A 2D diffusion coefficient determination problem for the time-fractional equation

D.K. Durdiev, A.A. Rahmonov, Z.R. Bozorov

Abstract. In this paper, we consider two dimensional inverse problem for a fractional diffusion equation. The inverse problem is reduced to the equivalent integral equation. For solving this equation the contracted mapping principle is applied. The local existence and global uniqueness results are proven. Also the stability estimate is obtained.

Keywords: Diffusion equation, Inverse problem, Gerasimov-Caputo fractional derivative, Hölder space, Integral equation.

1 Introduction and set up the problem

Fractional differential equations have excited, in recent years, a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and engineering (see, e.g., [1]-[6]). In [7]-[9] demonstrate a number of interesting features of the fractional diffusion equations, which represent a peculiar union of properties typical for second order parabolic differential equations.

The direct problems for fractional diffusion equations such as an initial or boundary value problems have been studied extensively in [1]-[4] and references therein. In contrast of direct problem, the mathematical analysis of inverse problem for the fractional diffusion equation is not satisfactorily investigated. The first mathematical results for the inverse problem of finding diffusion coefficient for a fractional differential equation are obtained in [5].

Inverse problems for classical differential equations of heat conduction have been studied quite widely. In literature are most often found the linear inverse source and nonlinear inverse coefficient problems with different type of over determination conditions (see, for example [12]-[16] and references there). In these works authors discussed the unique solvability and stability estimates of solution as well a numerical approach for solving such problems. The works [17]-[20] deal with a memory recovery problems from parabolic integro-differential equations of the second order with integral term of convolution type.

The main results of this paper are local existence, global uniqueness and the stability estimate in inverse problem of determining time-dependent reaction coefficient in the time-fractional diffusive equation by a single observation at the point $y = 0$ of the diffusion process.

Consider the following time-fractional diffusion equation:

$$\left({}^C\mathcal{D}_t^\alpha u\right)(\bar{x}, t) - \triangle_{\bar{x}} u + q(x,t)u(\bar{x}, t) = f(\bar{x}, t), \quad \bar{x} = (x, y),\ (\bar{x}, t) \in \mathbb{R}^2 \times \{t > 0\} \tag{1}$$

at condition

$$u\Big|_{t=0} = \varphi(\bar{x}), \quad \bar{x} \in \mathbb{R}^2, \tag{2}$$

where ${}^C\mathcal{D}_t^\alpha$, $0 < \alpha < 1$, is a regularized fractional derivative (the Gerasimov-Caputo derivative), that is

$$\left({}^C\mathcal{D}_t^\alpha u\right)(\bar{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{u_\tau(\bar{x}, \tau)d\tau}{(t-\tau)^\alpha},$$

and $f(\bar{x}, t)$, $\varphi(\bar{x})$ are given smooth functions.

Inverse problem. *Find the function $q(x,t)$, $x \in \mathbb{R}$, $t > 0$ in (1), if the solution to Cauchy problem (1), (2) satisfies*

$$u\Big|_{y=0} = g(x,t), \quad x \in \mathbb{R},\, t > 0, \tag{3}$$

where $g(x,t)$ is given.

We call a function $u(\bar{x},t)$ a classical solution to Cauchy problem (1) and (2), if:

(i) $u(\bar{x},t)$ is twice continuously differentiable in $\bar{x}$ for each $t > 0$;

(ii) for each $\bar{x} \in \mathbb{R}^2$ the Caputo derivative ${}^C\mathcal{D}_t^\alpha u(\bar{x},t)$ is continuous in t on $[0,T]$;

(iii) $u(\bar{x},t)$ satisfies (1) and (2).

Let $u(\bar{x},t)$ be a classical solution to Cauchy problem (1),(2) and f, φ, g be enough smooth functions. We carry out the next converting of the inverse problem (1)-(3). Denote for this purpose the second derivative of $u(\bar{x},t)$ with respect to y, by $v(\bar{x},t)$, i.e. $v(\bar{x},t) := u_{yy}(\bar{x},t)$. Differentiating (1) and (2) twice in y, we get

$$\left({}^C\mathcal{D}_t^\alpha v\right)(\bar{x},t) - \triangle_{\bar{x}} v + q(x,t)v(\bar{x},t) = f_{yy}(\bar{x},t), \quad t > 0,\, \bar{x} \in \mathbb{R}^2, \tag{4}$$

$$v\Big|_{t=0} = \varphi_{yy}(\bar{x}), \quad \bar{x} \in \mathbb{R}^2, \tag{5}$$

