# Magnetoconvection in a Bidispersive Porous Media

G. Shiva Kumar Reddy, Ravi Ragoju

*Department of Applied Sciences, National Institute of Technology Goa, Goa, India, 403401.*

**Abstract**

Thermal instability of magnetoconvection in a horizontal bidispersive porous layer, uniformly heated from below, is analyzed. To study the linear stability theory, we perturbed the basic state with small-amplitude disturbances. Then, the governing dimensionless equations are solved using the normal modes. By employing the one-term Galerkin weighted residuals method, the critical values of Rayleigh numbers for the onset of stationary and oscillatory instability, have been determined. The effect of Chandrasekhar number on the system was analyzed.

**Keywords:** Bidispersive Porous Media, Thermal convection, Magnetic field, Linear stability analysis.

## 1. Introduction

Recently, many researchers are interested to study the convection bidispersive porous media (BDPM). A BDPM is an extension of a regular porous medium[1]. In general, a BDPM as a regular porous medium, where the solid phase is replaced by another porous medium. BDPM is composed of clusters of large particles that are agglomerations of small particles [1, 2]. In a BDPM, the voids between the clusters are known as macropores and the voids within the clusters are known as micropores. In other word, A BDPM is a porous medium in which fractures or tunnels have been introduced. In the present model, the f-phase and p-phase are represented by 'fracture phase' and 'porous phase' respectively.

Understanding convection in a BDPM is of considerable interest for geophysical applications [3, 4]. A general model for the convection in BDPM has received from

Nield & Kuznetsov [5] and Straughan [6]. In their analysis, they found that in a BDPM the critical values of Rayleigh number are much larger than that in the regular porous medium. Later, many research made an effort to investigate the convective instability in a BDPM. Very recently, Falsaperla, P., Mulone, G., and Straughan B [7] have investigated the linear and non linear stability analyses of Bidispersive-inclined convection. Gentile, M., and Straughan[8] have studied the problem of Bidispersive vertical convection. Straughan B. [9, 10] has investigated the double diffusive convection in a BDPM. Capone F, De Luca R, and Gentile M [11] have showed that the linear instability and nonlinear stability thresholds for a problem of thermal instability in a rotating BDPM. Later, Capone F and De Luca R [12] extended their work by considering inertia term and they showed that the effect of the Vadasz number can give rise to oscillatory motion at the loss of stability of thermal conduction solution.

In this paper, we reconsider the problem studied in [13], on taking into account the effect of an external magnetic field. The plan of the paper is as follows. Section 2 is devoted to the introduction of the mathematical problem. In Section 2, the governing dimensionless equations are solved using the normal modes to perform the linear stability analysis. The critical values of Rayleigh number at the onset of stationary and oscillatory convection, is determined by employing one-term Galerkin weighted residuals method. The paper ends with a Discussion and Conclusion section, which containing a table to show some examples in which stationary or oscillatory instability sets in and the figures showing the behavior of critical Rayleigh numbers and critical wave numbers for the steady and oscillatory instability versus Chandrasekhar number.

## 2. Mathematical formulation

Let us consider a horizontal BDPM of depth $d$. $T_L^0 C$, $T_U^0 C$ with $T_L > T_U > 0$ are the fixed temperatures at $z = 0$ and at $z = d$ respectively. $V_i^f$, $V_i^p$ and $T$ are the velocity of the fluid in the macro pores, the velocity of the fluid in the micro pores and the temperature. A constant magnetic field $H = (0, 0, H_0)$ is applied. The governing equations are

$$\nabla \cdot \mathbf{V}^f = 0, \nabla \cdot \mathbf{V}^p = 0, \nabla \cdot H = 0, \tag{1}$$

$$-\frac{\mu}{\kappa_f}\mathbf{V}^f - \delta\left(\mathbf{V}^f - \mathbf{V}^p\right) - \nabla P^f + \rho_0 g\beta T\hat{e}_z + \frac{\mu_m}{\rho_0}\left(\nabla \times H\right) \times H = 0, \tag{2}$$

