tilde{x}_j$ may coalesce at the endpoints ± 1(see Figure 1).

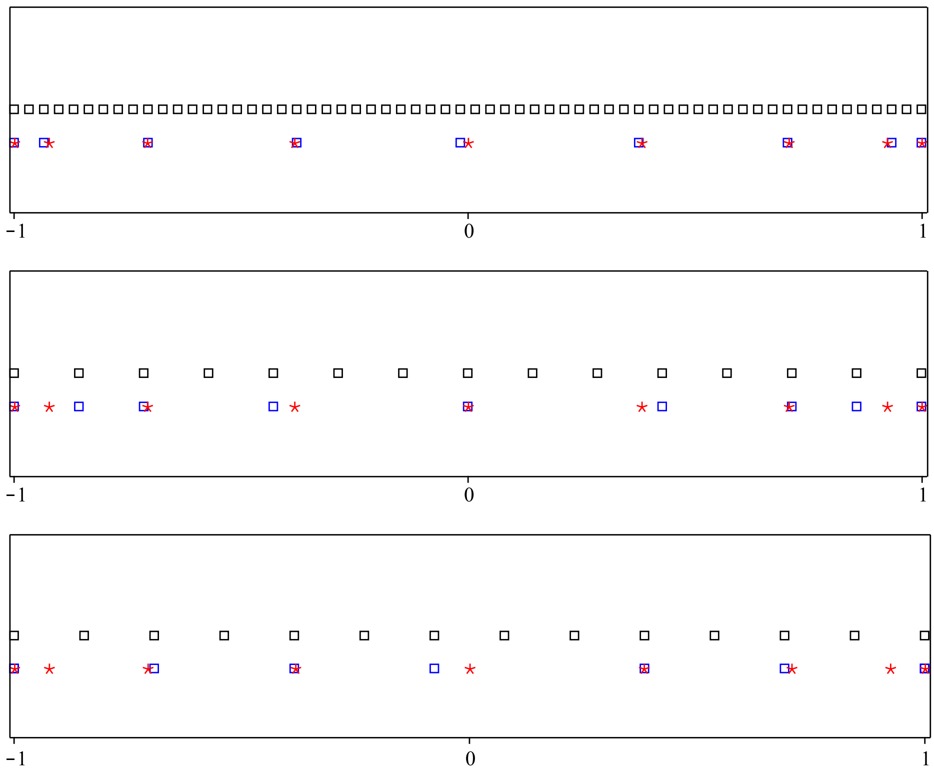

FIGURE 1 Graphs of 9 Chebyshev-Lobatto points ($*$) with their mock-Chebyshev points ($\square$) selected from equispaced grid ($\square$) for $m = 61, 14, 13$ from top to bottom, respectively. For $m = 13$, the endpoints -1 and 1 repeat due to $m < \left\lceil \frac{2n^2}{\pi^2} \right\rceil + 1$.

3.2 | Computing the mock-Chebyshev nodes

Boyd[5] first introduced a nice strategy for computing a "mock-Chebyshev" grid choosing the nodes from a uniform satisfactory grid, which is as follows (see Figure 2):

$$\begin{aligned} &\tilde{x}_j = \bar{x}_l, \quad \text{such that } |x_j - \bar{x}_l| = \min_k |x_j - \bar{x}_k|, \\ &\forall k = 0, 1, \dots, m, \quad j = 0, 1, \dots, n, \quad l \in \{0, 1, \dots, m\} \end{aligned} \tag{16}$$

where x_j, $\bar{x}_k$ and $\tilde{x}_j$ are defined as in Section 3.1. Since this strategy is based on comparing each x_j to all points $\bar{x}_k$, however, the computational cost is $\mathcal{O}(nm)$. That is, this is about $\mathcal{O}(n^3)$, where $n + 1$ is the number of the Chebyshev-Lobatto nodes. On the other hand, in the recent paper[6], we have presented a fast algorithm for computing the mock-Chebyshev nodes for a given set of $(n + 1)$ Chebyshev-Lobatto points with only an $\mathcal{O}(n)$ computational cost. The following explains the algorithm:

$$\begin{aligned} &\tilde{x}_j = \tilde{x}_0 + S_j \tilde{h}, \quad \tilde{h} = \frac{2}{S_n}, \quad \tilde{x}_0 = x_0, \quad S_0 = 0, \quad j = 1, 2, \dots, n, \\ &S_j = \left\lceil \frac{h_j(X_n)}{\min h_j(X_n)} \right\rceil + S_{j-1}, \quad h_j(X_n) = x_j - x_{j-1}, \end{aligned} \tag{17}$$

where x_j, $\bar{x}_k$ and $\tilde{x}_j$ are defined as in Section 3.1, and $\lceil \cdot \rceil$ denotes the ceil function. Note that, for this technique, it is not necessary to give a satisfactory uniform grid. In the sequel, $\tilde{x}^C$ (or $\tilde{x}_j^C$) denotes points obtained by this procedure.

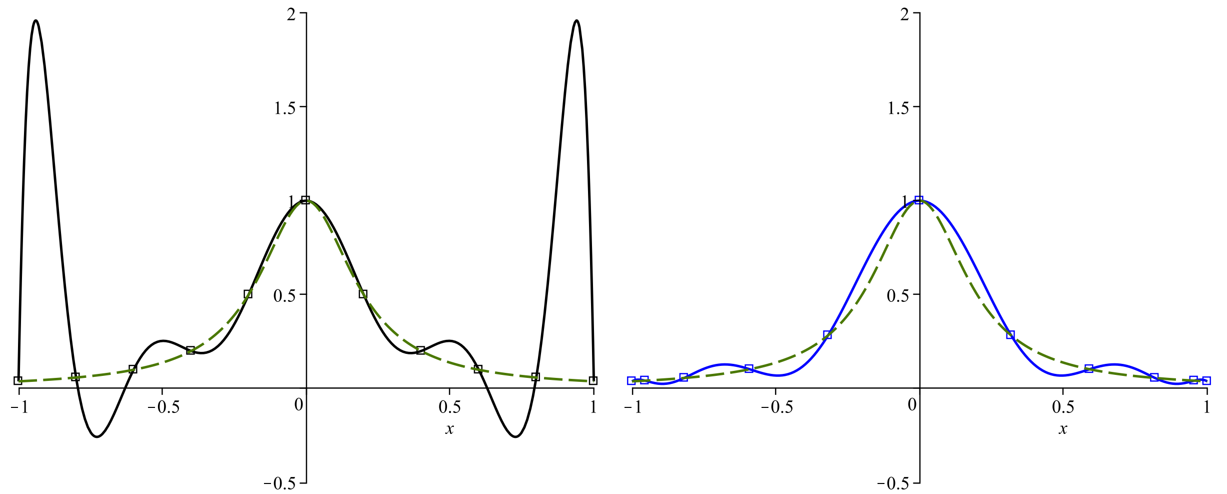

FIGURE 2 Graphs of Runge's function (dashed line) $f(x) = \frac{1}{1+25x^2}$, $-1 \le x \le 1$, and its polynomial interpolant (solid line) with equidistant points (□) (left) and with mock-Chebyshev points (□) (right) for degree $n = 10$.

4 | A NEW APPROACH FOR COMPUTING THE MOCK-CHEBYSHEV NODES

In section 3, we introduce the properties of a mock-Chebyshev grid. The following proposition explains these properties. Also, using the proposition, we can compute a family of mock-Chebyshev grids which satisfy the properties of a mock-Chebyshev grid.

Proposition 1. Let n and m be two positive integers with $m \ge \left\lceil \frac{2n^2}{\pi^2} \right\rceil + 1$, and let $\bar{X}_m = \{\bar{x}_k = -1 + \frac{2k}{m}, \quad k = 0, 1, \dots, m\}$ and $X_n^{CL} = \{x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, \dots, n\}$. Also, let $\bar{X}_{m,0} = \{\bar{x}_0 = -1\}$, $\bar{X}_{m,n} = \{\bar{x}_n = 1\}$, and $\bar{X}_{m,j} = \{\bar{x}_l : \frac{x_{j-1}+x_j}{2} < \bar{x}_l < \frac{x_j+x_{j+1}}{2}, \quad 1 \le j \le n-1, \ l \in \{0, 1, \dots, m\}\}$. If we choose the points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ such that

$$\tilde{x}_j \in \bar{X}_{m,j}, \quad j = 0, 1, \dots, n,$$

then we obtain a set of $(n+1)$ mock-Chebyshev points which satisfies the following properties:

1. The points $\tilde{x}_j$ are necessarily distinct.
2. The points $\tilde{x}_j$ are distributed like Chebyshev-Lobatto points in the interval $[-1, 1]$, i.e., the distance of the points $\tilde{x}_j$ closest to the endpoints ± 1 is $\mathcal{O}(n^{-2})$, while the distance between two interior points near the center is $\mathcal{O}(n^{-1})$.