To obtain an additional condition for the function $v(\bar{x},t)$, we note that the second term of Laplacian in (1) is $v(\bar{x},t)$. Setting $y = 0$ in (1) and using equalities (2) and (3), we obtain

$$v\Big|_{y=0} = \left({}^C\mathcal{D}_t^\alpha g\right)(x,t) - g_{xx}(x,t) + q(x,t)g(x,t) - f(x,0,t), \quad t > 0,\, x \in \mathbb{R}. \tag{6}$$

When the matching condition $\varphi(x,0) = g(x,0)$ is fulfilled, it is easy to derive from (4)-(6) the equations (1)-(3).

For the given functions $q(x,t)$, $f(x,y,t)$, $\varphi(x,y)$ and a number $\alpha \in (0,1)$, the problem of determining the solution to Cauchy problem (4) and (5) we call as the direct problem.

By $\Pi_T := \{(\bar{x},t) : \bar{x} \in \mathbb{R}^2,\, 0 < t \le T\}$ we denote a strip with the thickness T, where $T > 0$ is any fixed number.

Let $C^{\alpha,m}(\Pi_T)$ be the class of the m times continuously differentiable, bounded with all derivatives of order up to m with respect to $\bar{x} \in \mathbb{R}^2$ variable and its fractional derivative ${}^C\mathcal{D}_t^\alpha$ is continuous in t on $[0,T]$ functions.

Everywhere in this paper we will denote by $H^l(\mathbb{R}^n)$ locally Hölder continuous functions with exponent $l \in (0,1)$. The norms in $H^l(\mathbb{R}^n)$ are determined in [21, pp. 15-20].

By $C(H^l(\mathbb{R}^n),[0,T])$ we denote the class of continuous with respect to t variable on the segment $[0,T]$ with values in $H^l(\mathbb{R}^n)$ functions. For a fixed t, the norm of the function $\phi(x,t)$ in $H^l(\mathbb{R}^n)$ will be denoted by $|\phi|^l(t)$. The norm of a function $\phi(x,t)$ in $C(H^l(\mathbb{R}^n),[0,T])$ is defined by the equality

$$\|\phi\|^l := \max_{t\in[0,T]} \big||\phi|^l(t)\big|.$$

2 Investigation of direct problem (4), (5)

In the paper [9] it was found the representation of the solution in terms of the fundamental solution to the following Cauchy problem

$${}^C\mathcal{D}_t^\alpha u - Bu(x,t) = F(x,t), \quad x \in \mathbb{R}^n,\, t \in (0,T],$$

$$u\Big|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n,$$

where

$$B := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j} + c(x)$$

is a uniformly second order elliptic differential operator with bounded continuous realvalued coefficients. In the case $B \equiv \triangle$, where $\triangle$ is $n-$dimensional laplacian, for any bounded continuous function $u_0(x)$ (locally Hölder continuous, if $n > 1$) and any bounded continuous with respect to the both variables $x,\ t$ and locally Hölder continuous in x function $F(x,t)$, it has the form

$$u(x,t) = \int_{\mathbb{R}^n} Z(x-\xi,t)u_0(\xi)d\xi + \int_0^t \int_{\mathbb{R}^n} Y(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau, \tag{7}$$

with

$$Z(x,t) = \pi^{-n/2}|x|^{-n}H^{2,0}_{1,2}\Big[\frac{1}{4}t^{-\alpha}|x|^2\Big|^{(1,\alpha)}_{(n/2,1),(1,1)}\Big],$$

$$Y(x,t) = \pi^{-n/2}|x|^{-n}t^{\alpha-1}H^{2,0}_{1,2}\Big[\frac{1}{4}t^{-\alpha}|x|^2\Big|^{(\alpha,\alpha)}_{(n/2,1),(1,1)}\Big],$$

where H is Fox's $H-$function (see, [10, pp. 2-6]). Actually, $Y(x,t)$ is the Riemann-Liouville derivative of $Z(x,t)$ with respect to t of the order $1-\alpha$ (for $x \neq 0$, $Z(x,t) \to 0$ as $t \to 0$, so that the Riemann-Liouville derivative coincides in this case with Grasimov-Caputo derivative, i.e. $Y(x,t) = \left({}^C\mathcal{D}_t^{1-\alpha}Z\right)(x,t))$ [9].