$$-\frac{\mu}{\kappa_p}\mathbf{V}^p - \delta\left(\mathbf{V}^p - \mathbf{V}^f\right) - \nabla P^p + \rho_0 g\beta T\hat{e}_z + \frac{\mu_m}{\rho_0}\left(\nabla \times H\right) \times H = 0, \tag{3}$$

$$\sigma\frac{\partial T}{\partial t} + \left(\mathbf{V}^f + \mathbf{V}^p\right) \cdot \nabla T = \alpha\nabla^2 T, \tag{4}$$

$$\frac{\partial H}{\partial t} + \left(\mathbf{V}^f + \mathbf{V}^p\right) \cdot \nabla H - H \cdot \nabla\left(\mathbf{V}^f + \mathbf{V}^p\right) = \eta\nabla^2 H. \tag{5}$$

Subject to the boundary conditions

$$\mathbf{V}^f \cdot \hat{e}_z = \mathbf{V}^p \cdot \hat{e}_z = 0,\ H = H_0 \text{ on } z = 0,\ d. \tag{6}$$

$$T(x, y, 0, t) = 1,\ T(x, y, d, t) = 0. \tag{7}$$

Here, $\hat{e}_z = (0, 0, 1)$, $c_a,\ \kappa_f,\ \kappa_p,\ \mu,\ \delta,\ g,\ \beta,\ \sigma,\ \alpha,\ \mu_m$ are acceleration coefficient, permeability in the macro pores, permeability in the micro pores, dynamic viscosity, an interaction coefficient, gravity, coefficient of thermal expansion of the fluid, heat capacity ratio, thermal conductivity and magnetic permeability. In addition, $\rho_0$ is a reference density, $\nabla P^f$, $\nabla P^p$ are the reduced pressures in the macro and micro pores, $H_0$ is the external magnetic field and $k_m$ is the thermal conductivity. We introduce dimensionless quantities such that

$$\begin{aligned}
&x = x^* d, && y = y^* d, && z = z^* d,\\
&\mathbf{V}^f = \frac{\alpha}{d}\mathbf{V}^{f*}, && \mathbf{V}^p = \frac{\alpha}{d}\mathbf{V}^{p*}, && t = \frac{\sigma d^2}{\alpha}t^*,\\
&T = \Delta T T^*, && H = H_0 H^*
\end{aligned}$$

The non-dimensional equations (after omitting the asterisks) governing the system are

$$\nabla \cdot \mathbf{V}^f = 0, \nabla \cdot \mathbf{V}^p = 0, \nabla \cdot H = 0, \tag{8}$$

$$-\mathbf{V}^f - \gamma \left(\mathbf{V}^f - \mathbf{V}^p\right) - \nabla P^f + RaT\hat{e}_z + QPm\left(\nabla \times H\right) \times H = 0, \tag{9}$$

$$-\kappa_r \mathbf{V}^p - \gamma \left(\mathbf{V}^p - \mathbf{V}^f\right) - \nabla P^p + RaT\hat{e}_z + QPm\left(\nabla \times H\right) \times H = 0, \tag{10}$$

$$\frac{\partial T}{\partial t} + \left(\mathbf{V}^f + \mathbf{V}^p\right) \cdot \nabla T = \nabla^2 T, \tag{11}$$

$$\frac{1}{\sigma}\frac{\partial H}{\partial t} + \left(\mathbf{V}^f + \mathbf{V}^p\right) \cdot \nabla H - H \cdot \nabla \left(\mathbf{V}^f + \mathbf{V}^p\right) = Pm\nabla^2 H, \tag{12}$$

$$\mathbf{V}^f \cdot \hat{e}_z = \mathbf{V}^p \cdot \hat{e}_z = 0,\ H = \hat{e}_z \text{ on } z = 0,\ 1, \tag{13}$$

$$T(x, y, 0, t) = 1,\ T(x, y, 1, t) = 0. \tag{14}$$

Where

$$Ra = \frac{\rho_0 \Delta T g \beta d \kappa_f}{\mu \alpha}, \qquad \kappa_r = \frac{\kappa_f}{\kappa_p}, \qquad \gamma = \frac{\delta \kappa_f}{\mu},$$

$$Q = \frac{\mu_m H_0^2 \kappa_f}{\rho_0 \mu \eta}, \qquad Pm = \frac{\eta}{\alpha}.$$

Here $Ra$ is the Rayleigh number, $Q$ is the Chandrasekhar number, and $Pm$ is the Magnetic Prandtl number.

