OBTAINING THE EQUATIONS OF MOTION OF A MECHANICAL BODY THROUGH ANALYTICAL METHODS

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The aim of this paper is to obtain the equations of motion in n-dimensional space for the case where no external forces act on a mechanical system using analytical methods.

One such method is known as Lagrangian Mechanics. Lagrangian Mechanics is founded on the principle of least action which states that the spontaneous change from one configuration to another of a dynamical system has a minimum action value if the law of conservation of energy holds.

We define the Lagrangian to be a smooth function $L(q, v, t)$ and write the non-relativistic Lagrangian:

$$L = T - V$$

The kinetic energy of the system is denoted by T while the potential energy of the system is denoted by V. The equation can therefore be rewritten as:

$$L = \frac{1}{2}mv^2 - mgh$$

Furthermore, T and V will be generalised for a system of particles and will have form:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i {v_i}^2$$

And

$$V = \sum_{i=1}^{N} m_i g_i (y_c^0)_i$$

However, a more formal introduction to the above notation will be presented later since we are currently only concerned about one particle with uniform mass distribution.

APPLYING THE LAGRANGIAN TO A MECHANICAL SYSTEM IN THE ABSENCE OF EXTERNAL FORCES

Let us suppose that there exists some particle p centred at its origin O' relative to some stationary point O such that $O, O' \in \mathbb{R}^2$. We also let the respective axes of O be parallel to that of O'.

We define a position vector $\vec{r}$ pointing from O to O' at some angle φ, then

$$\vec{r} = |\vec{r}| \cos\varphi \,\hat{\imath} + |\vec{r}| \sin\varphi \,\hat{\jmath}$$

denotes the position of p with uniform mass m relative to O. From this result, we can separate $\vec{r}$ into its $\hat{\imath}$ and $\hat{\jmath}$ components, respectively.

The component with respect to the $\hat{\imath}$ direction denoted by r_x , is given by:

$$r_x = |\vec{r}| \cos\varphi$$

The component with respect to the $\hat{\jmath}$ direction denoted by r_y , is given by:

$$r_y = |\vec{r}| \sin\varphi$$

We then calculate the Lagrangian, $L = T - V$ for the system by using the equations from above. The kinetic energy term can be written and expanded as:

$$T = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}m\dot{r}^2$$

$$T = \frac{1}{2}m({\dot{r}_x}^2 + {\dot{r}_y}^2)$$

Note that r is the magnitude of $\vec{r}$ and that $\dot{r}$ denotes the time derivative of r.

Similary, the potential energy term can be expressed as:

$$V = mgh$$

$$V = mgr_y$$

We can then write the Lagrangian for the kinetic energy and potential energy of the system as:

$$L = \frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y$$

The Euler – Lagrange Equation can then be written for the Lagrangian:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

Since our system is defined by two spatial dimensions we expect two solutions to the Euler – Lagrange equation.

We first find the solution obtained from r_x:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r}_x}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right) = \frac{\partial}{\partial r_x}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right)$$

$$\frac{d}{dt}(m\dot{r}_x) = 0$$

$$\ddot{r}_x = 0$$

Which is valid since the acceleration in the $\hat{\imath}$ direction is zero. By a similar manner, we find the solution obtained from r_y:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r}_y}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right) = \frac{\partial}{\partial r_y}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right)$$

$$\frac{d}{dt}\left(m\dot{r}_y\right) = -mg$$

$$\ddot{r}_y = -g$$

Which is also a valid result since we expect that p accelerates due to gravity in the $\hat{\jmath}$ direction.

We assume that p has some initial velocity $\vec{v}_0$ at some angle θ relative to O'. Then $\vec{v}_0$ can be written in terms of its $\hat{\imath}'$ and $\hat{\jmath}'$. Since O is parallel to O' we can say that $\hat{\imath} \equiv \hat{\imath}'$ and also that $\hat{\jmath} \equiv \hat{\jmath}'$ such that:

$$\vec{v}_0 = |\vec{v}_0| \cos\theta\, \hat{\imath} + |\vec{v}_0| \sin\theta\, \hat{\jmath}$$

Note that $\vec{v}_0 = \dot{\boldsymbol{r}}_0$.