Proof. In the following proofs we use the expansion of x_j in Taylor series to express the distance of the points x_j.

1. We begin by showing that

$$\bar{X}_{m,j} \ne \emptyset, \quad 1 \le j \le n-1.$$

Since $\bar{x}_1 - x_1 < x_2 - \bar{x}_1$ for a satisfactory uniform grid,

$$\bar{X}_{m,1} \ne \emptyset.$$

Using the upper bound of $\frac{2}{m}$ and the distance between $\frac{x_1+x_2}{2}$ and $\frac{x_2+x_3}{2}$, we obtain

$$\frac{2}{m} < \frac{\pi^2}{n^2} - \frac{1}{12}\frac{\pi^4}{n^4} < \frac{2\pi^2}{n^2} - \frac{27}{16}\frac{\pi^4}{n^4} < \frac{x_3 - x_1}{2}.$$

This implies that

$$\bar{X}_{m,2} \ne \emptyset.$$

Due to the distribution of Chebyshev-Lobatto points in the interval, we have

$$\frac{x_{j+1} - x_{j-1}}{2} > \frac{x_3 - x_1}{2}, \quad j = 3, \dots, n-3.$$

From this, we deduce that

$$\bar{X}_{m,j} \neq \emptyset, \quad 3 \leq j \leq n-3.$$

In conclusion, since $\bar{X}_{m,j} \cap \bar{X}_{m,j+1} = \emptyset, \quad j = 0, \ldots, n-1$ and $\bar{X}_{m,j} \neq \emptyset, \quad j = 0, \ldots, n$, one concludes that the points $\tilde{x}_j$ are necessarily distinct.

2. It is well-known that the Chebyshev-Lobatto points are distributed on $[-1, 1]$ with the density $\frac{n}{\pi\sqrt{1-x^2}}$, which implies that the distance between adjacent points is $\mathcal{O}(n^{-2})$ near the endpoints ± 1 and $\mathcal{O}(n^{-1})$ near the center.

   Since $\tilde{x}_j \in \bar{X}_{m,j}, \quad j = 0, 1, \ldots, n$, it follows that

$$\tilde{x}_0 = -1, \tilde{x}_n = 1, \frac{x_{j-1} + x_j}{2} < \tilde{x}_j < \frac{x_j + x_{j+1}}{2}, \quad 1 \leq j \leq n-1.$$

   From this, it is easy to verify that

$$\tilde{x}_{j+1} - \tilde{x}_j < 2(x_{j+1} - x_j), \quad j = 0, 1, \ldots, n-1.$$

   Using this, one concludes that the points $\tilde{x}_j$ are distributed like Chebyshev-Lobatto points in the interval $[-1, 1]$, i.e., the distance of the points $\tilde{x}_j$ closest to the endpoints ± 1 is $\mathcal{O}(n^{-2})$, while the distance between two interior points near the center is $\mathcal{O}(n^{-1})$.

□

Proposition 1 ensures that for a given set of $(n+1)$ Chebyshev-Lobatto points, the points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ satisfy the conditions of a mock-Chebyshev grid when they are chosen from a satisfactory uniform grid such that

$$\tilde{x}_0 = -1, \tilde{x}_n = 1, \frac{x_{j-1} + x_j}{2} < \tilde{x}_j < \frac{x_j + x_{j+1}}{2}, \quad 1 \leq j \leq n-1.$$

Note that this may give more than one configuration of the points in which the Chebyshev-Lobatto points are mimicked. But the question remains how to choose each $\tilde{x}_j$ from its associated $\bar{X}_{m,j}$ in order to make the maximum deviation between the points x_j and $\tilde{x}_j$, $\max_{0 \leq j \leq n} |x_j - \tilde{x}_j|$, is minimal. This can be done in the following way:

$$\tilde{x}_j = \bar{x}_l, \quad \text{such that } |x_j - \bar{x}_l| \text{ is minimum for all } \bar{x}_l \in \bar{X}_{m,j}.$$