### *2.1. Basic State*

The basic stationary flow of Eqs. (8)-(12) is given by,

$$\mathbf{V}_b^f = 0,\ \mathbf{V}_b^p = 0,\ T_b = 1 - z,\ H_b = \hat{e}_z, \tag{15}$$

where the subscript $b$ indicates for basic state. Let us introduce the perturbation of the basic state in the form of

$$\mathbf{V}^f = \mathbf{V}_b^f + \epsilon \mathbf{V}^{f'},\ \mathbf{V}^p = \mathbf{V}_b^p + \epsilon \mathbf{V}^{p'},\ P^f = P_b^f + P^{f'},\ P^b = P_b^p + P^{p'},$$
$$T = T_b + \epsilon T',\ H = H_b + \epsilon H'. \tag{16}$$

where $\epsilon << 1$. By substituting Eq. (16) into Eqs. (8)-(14) and by neglecting terms $O(\epsilon^2)$ or higher, we obtain

$$\nabla \cdot \mathbf{V}^{f'} = 0, \nabla \cdot \mathbf{V}^{p'} = 0, \nabla \cdot H' = 0, \tag{17}$$

$$-\mathbf{V}^{f'} - \gamma \left( \mathbf{V}^{f'} - \mathbf{V}^{p'} \right) - \nabla P^{f'} + RaT'\hat{e}_z + QPm \left( \nabla \times H \right) \times \hat{e}_z = 0, \tag{18}$$

$$-\kappa_r \mathbf{V}^{p'} - \gamma \left( \mathbf{V}^{p'} - \mathbf{V}^{f'} \right) - \nabla P^{p'} + RaT'\hat{e}_z + QPm \left( \nabla \times H \right) \times \hat{e}_z = 0, \tag{19}$$

$$\frac{\partial T'}{\partial t} - \left( W^{f'} + W^{p'} \right) = \nabla^2 T', \tag{20}$$

$$\frac{1}{\sigma} \frac{\partial H'}{\partial t} = \frac{\partial}{\partial z} \left( \mathbf{V}^{f'} + \mathbf{V}^{p'} \right) + Pm \nabla^2 H'. \tag{21}$$

$$W^{f'} = W^{p'} = T' = \frac{\partial H'_z}{\partial z} = 0 \text{ on } z = 0,\ 1. \tag{22}$$

By taking the third component of double curl of Eqs. (18) and (19) and third component of Eq. (21), one obtains

$$\nabla \cdot \mathbf{V}^{f'} = 0, \nabla \cdot \mathbf{V}^{p'} = 0, \nabla \cdot H' = 0, \tag{23}$$

$$\nabla^2 W^{f'} + \gamma \left( \nabla^2 W^{f'} - \nabla^2 W^{p'} \right) - Ra \nabla_h^2 T' - QPm \frac{\partial}{\partial z} \nabla^2 H'_z = 0, \tag{24}$$

$$\kappa_r \nabla^2 W^{p'} + \gamma \left( \nabla^2 W^{p'} - \nabla^2 W^{f'} \right) - Ra \nabla_h^2 T' - QPm \frac{\partial}{\partial z} \nabla^2 H'_z = 0, \tag{25}$$

$$\frac{\partial T'}{\partial t} - \left( W^{f'} + W^{p'} \right) = \nabla^2 T', \tag{26}$$