Finally, the acceleration vector for p can be written as the sum of its components, which were obtained by solving the Euler – Lagrange equation:

$$\ddot{\boldsymbol{r}} = -g\hat{\jmath}$$

Since we know that $\ddot{\boldsymbol{r}} = \frac{d^2\boldsymbol{r}}{dt^2}$, we can show that:

$$\frac{d^2\boldsymbol{r}}{dt^2} = -g\hat{\jmath}$$

$$\int d^2\boldsymbol{r} = \int -g\hat{\jmath}\, dt^2$$

$$\frac{d\boldsymbol{r}}{dt} = -gt\hat{\jmath} + \dot{\boldsymbol{r}}_0$$

$$\dot{\boldsymbol{r}}(t) = |\dot{\boldsymbol{r}}_0| \cos\theta\, \hat{\imath} + (|\dot{\boldsymbol{r}}_0| \sin\theta - gt)\hat{\jmath}$$

The result obtained from integrating $\ddot{\boldsymbol{r}}$ is the velocity function of particle p. It is important to note that integrating $\dot{\boldsymbol{r}}(t)$ with respect to time gives us two solutions, one which gives the position vector as a function of time for O' relative to O and the other gives the position vector as a function of time for p with respect to O'.

Since we have already defined the position vector of O' with respect to O, we will determine the first solution first. We know that $\boldsymbol{r}_0 = \vec{r} = |\vec{r}| \cos\varphi\, \hat{\imath} + |\vec{r}| \sin\varphi\, \hat{\jmath}$, so the position vector of O' relative to O as a function of time can be calculated by:

$$\int \dot{\boldsymbol{r}}(t)\, dt = \int |\dot{\boldsymbol{r}}_0| \cos\theta\, \hat{\imath} + (|\dot{\boldsymbol{r}}_0| \sin\theta - gt)\hat{\jmath}\, dt$$

$$\boldsymbol{r}(t) = |\dot{\boldsymbol{r}}_0| \cos\theta t\, \hat{\imath} + \left(|\dot{\boldsymbol{r}}_0| \sin\theta t - \frac{1}{2}gt^2\right)\hat{\jmath} + \boldsymbol{r}_0$$

$$\boldsymbol{r}(t) = (|\dot{\boldsymbol{r}}_0| \cos\theta t + |\vec{r}| \cos\varphi)\hat{\imath} + \left(|\dot{\boldsymbol{r}}_0| \sin\theta t - \frac{1}{2}gt^2 + |\vec{r}| \sin\varphi\right)\hat{\jmath}$$

We obtain a similar result for the position vector as a function of time for p with respect to O' since $\boldsymbol{r}'_0 = \vec{0}$.

$$\int \dot{\boldsymbol{r}}(t)\, dt = \int |\dot{\boldsymbol{r}}_0| \cos\theta\, \hat{\imath} + (|\dot{\boldsymbol{r}}_0| \sin\theta - gt)\hat{\jmath}\, dt$$

$$\boldsymbol{r}(t) = |\dot{\boldsymbol{r}}_0| \cos\theta t\, \hat{\imath} + \left(|\dot{\boldsymbol{r}}_0| \sin\theta t - \frac{1}{2}gt^2\right)\hat{\jmath}$$

We have obtained the equations of motion for a mechanical body, however we are able to obtain one final equation where motion in the $y - axis$ becomes a function for motion in the $x - axis$. We do this by resolving $\boldsymbol{r}(t)$ into its respective components, and solving for time within the $\hat{\imath}$ direction.