In (4), introducing the notation $f_{yy}(x,y,t) - q(x,t)v(x,y,t) =: F(x,y,t)$ and applying the formula (7) to direct problem (4), (5) for $n = 2$, we obtain the integral equation for determining $v(\bar{x},t)$:

$$v(\bar{x},t) = v_0(\bar{x},t) - \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)q(\xi_1,\tau)v(\bar{\xi},\tau)d\bar{\xi}d\tau, \tag{8}$$

where

$$v_0(\bar{x},t) := \int_{\mathbb{R}^2} Z(\bar{x}-\bar{\xi},t)\varphi_{\xi_2\xi_2}(\bar{\xi})d\bar{\xi} + \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)f_{\xi_2\xi_2}(\bar{\xi},\tau)d\bar{\xi}d\tau. \tag{9}$$

It is hold the following assertion:

Lemma 2.1. *If $q(x,t) \in C(H^{\alpha}(\mathbb{R}),[0,T])$, $f(\bar{x},t) \in C(H^{\alpha+2}(\mathbb{R}^2),[0,T])$, $\varphi(\bar{x}) \in H^{\alpha+2}(\mathbb{R}^2)$, then there exists a unique solution of the integral equation* (8) *$v(\bar{x},t) \in C^{\alpha,2}(\Pi_T)$, where $\alpha \in (0,1)$.*

Proof. For proof we use the method of successive approximations and consider the sequence of functions defined recursively by the formulas:

$$v_n(\bar{x},t) = -\int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)q(\xi_1,\tau)v_{n-1}(\bar{\xi},\tau)d\bar{\xi}d\tau,\ n = 1,2,..., \tag{10}$$

where $v_0(\bar{x},t)$ is determined by the equality (9). Further we use the following estimates [9] (for $n = 2$):

$$\begin{aligned} \left|D^m_{\bar{x}}Z(\bar{x},t)\right| &\le Ct^{-\frac{\alpha(2+m)}{2}}e^{-\mu_m t^{-\frac{\alpha}{2-\alpha}}|\bar{x}|^{\frac{2}{2-\alpha}}}, \\ \left|D^m_{\bar{x}}Y(\bar{x},t)\right| &\le Ct^{-\frac{\alpha(2+m)}{2}-1+\alpha}e^{-\mu_m t^{-\frac{\alpha}{2-\alpha}}|\bar{x}|^{\frac{2}{2-\alpha}}}, \\ \left|{}^C\mathcal{D}_t^{\alpha}Z(\bar{x},t)\right| &\le Ct^{-2\alpha}e^{-\mu_m t^{-\frac{\alpha}{2-\alpha}}|\bar{x}|^{\frac{2}{2-\alpha}}}, \end{aligned} \tag{11}$$

for $|\bar{x}|^2 \geq t^\alpha$, $|m| \leq 3$; where $\mu_0 := (2-\alpha)\alpha^{\alpha/(2-\alpha)}$, and as μ_m it can be taken any positive number less than μ_0;

$$|Z(\bar{x},t)| \leq Ct^{-\alpha}\Big[1+\big|\ln\big(t^{-\alpha}|\bar{x}|^2\big)\big|\Big], \tag{12}$$

$$|D_x^m Z(\bar{x},t)| \leq Ct^{-\alpha}|\bar{x}|^{-|m|}, \quad |m| \leq 3, \tag{13}$$

$$\big|{}^C\mathcal{D}_t^\alpha Z(\bar{x},t)\big| \leq Ct^{-\alpha}\Big[1+\big|\ln\big(t^{-\alpha}|\bar{x}|^2\big)\big|\Big], \tag{14}$$

$$|Y(\bar{x},t)| \leq Ct^{-1}, \tag{15}$$

$$|D_{\bar{x}}Y(\bar{x},t)| \leq Ct^{-\frac{\alpha}{2}-1}, \tag{16}$$

$$|D_{\bar{x}}^m Y(\bar{x},t)| \leq Ct^{-\alpha-1}\Big[1+\big|\ln\big(t^{-\alpha}|\bar{x}|^2\big)\big|\Big], \quad |m|=2, \tag{17}$$

$$|D_{\bar{x}}^m Y(\bar{x},t)| \leq Ct^{-\alpha-1}|\bar{x}|^{-1}\Big[1+\big|\ln\big(t^{-\alpha}|\bar{x}|^2\big)\big|\Big], \quad |m|=3, \tag{18}$$