$$\frac{1}{\sigma} \frac{\partial H'_z}{\partial t} = \frac{\partial}{\partial z} \left( W^{f'} + W^{p'} \right) + Pm \nabla^2 H'_z, \tag{27}$$

$$W^{f'} = W^{p'} = T' = \frac{\partial H'_z}{\partial z} = 0 \text{ on } z = 0,\ 1, \tag{28}$$

where $\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

### 3. Linear Stability Analysis

Let us introduce the normal modes by writing the perturbations in the form of

$$\left( W^{f'}, W^{p'}, T', H'_z \right) = \left( w^f(z), w^p(z), \theta(z), h(z) \right) e^{i(lx + my - \omega t)}. \tag{29}$$

Substituting the above normal mode solution into the Eqs. (23)-(28),

$$(1+\gamma)\left(D^2-q^2\right)w^f-\gamma\left(D^2-q^2\right)w^p+Raq^2\theta-QPmD\left(D^2-q^2\right)h=0, \tag{30}$$

$$(\kappa_r+\gamma)\left(D^2-q^2\right)w^p-\gamma\left(D^2-q^2\right)w^f+Raq^2\theta-QPmD\left(D^2-q^2\right)h=0, \tag{31}$$

$$\left(D^2-q^2+i\omega\right)\theta+\left(w^f+w^p\right)=0, \tag{32}$$

$$Pm\left(D^2-q^2\right)h+\frac{1}{\sigma}i\omega h+D\left(w^f+w^p\right)=0, \tag{33}$$

$$w^f=w^p=\theta=Dh=0 \text{ on } z=0,\ 1, \tag{34}$$

where $D=\frac{d}{dz}$ and $q^2=l^2+m^2$. We assume the solution to $w^f, w^p, \theta$ and $h$ in the form

$$\begin{pmatrix} w^f(z) \\ w^p(z) \\ \theta(z) \\ h(z) \end{pmatrix} = \begin{pmatrix} w_0^f\sin n\pi z \\ w_0^p\sin n\pi z \\ \theta_0\sin n\pi z \\ h_0\cos n\pi z \end{pmatrix}, \tag{35}$$

which satisfy the boundary conditions (34). On substituting Eq. (35) with $n=1$ into Eqs. (30)-(33), one obtains

$$\begin{pmatrix} -\delta^2(1+\gamma) & \gamma\delta^2 & Raq^2 & -QPm\pi\delta^2 \\ \gamma\delta^2 & -\delta^2(\kappa_r+\gamma) & Raq^2 & -QPm\pi\delta^2 \\ 1 & 1 & \left(-\delta^2+i\omega\right) & 0 \\ \pi & \pi & 0 & \left(-Pm\delta^2+\frac{i\omega}{\sigma}\right) \end{pmatrix} \begin{pmatrix} w_0^f \\ w_0^p \\ \theta_0 \\ h_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{36}$$

where $\delta^2=\pi^2+q^2$. For the nontrivial solution of the above matrix Eq. (36) the determinant of above matrix is zero, from which one can get the expression of the Rayleigh numbers $Ra_{sc}$ and $Ra_{oc}$ respectively for the stationary and also oscillatory modes of convection in the form:

$$Ra_{sc} = \frac{\delta_{sc}^4 \left(\gamma + \kappa_r + \gamma\kappa_r\right)}{q_{sc}^2 \left(1 + \kappa_r + 4\gamma\right)} + \frac{Q\pi^2\delta^2}{q_{sc}^2}, \quad Ra_{oc} = \frac{I_1 + \omega^2 I_2}{q_{oc}^2 \chi_2 \left(\omega^2 + Pm^2\delta_{oc}^4\sigma^2\right)}, \tag{37}$$

where $I_1 = Pm^2\delta_{oc}^6(\chi_2\pi^2 Q + \chi_1\delta_{oc}^2)\sigma^2$, $I_2 = \delta_{oc}^2(\chi_1\delta_{oc}^2 + \chi_2\pi^2 PmQ\sigma)$,
$\omega^2 = Pm\delta^2\sigma(Pm\delta^2\sigma + \frac{\chi_2}{\chi_1}\pi^2 Q(-1 + Pm\sigma))$, $\chi_1 = \kappa_r + \gamma + \gamma\kappa_r$ and $\chi_2 = 1 + \kappa_r + 4\gamma$.
and the subscripts $sc$ and $oc$ denote stationary and oscillatory convections.