$$r_x = |\dot{\boldsymbol{r}}_0| \cos\theta t \text{ and } r_y = |\dot{\boldsymbol{r}}_0| \sin\theta t - \tfrac{1}{2} g t^2$$

Where

$$t = \frac{r_x}{|\dot{\boldsymbol{r}}_0| \cos\theta}$$

We may then write:

$$y(x) = |\dot{\boldsymbol{r}}_0| \sin\theta \left(\frac{x}{|\dot{\boldsymbol{r}}_0| \cos\theta}\right) - \frac{1}{2} g \left(\frac{x}{|\dot{\boldsymbol{r}}_0| \cos\theta}\right)^2$$

Which simplifies to:

$$y(x) = (\tan\theta)x - \left(\frac{g}{2|\dot{\boldsymbol{r}}_0|^2 \cos^2\theta}\right) x^2$$

We may also find the equation for the range of p by letting $y(x) = 0$:

$$(\tan\theta)x = \left(\frac{g}{2|\dot{\boldsymbol{r}}_0|^2 \cos^2\theta}\right) x^2$$

$$x_R = \frac{|\dot{\boldsymbol{r}}_0|^2 \sin 2\theta}{g}$$

In summary we see that the equations of motion can easily be obtained from the Euler – Lagrange equation by solving for the Lagrangian.

We will further study the effects on a mechanical body where external forces such as drag are present by the same methods as above. A more formal introduction to the Euler – Lagrange equation will be presented in which we account for external forces or forces of constraint.

It is important to note that the gravitational acceleration due to the force of gravity on a particle remains unchanged and present for every particle in every study of a mechanical system throughout this paper and will therefore not be considered as an external force.

APPLYING THE LAGRANGIAN TO A MECHANICAL SYSTEM WHERE EXTERNAL FORCES ACT ON A MECHANICAL BODY

By a more formal definition of the Euler – Lagrange equation, we are able to study the effects of external forces, which we will call the forces of constraint acting on a mechanical system, Q_j.

We write the derived equation for the Lagrangian as stated before:

$$\sum_{j=1}^{N}\left(Q_j-\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}\right)\right)\delta q_j=0$$

For which we neglect the independent displacement term such that we may write:

$$Q_j=\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}$$

Where $Q_j=\{\vec{F}_{Dx},\vec{F}_{Dy}\}$ and $q_j=\{x,y\}$. Note that for now we specify specific variables x,y to denote different spaces, however I will introduce a more generalised way of specifying spaces to only one variable q_j to simplify notation.

We assume the same mechanical system as before for a particle p centred at the origin of its reference frame O' relative to some other reference frame O whose axes are parallel to that of O'. Since we are only concerned about the drag force $\vec{F}_D$, we say that it is proportional to and acting in the opposite direction of the velocity vector of p, such that we may write:

$$\vec{F}_D=-c|\vec{v}_0|\hat{v}$$

Which simplifies to:

$$\vec{F}_D=-c\dot{r}_j$$

The Lagrangian remains the same as before and we write it as:

$$L = \frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y$$

Now we will solve the new Euler – Lagrange equation for both the x and y components. Firstly for $j = x$

$$-c\dot{r}_x = \frac{d}{dt}\frac{\partial}{\partial \dot{r}_x}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right) - \frac{\partial}{\partial r_x}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right)$$

Which we know simplifies to:

$$\ddot{r}_x = -\frac{c}{m}\dot{r}_x$$

Differently than before, we will integrate the equation above to get the velocity function acting in the $\hat{\imath}$ direction before combining it with the velocity function acting in the $\hat{\jmath}$ direction.