for $t^\alpha \geq |\bar{x}|^2$, $(\bar{x},t) \in \Pi_T$. In (11)-(18) the letter C denotes various positive constants. We also note that it follows from the construction of the function $Z(\bar{x},t)$:

$$\int_{\mathbb{R}^2} Z(\bar{\xi},t)d\bar{\xi} = 1, \tag{19}$$

and it is true the equality [9]

$$\int_{\mathbb{R}^2} Y(\bar{\xi},t)d\bar{\xi} = C_0 t^{1-\alpha}, \quad t \in (0,T], \tag{20}$$

where C_0 depends only on α.

Set $q_0 := \|q\|^\alpha$, $\varphi_0 := |\varphi|^{\alpha+2}$ and $f_0 := \|f\|^{\alpha+2}$. Using (10), (19) and (20) we estimate the modulus of $v_n(\bar{x},t)$ in the domain Π_T as

$$|v_0(\bar{x},t)| \leq \varphi_0 + C_0 f_0 \frac{T^\alpha}{\alpha} =: \lambda_0,$$

$$|v_1(\bar{x},t)| \leq C_0 q_0 \lambda_0 \int_0^t (t-\tau)^{\alpha-1}d\tau = C_0 q_0 \lambda_0 \frac{t^\alpha}{\alpha} = \lambda_0 \frac{C_0 q_0 \Gamma(\alpha)}{\Gamma(1+\alpha)} t^\alpha,$$

$$|v_2(\bar{x},t)| \leq \lambda_0 (C_0 q_0 \Gamma(\alpha))^2 \frac{1}{\Gamma(1+\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\tau^\alpha d\tau}{(t-\tau)^{1-\alpha}} = \lambda_0 \frac{\big(C_0 q_0 \Gamma(\alpha)\big)^2}{\Gamma(1+\alpha)} I_{0+}^\alpha t^\alpha,$$

where $I_{0+}^\alpha t^\alpha$ is the Riemann-Liouville fractional integral of the power function t^α and $\Gamma(\cdot)$ is the Euler's gamma function. It is not difficult note (see, [11, p. 15]) that the formula

$$I_{0+}^\alpha t^{n\alpha} = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} t^{(1+n)\alpha}, \ n = 0,1,2,...$$

is valid. In accordance with this formula we continue to estimate $v_2(\bar{x},t)$:

$$|v_2(\bar{x},t)| \leq \lambda_0 \frac{\big(C_0 q_0 \Gamma(\alpha)\big)^2}{\Gamma(1+\alpha)} I_{0+}^\alpha t^\alpha = \lambda_0 \frac{\big(C_0 q_0 \Gamma(\alpha)\big)^2}{\Gamma(1+2\alpha)} t^{2\alpha},$$

For arbitrary $n = 0,1,2,...$ we have

$$|v_n(\bar{x},t)| \leq \lambda_0 \frac{\big(C_0 q_0 \Gamma(\alpha)\big)^n}{\Gamma(1+n\alpha)} t^{n\alpha}.$$

It follows from the above estimates that the series

$$v(\bar{x},t) = \sum_{n=0}^{\infty} v_n(\bar{x},t)$$

converges uniformly in Π_T, since it can be majorized in Π_T by the convergent numerical series

$$\lambda_0 \sum_{n=0}^{\infty} \frac{\big(C_0 q_0 \Gamma(\alpha) T^{\alpha}\big)^n}{\Gamma(1+n\alpha)}.$$

This means the following estimate for the solution of the integral equation (8) takes place:

$$|v(\bar{x},t)| \le \lambda_0 \sum_{n=0}^{\infty} \frac{\big(C_0 q_0 \Gamma(\alpha) T^{\alpha}\big)^n}{\Gamma(1+n\alpha)} = \lambda_0 E_{\alpha}\big(C_0 q_0 \Gamma(\alpha) T^{\alpha}\big), \quad (\bar{x},t) \in \Pi_T, \tag{21}$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function of a nonnegative real argument (see, [11, pp. 40-45]).