In the absence of magnetic field, above stationary Rayleigh number reduces to

$$Ra_{sc} = \frac{\delta^4 \left(\gamma + \kappa_r + \gamma\kappa_r\right)}{q^2 \left(1 + \kappa_r + 4\gamma\right)}, \tag{38}$$

which is well agree with Eq. (31) in Gentile and Straughan [13].

## 4. Discussion and Conclusions

The numerical results and conclusions are presented in this section. Many theoretical and experimental studies have made an effort to model the convective instability problem that arises in the study of geophysical fluid dynamics. In the present analysis, a mathematical model of magnetoconvection in a bidispersive porous layer has been modeled to investigate the linear stability analysis. To the best of author's knowledge, this is the first study in the hydrodynamic stability. Darcy's law and the Oberbeck-Boussinesq approximation are adopted for modeling the buoyant flow. The one-term Galerkin weighted residuals method, has been employed to determine the critical Rayleigh numbers for the onset of stationary and oscillatory instability.

In the present analysis, we fix $\sigma = 1$ and $\gamma = 1$. In Table 1, some values of the physical parameters have been fixed, in order to show some examples in which stationary or oscillatory instability sets in. For example, when $Q = 5$, the onset of convection is more likely to be via oscillatory convection when $Pm < 0.82$, whereas the stationary convection dominates when $Pm \geq 0.82$. Furthermore, one can observe from this Table that $Pm$ does not show any effect on $Ra_{sc}^c$, since $Ra_{sc}$ is independent of $Pm$.

| $Q$ | $Pm$ | Stationary | | Oscillatory | | Instability |
|---|---|---|---|---|---|---|
| | | $q_{sc}^c$ | $Ra_{sc}^c$ | $q_{oc}^c$ | $Ra_{oc}^c$ | |
| 5 | 0.4 | 6.82311 | 75.9007 | 4.58621 | 27.3207 | Oscillatory |
| 5 | 0.6 | 6.82311 | 75.9007 | 5.12303 | 43.5326 | Oscillatory |
| 5 | 0.81 | 6.82311 | 75.9007 | 5.65649 | 75.6315 | Oscillatory |
| 5 | 0.82 | 6.82311 | 75.9007 | 5.66515 | 76.4059 | Stationary |
| 5 | 1.2 | 6.82311 | 75.9007 | 5.91785 | 105.693 | Stationary |
| 5 | 1.4 | 6.82311 | 75.9007 | 6.01025 | 121.032 | Stationary |
| 10 | 0.4 | 8.06812 | 133.973 | 5.98167 | 69.5537 | Oscillatory |
| 10 | 0.6 | 8.06812 | 133.973 | 6.37327 | 97.2322 | Oscillatory |
| 10 | 0.81 | 8.06812 | 133.973 | 6.69704 | 132.775 | Oscillatory |
| 10 | 0.82 | 8.06812 | 133.973 | 6.70692 | 134.136 | Stationary |
| 10 | 1.2 | 8.06812 | 133.973 | 6.9666 | 178.9 | Stationary |
| 10 | 1.4 | 8.06812 | 133.973 | 7.07978 | 205.931 | Stationary |

Table 1: Onset of stationary and oscillatory instability for the fixed values of $\kappa_r = 0.1$.