This is probably the best case in the sense that it minimizes the error $\max_{0 \leq j \leq n} |x_j - \tilde{x}_j|$ with the maximum deviation of size $\mathcal{O}(n^{-2})$. In the sequel, we denote these points by $\tilde{x}^B$ (or $\tilde{x}^B_j$). Note that these points coincide with the points obtained by 16, but here, the computational cost is $\mathcal{O}(n^2)$ instead of $\mathcal{O}(n^3)$.
On the other hand, if the points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ are chosen in the following way

$$\tilde{x}_j = \bar{x}_l, \quad \text{such that } |x_j - \bar{x}_l| \text{ is maximum for all } \bar{x}_l \in \bar{X}_{m,j},$$

then this is probably the worst case in the sense of small maximum deviation (see Figure 3). We denote these points by $\tilde{x}^W$ (or $\tilde{x}^W_j$) in the sequel. So, it is clear that for a given set of $(n+1)$ Chebyshev-Lobatto points if we choose the $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ from a satisfactory uniform grid with $m \gg \left\lceil \frac{2n^2}{\pi^2} \right\rceil + 1$, then the maximum deviation decreases (see Figure 1).

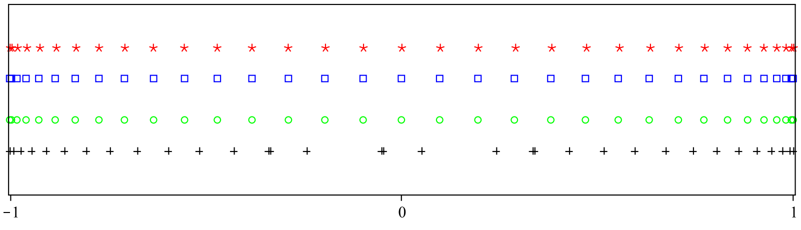

FIGURE 3 Graphs of sets 33 nodes; $(*)x^{CL}$, $(\square)\tilde{x}^B$, $(\circ)\tilde{x}^C$, $(+)\tilde{x}^W$ from top to bottom.

5 | NUMERICAL EXPERIMENTS

In this section, we give some numerical experiments using the points obtained by our method given in Section 4. To do this, as a standard quality measure of a set of interpolation points, we compare Lebesgue constants.

TABLE 1 Lebesgue constants of the Chebyshev-Lobatto points and mock-Chebyshev points

n	Chebyshev-Lobatto points, x^{CL}	points $\tilde{x}^B$	points $\tilde{x}^C$	points $\tilde{x}^W$
5	1.99	2.25	2.25	5.63
10	2.42	2.58	2.58	6.95
20	2.87	2.87	2.87	37.79
40	3.31	3.36	3.33	124.95
100	3.89	3.98	3.80	535.13

Table 1 presents Lebesgue constants for the Chebyshev-Lobatto points and their mock-Chebyshev points. From Table 1, one concludes that, although the points $\tilde{x}^B$ have the smallest maximum deviation, the points $\tilde{x}^C$ have smaller Lebesgue constants. Moreover, the Lebesgue constants of the points $\tilde{x}^B$ and $\tilde{x}^C$ grow like that of Chebyshev-Lobatto points, which has the minimal order of growth $\mathcal{O}(\log n)$. On the other hand, the points $\tilde{x}^W$ differ from the other mock-Chebyshev sets with Lebesgue constants of size about $\mathcal{O}(n)$.

6 | CONCLUSION

For a given set of $(n + 1)$ Chebyshev-Lobatto points, we presented a method based on subsetting an associated satisfactory uniform grid for constructing a subset of uniformly-distributed points $(n+1)$ which satisfies the conditions of a mock-Chebyshev grid. Using our method, we show that it is possible to obtain different configurations of the sets in which the given Chebyshev-Lobatto points are mimicked. This gives us the opportunity to compare different configurations of mock-Chebyshev sets. For a given set of Chebyshev-Lobatto points using our method, numerical evidence is presented suggesting that when the maximum deviation is small, the Lebesgue constant associated with these points grows like that of the given points.

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