Note that $v_0(\bar{x},t)$ is the solution to the problem (4), (5) for $q(x,t) \equiv 0$. Under the assumptions of Lemma 2.1 it is true inclusion $v_0(\bar{x},t) \in C^{\alpha,2}(\Pi_T)$. Indeed, in accordance with the estimates (11)-(18), the first derivatives in x of function v_0, given by formula (9), can be calculated by differentiating the sub-sign of the integral. Calculating the second derivatives by definition and using the locally Hölder continuous in x, φ_{yy}, f_{yy}, as well as estimates of the third derivatives of Z, Y from (11)-(18) with respect to x, we have that v_0 has continuous derivatives up to the second order, inclusive [9]. The third estimates of (11), (14) and $Y(\bar{x},t) = \big({}^C\mathcal{D}_t^{1-\alpha} Z(\bar{x},t)\big)(\bar{x},t)$ implies the continuity of ${}^C\mathcal{D}_t^{\alpha} v_0$ in t on $[0,T]$.

From (10) it follows $v_n(\bar{x},t) \in C^{\alpha,2}(\Pi_T)$ for all $n = 1,2,\ldots$. Then, according to the general theory of functional series, this implies that the same property will be possessed the function $v(\bar{x},t)$. The function thus constructed is a classical solution to the problem (4), (5).

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (8) and the solution of this equation with perturbed functions $\tilde{q}$, $\tilde{f}_{yy}$ and $\tilde{\varphi}_{yy}$. Let $\tilde{v}(\bar{x},t)$ be a solution of the integral equation (8) corresponding to the functions $\tilde{q}$, $\tilde{f}_{yy}$ and $\tilde{\varphi}_{yy}$, i.e., determining $v(\bar{x},t)$:

$$\tilde{v}(\bar{x},t) = \tilde{v}_0(\bar{x},t) - \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\tilde{q}(\xi_1,\tau)\tilde{v}(\bar{\xi},\tau)d\bar{\xi}d\tau, \tag{22}$$

where

$$\tilde{v}_0(\bar{x},t) := \int_{\mathbb{R}^2} Z(\bar{x}-\bar{\xi},t)\tilde{\varphi}_{\xi_2\xi_2}(\bar{\xi})d\bar{\xi} + \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\tilde{f}_{\xi_2\xi_2}(\bar{\xi},\tau)d\bar{\xi}d\tau. \tag{23}$$

Composing the difference $v - \tilde{v}$ with the help of the equations (8) and (22), for it we obtain the integral equation

$$\begin{aligned} v(\bar{x},t) - \tilde{v}(\bar{x},t) = v_0(\bar{x},t) - \tilde{v}_0(\bar{x},t) - \\ - \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\big(q(\xi_1,\tau) - \tilde{q}(\xi_1,\tau)\big)v(\bar{\xi},\tau)d\bar{\xi}d\tau - \\ - \int_0^t \int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\tilde{q}(\xi_1,\tau)\big(v(\bar{\xi},\tau) - \tilde{v}(\bar{\xi},\tau)\big)d\bar{\xi}d\tau, \end{aligned} \tag{24}$$

from which, is derived the following linear integral inequality in $|v(\bar{x},t) - \tilde{v}(\bar{x},t)|$:

$$\big|v(\bar{x},t) - \tilde{v}(\bar{x},t)\big| \le \big|v_0(\bar{x},t) - \tilde{v}_0(\bar{x},t)\big| + \lambda_0 C_0 \frac{T^{\alpha}}{\alpha} E_{\alpha}\big(C_0 q_0 \Gamma(\alpha) T^{\alpha}\big)\|q - \tilde{q}\|^{\alpha} +$$

$$+\tilde{q}_0\int_0^t\int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\big|v(\bar{\xi},\tau)-\tilde{v}(\bar{\xi},\tau)\big|d\bar{\xi}d\tau, \tag{25}$$

where $\tilde{q}_0:=\|\tilde{q}\|^\alpha$ It follows from the equalities (9) and (23) the estimate

$$\big|v_0(\bar{x},t)-\tilde{v}_0(\bar{x},t)\big|\leq\|\varphi_{yy}-\tilde{\varphi}_{yy}\|^\alpha+C_0\frac{T^\alpha}{\alpha}\|f_{yy}-\tilde{f}_{yy}\|^\alpha.$$