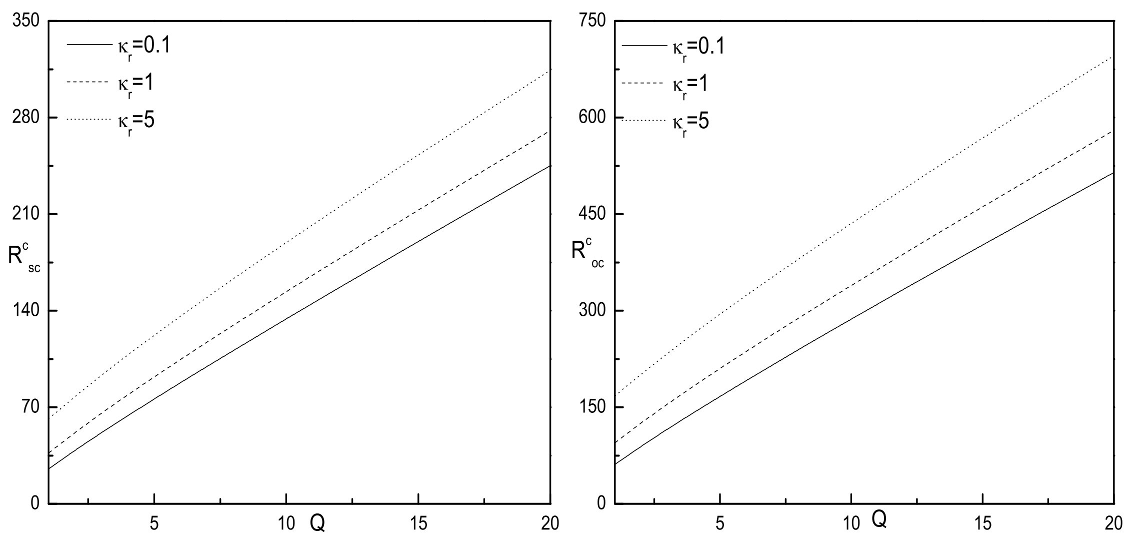


Figure 1: Variation of critical Rayleigh number with $Q$ at the onset of (a) stationary convection and (b) oscillatory convection.

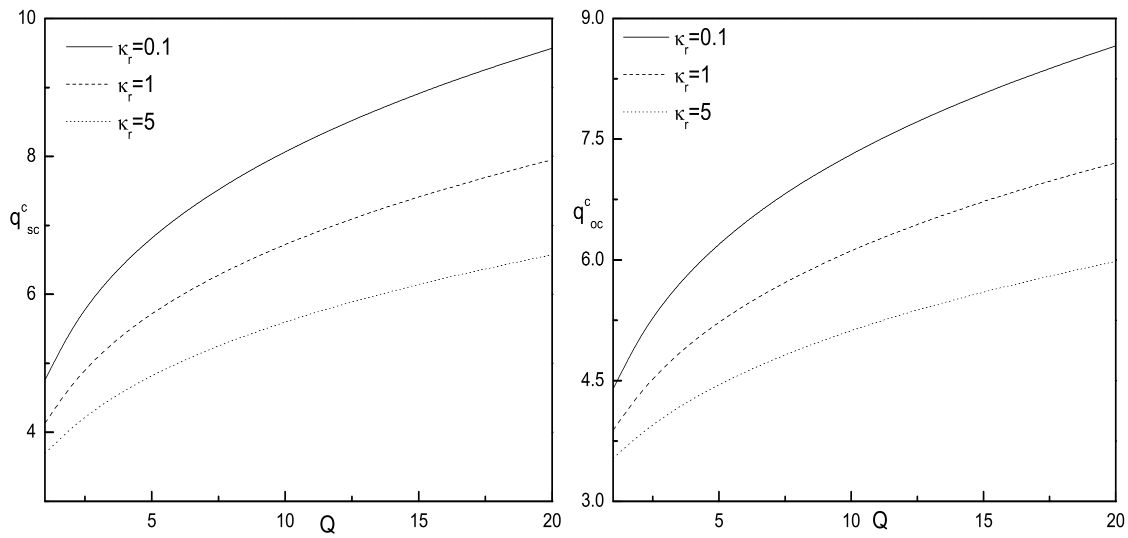


Figure 2: Variation of critical wave number with $Q$ at the onset of (a) stationary convection and (b) oscillatory convection.