Let $\sigma=\sigma(\alpha,T,q_0,\tilde{q}_0,\varphi_0,f_0)=\max\Big\{1,\tilde{q}_0,C_0\frac{T^\alpha}{\alpha},\lambda_0C_0\frac{T^\alpha}{\alpha}E_\alpha\big(C_0q_0\Gamma(\alpha)T^\alpha\big)\Big\}$. Applying the successive approximation method to inequality (25) with the help of the scheme

$$\big|v(\bar{x},t)-\tilde{v}(\bar{x},t)\big|_0\leq\sigma\Big(\|\varphi_{yy}-\tilde{\varphi}_{yy}\|^\alpha+\|f_{yy}-\tilde{f}_{yy}\|^\alpha+\|q-\tilde{q}\|^\alpha\Big),$$

$$\big|v(\bar{x},t)-\tilde{v}(\bar{x},t)\big|_n\leq\tilde{q}_0\int_0^t\int_{\mathbb{R}^2} Y(\bar{x}-\bar{\xi},t-\tau)\big|v(\bar{\xi},\tau)-\tilde{v}(\bar{\xi},\tau)\big|_{n-1}d\bar{\xi}d\tau,\ n=1,2,...,$$

we arrive at the estimate

$$\big|v(\bar{x},t)-\tilde{v}(\bar{x},t)\big|\leq\sigma\lambda_0E_\alpha\big(C_0q_0\Gamma(\alpha)T^\alpha\big)\Big(\|\varphi_{yy}-\tilde{\varphi}_{yy}\|^\alpha+\|f_{yy}-\tilde{f}_{yy}\|^\alpha+\|q-\tilde{q}\|^\alpha\Big), \tag{26}$$

which will be used in the next section of the paper. Indeed the expression (26) is the stability estimate for the solution to the Cauchy problem (4) and (5). The uniqueness for this solution follows from (26).

3 Investigation of the inverse problem (4)-(6)

Setting in (8) $x=0$ and using additional condition (6), after simple converting, we get the following integral equation for determining $q(x,t)$:

$$q(x,t)=q_0(x,t)-\frac{1}{g(x,t)}\int_0^t\int_{\mathbb{R}^2} Y(x-\xi_1,\xi_2,t-\tau)q(\xi_1,\tau)v(\xi_1,\xi_2,\tau)d\xi_1d\xi_2d\tau, \tag{27}$$

where

$$q_0(x,t):=\frac{1}{g(x,t)}\Bigg[f(x,0,t)+g_{xx}(x,t)-\left({}^C\mathcal{D}_t^\alpha g\right)(x,t)+$$
$$+\int_0^t\int_{\mathbb{R}^2} Z(x-\xi_1,\xi_2,t)\varphi_{\xi_2\xi_2}(\xi_1,0)d\xi_1d\xi_2+$$
$$+\int_0^t\int_{\mathbb{R}^2} Y(x-\xi_1,\xi_2,t-\tau)f_{\xi_2\xi_2}(\xi_1,\xi_2,\tau)d\xi_1d\xi_2d\tau\Bigg].$$

We introduce an operator A defining it by the right hand side of (27)

$$A[q](x,t)=q_0(x,t)-\frac{1}{g(x,t)}\int_0^t\int_{\mathbb{R}^2} Y(x-\xi_1,\xi_2,t-\tau)q(\xi_1,\tau)v(\xi_1,\xi_2,\tau)d\xi_1d\xi_2d\tau.$$

Then the equation (27) is written in a more convenient form as

$$q(x,t)=A[q](x,t). \tag{28}$$

Let $q_{00}:=\|q_0\|^\alpha$. Fix a number $\rho>0$ and consider the ball

$$B_T^\alpha(q_0,\rho):=\big\{q(x,t):\ q(x,t)\in C\big(H^\alpha(\mathbb{R}),[0,T]\big),\ \|q-q_0\|^\alpha\leq\rho\big\},\ \alpha\in(0,1).$$

Theorem 3.1. *If $f(\bar{x},t) \in C\big(H^{\alpha+2}(\mathbb{R}^2),[0,T]\big)$, $\varphi(\bar{x}) \in H^{\alpha+2}(\mathbb{R}^2)$, $g(x,t) \in C^1(H^\alpha(\mathbb{R}),[0,T])$, $\|g(x,t)\|^\alpha \geq g_0 > 0$, $g(0,0) = \varphi(0,0)$, then there exists a number $T^* \in (0,T)$, such that there exists a unique solution $q(x,t) \in C\big(H^\alpha(\mathbb{R}),[0,T^*]\big)$ of the inverse problem (1)-(3).*

Let us first prove that for an enough small $T > 0$ the operator A maps the ball $B_T^\alpha(q_0,\rho)$ into itself; i.e., the condition $q(x,t) \in B_T^\alpha(q_0,\rho)$ implies that $A[q](x,t) \in B_T^\alpha(q_0,\rho)$. Indeed, for any function $q(x,t)$ in $C\big(H^\alpha(\mathbb{R}),[0,T]\big)$ be continuous, implies that function $A[q](x,t)$ calculated using formula (28) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|A[q]-q_0\|^\alpha \leq \frac{C_0 q_0 \lambda_0}{\alpha g_0} T^\alpha E_\alpha\big(C_0 q_0 \Gamma(\alpha) T^\alpha\big).$$

Here we have used the estimate (21). Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(x,t)$ belongs to the ball $B_T^\alpha(q_0,\rho)$ implies the inequality

$$\|q\|^\alpha \leq \rho + q_{00}. \tag{29}$$

Therefore, we only strengthen the inequality if we replace $\|q\|^\alpha$ in this inequality with the expression $\rho + q_{00}$. Performing these replacements, we obtain the estimate

$$\|A[q]-q_0\|^\alpha \leq \frac{C_0 \lambda_0(\rho+q_{00})}{\alpha g_0} T^\alpha E_\alpha\big((\rho+q_{00}) C_0 \Gamma(\alpha) T^\alpha\big).$$

Let T_1 be a positive root of the equation

$$r(T_1) = \frac{C_0 \lambda_0(\rho+q_{00})}{\alpha g_0} T^\alpha E_\alpha\big((\rho+q_{00}) C_0 \Gamma(\alpha) T^\alpha\big) = \rho.$$

Then for $T \in [0,T_1]$ we have $A[q](x,t) \in B_T^\alpha(q_0,\rho)$.

Now consider two functions $q(x,t)$ and $\tilde{q}(x,t)$ belonging to the ball $B_T^\alpha(q_0,\rho)$ and estimate the distance between their images $A[q](x,t)$ and $A[\tilde{q}](x,t)$ in the space $C\big(H^\alpha(\mathbb{R}),[0,T]\big)$. The function $\tilde{v}(\bar{x},t)$ corresponding to $\tilde{q}(x,t)$ satisfies the integral equation (22) with the functions $\partial_y^2\varphi = \partial_y^2\tilde{\varphi}$ and $\partial_y^2 f = \partial_y^2\tilde{f}$. Composing the difference $A[q](x,t) - A[\tilde{q}](x,t)$ with the help of equations (8), (22) and then estimating its norm, we obtain

$$\|A[q](x,t) - A[\tilde{q}](x,t)\|^\alpha \leq \frac{C_0 T^\alpha}{\alpha g_0}\Big[\|v\|\|q-\tilde{q}\|^\alpha + \|q\|^\alpha\|v-\tilde{v}\|\Big].$$

Using inequality (21) and the estimate (26) with $\partial_y^2\varphi = \partial_y^2\tilde{\varphi}$ and $\partial_y^2 f = \partial_y^2\tilde{f}$, we continue the previous inequality in the following form:

$$\|A[q](x,t) - A[\tilde{q}](x,t)\|^\alpha \leq \frac{C_0 T^\alpha}{\alpha g_0}\lambda_0 E_\alpha\big(C_0 q_0 \Gamma(\alpha) T^\alpha\big)\big(1+\sigma\tilde{q}_0\big)\|q-\tilde{q}\|^\alpha. \tag{30}$$

The functions $q(x,t)$ and $\tilde{q}(x,t)$ belong to the ball $B_T^\alpha(q_0,\rho)$, and hence for each of these functions one has inequality (29). Note that the function on the right-hand side in inequality (30) at the factor $\|q-\tilde{q}\|^\alpha$ is monotone increasing with $\|q\|^\alpha$, $\|\tilde{q}\|^\alpha$ and T.