The effect of the Chandrasekhar number, $Q$, for the onset of steady and oscillatory convection, has been examined. In Fig. 1, the effect of the Chandrasekhar number, $Q$, on the onset of instability, is showed. The curves displayed are included in the range $1 \leq Q \leq 20$. Fig. 1 depicts an increasing trend of critical Rayleigh number at the onset of steady and oscillatory instability versus $Q$, and also a increasing function of $\kappa_r$. In other words, an increase in the value of $Q$ makes the system stable.

Fig. 2 gives a variation of critical wave number (corresponding to critical Rayleigh number from Fig. 1) at the onset of steady and oscillatory convection versus $Q$ for different values of $\kappa_r$. Fig. 2 depicts an increasing trend of critical wave number versus $Q$, and a decreasing function of $\kappa_r$.

The principal results found from the linear stability analysis can be summarized as follows:

- The eigenvalue problem for linear stability analysis is solved analytically by using normal mode technique.

- The critical Rayleigh number corresponding wave number are analyzed for the different values of other physical parameters at the onset of stationary and oscillatory convection.

- The Chandrasekhar number has a stabilizing effect on the stationary and oscillatory convection.
- $\kappa_r$ has a stabilizing effect on the stationary and oscillatory convection.
- The critical wave number is an increasing function of $Q$ and a decreasing function of $\kappa_r$ at the onset of stationary and oscillatory convection.


[1] Z.Q. Chen, P. Cheng, C.T. Hsu, 2000 A theoretical and experimental study on stagnant thermal conductivity of bi-dispersed porous media, *Int. Commun. Heat Mass Transfer*. 27, 601610.

[2] Z.Q. Chen, P. Cheng, T.S. Zhao, 2000 An experimental study of two phase flow and boiling heat transfer in bi-disperse porous channels, *Int. Commun. Heat Mass Transfer* 27, 293302.

[3] GRARD, A., GENTER, A., KOHL, T., LUTZ, P., ROSE, P. & RUMMEL, F. 2006 The deep EGS (Enhanced Geothermal System) project at Soultz-sous-Forłts (Alsace, France). *Geothermics* 35 (5), 473483.

[4] NIE, R. S., MENG, Y. F., JIA, Y. L., ZHANG, F. X., YANG, X. T. & NIU, X. N. 2012 Dual porosity and dual permeability modeling of horizontal well in naturally fractured reservoir. *Transp. Porous Med.* 92 (1), 213235.

[5] Nield DA, Kuznetsov AV. 2006 The onset of convection in a bidispersive porous medium. *Int. J. Heat Mass Transf.* 49, 30683074.

[6] Straughan B. 2009 On the NieldKuznetsov theory for convection in bidispersive porous media. *Transp. Porous Med.* 77, 159168.

[7] Falsaperla, P., Mulone, G., & Straughan, B. 2016. Bidispersive-inclined convection. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* , 472(2192), 20160480.

[8] Gentile, M., & Straughan, B. 2017. Bidispersive vertical convection. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 473(2207), 20170481.

[9] Straughan, B. (2018). Bidispersive double diffusive convection. *International Journal of Heat and Mass Transfer*, 126, 504-508.

[10] Straughan, B. (2019). Effect of inertia on double diffusive bidispersive convection. *International Journal of Heat and Mass Transfer*, 129, 389-396.

[11] Capone, F., De Luca, R., & Gentile, M. 2020. Coriolis effect on thermal convection in a rotating bidispersive porous layer. *Proceedings of the Royal Society A*, 476(2235), 20190875.

[12] Capone, F., & De Luca, R. (2020). The Effect of the Vadasz Number on the Onset of Thermal Convection in Rotating Bidispersive Porous Media. *Fluids*, 5(4), 173.

[13] Gentile, M., & Straughan, B. (2017). Bidispersive thermal convection. International Journal of Heat and Mass Transfer, 114, 837-840.