Consequently, replacing $\|q\|^\alpha$ and $\|\tilde{q}\|^\alpha$ in inequality (30)(including in σ) with $\rho + q_{00}$ will only strengthen the inequality. Thus, we have

$$\|A[q](x,t) - A[\tilde{q}](x,t)\|^\alpha \leq \frac{C_0 T^\alpha}{\alpha g_0}\lambda_0 E_\alpha\big((\rho+q_{00}) C_0 \Gamma(\alpha) T^\alpha\big)\big(1+\sigma(\rho+q_{00})\big)\|q-\tilde{q}\|^\alpha.$$

Let T_2 be a positive root of the equation

$$r_2(T) = \frac{C_0 T^\alpha}{\alpha g_0} \lambda_0 E_\alpha \big((\rho + q_{00}) C_0 \Gamma(\alpha) T^\alpha\big) \big(1 + \sigma(\rho + q_{00})\big) = 1.$$

Then for $T \in [0, T_2]$ we have that the distance between the functions $A[q](x,t)$ and $A[\tilde{q}](x,t)$ in the function space $C\big(H^\alpha(\mathbb{R}), [0,T]\big)$ is not greater than the distance between the functions $q(x,t)$ and $\tilde{q}(x,t)$ multiplied by $r_2(T) < 1$. Consequently, if we choose $T^* = \min(T_1, T_2)$, then the operator A is a contraction in the ball $B_T^\alpha(q_0, \rho)$. However, in accordance with the Banach theorem, the operator A has a unique fixed point in the ball $B_T^\alpha(q_0, \rho)$; i.e., there exists a unique solution of the equation (28). Theorem 3.1 is proven.

Let T be a positive fixed number. Consider the set $\Omega(\gamma_0)$ ($\gamma_0 > 0$ is some fixed number) of the given functions (f, φ, g) for which all conditions from Theorem 3.1 are fulfilled and so that $\max\big\{\|f\|^{\alpha+2}, \|\varphi\|^{\alpha+2}, \|g\|^\alpha\big\} \le \gamma_0$. By $Q(\gamma_1)$ we denote the class of functions $q(x,t) \in C\big(H^\alpha(\mathbb{R}), [0,T]\big)$, satisfying the inequality $\|q\|^\alpha \le \gamma_1$ with some fixed positive number γ_0.

Theorem 3.2. *Let $(f, \varphi, g) \in \Omega(\gamma_0)$, $(\tilde{f}, \tilde{\varphi}, \tilde{g}) \in \Omega(\gamma_0)$ and $(q, \tilde{q}) \in Q(\gamma_1)$. Then for the solution of the inverse problem (1)-(3) the following stability estimate is valid:*

$$\|q - \tilde{q}\|^\alpha \le c\Big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\Big), \tag{31}$$

where the constant c depends only on T, α, γ_0, γ_1.

To prove this theorem, using (27) we write down the equations for $\tilde{q}(x,t)$ and compose the difference $q(x,t) - \tilde{q}(x,t)$. Then, after evaluating this expression and using estimates (21), (26), we obtain

$$|q - \tilde{q}|^\alpha(t) \le c_0\Big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\Big) +$$

$$+ c_1 \int_0^t |q - \tilde{q}|^\alpha(\tau) d\tau, \quad t \in [0,T], \tag{32}$$

where c_0 and c_1 depend on the same constants as c. From (32) using Gronwall's inequality, we get the estimate

$$|q - \tilde{q}|^\alpha(t) \le c_0 \exp\big(c_1 t\big)\Big(\|f - \tilde{f}\|^{\alpha+2} + \|\varphi - \tilde{\varphi}\|^{\alpha+2} + \|g - \tilde{g}\|^\alpha\Big), \quad t \in [0,T].$$

This inequality implies the estimate (31), if we set $c = c_0 \exp\big(c_1 t\big)$.

From Theorem 3.2 readily follows the following uniqueness theorem for any $T > 0$:

Theorem 3.3. *Let the functions q, f, φ, g and $\tilde{q}$, $\tilde{f}$, $\tilde{\varphi}$, $\tilde{g}$ have the same meaning as in Theorem 3.2. Moreover, if $q = \tilde{q}$, $f = \tilde{f}$, $\varphi = \tilde{\varphi}$, $g = \tilde{g}$ for $(x,t) \in \Pi_T$, then $q(x,t) = q(x,t)$, $t \in \Pi_T$.*

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Durdimurod K. Durdiev
Bukhara Branch of the Institute of Mathematics at
the Academy of Sciences of the Republic of
Uzbekistan, Bukhara, Uzbekistan
E-mail: durdiev65@mail.ru,

Askar A. Rahmonov

Bukhara State University, Bukhara, Uzbekistan
E-mail: araxmonov@mail.ru,

Zavqiddin R. Bozorov
Bukhara Branch of the Institute of Mathematics at the Academy of Sciences of the Republic of Uzbekistan, Bukhara, Uzbekistan
E-mail: zavqiddinbozorov2011@mail.ru.