$$\frac{d\boldsymbol{v}_x}{dt} = -\frac{c}{m}v_x$$

$$\int -\frac{c}{m}\,dt = \int \frac{1}{v_x}\,dv_x$$

$$-\frac{c}{m}t = \ln v_x + v_c$$

Which by solving for v_c can then be written as:

$$-\frac{c}{m}t = \ln\left(\frac{v_x}{|\dot{r}_0|\cos\theta}\right)$$

$$v_x = \dot{r}_x(t) = |\dot{r}_0|\cos\theta\, exp\left(-\frac{c}{m}t\right)$$

Which upon further integration gives the position as a function of time in the $\hat{\imath}$ direction:

$$r_x = \int |\dot{r}_0|\cos\theta\, exp\left(-\frac{c}{m}t\right)\,dt$$

$$r_x(t) = \frac{m}{c}|\dot{r}_0|\cos\theta\left(1 - exp\left(-\frac{c}{m}t\right)\right)$$

Now we solve for $j = y$, and write the Euler – Lagrange equation as:

$$-c\dot{r}_y = \frac{d}{dt}\frac{\partial}{\partial \dot{r}_y}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right) - \frac{\partial}{\partial r_y}\left(\frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2\right) - mgr_y\right)$$

Which we then expect to be the same as before where:

$$m\ddot{r}_y = -c\dot{r}_y - mg$$

We do the same as before to get the velocity function acting in the $\hat{j}$ direction:

$$m\,\frac{d\boldsymbol{v}_y}{dt} = -cv_y - mg$$

$$\int -\frac{1}{m}\,dt = \int \frac{1}{cv_y + mg}\,dv_y$$

$$\int -\frac{1}{m}\,dt = \frac{1}{c}\int \frac{c}{cv_y + mg}\,dv_y$$

$$-\frac{1}{m}t = \frac{1}{c}\ln\left(cv_y + mg\right) + \dot{r}_c$$

$$-\frac{c}{m}t = \ln\left(\frac{cv_y + mg}{c|\dot{r}_0|\sin\theta + mg}\right)$$

$$v_y = \dot{r}_y(t) = \left(|\dot{r}_0|\sin\theta + \frac{mg}{c}\right)exp\left(-\frac{c}{m}t\right) - \frac{mg}{c}$$

Again, integrating once more will give us the position as a function of time in the $\hat{j}$ direction:

$$r_y = \int \left(|\dot{r}_0|\sin\theta + \frac{mg}{c}\right)exp\left(-\frac{c}{m}t\right) - \frac{mg}{c}\,dt$$

$$r_y(t) = -\frac{m}{c}\left(|\dot{r}_0|\sin\theta + \frac{mg}{c}\right)exp\left(-\frac{c}{m}t\right) - \frac{mg}{c}t + r_c$$

$$r_y(t) = \frac{m}{c}\left(|\dot{r}_0|\sin\theta + \frac{mg}{c}\right)\left(1 - exp\left(-\frac{c}{m}t\right)\right) - \frac{mg}{c}t$$

It is important to note that both the r_x and r_y position functions represent the position of p with respect to O' and not O. The next step would be to form a position vector for both components and then relate the result to O in order to generalise both frames.

We obtain the position vector for the particle p with respect to O' by adding its respective components in the form:

$$\boldsymbol{r}(t) = \left(\int |\dot{r}_0| \cos\theta \, exp\left(-\frac{c}{m}t\right) \, dt\right)\hat{\imath} + \left(\int \left(|\dot{r}_0| \sin\theta + \frac{mg}{c}\right) exp\left(-\frac{c}{m}t\right) - \frac{mg}{c} \, dt\right)\hat{\jmath}$$

Which becomes:

$$\boldsymbol{r}(t) = \left(\frac{m}{c}|\dot{r}_0| \cos\theta \left(1 - exp\left(-\frac{c}{m}t\right)\right)\right)\hat{\imath} + \left(\frac{m}{c}\left(|\dot{r}_0| \sin\theta + \frac{mg}{c}\right)\left(1 - exp\left(-\frac{c}{m}t\right)\right) - \frac{mg}{c}t\right)\hat{\jmath}$$

Similar to as before, we can easily model the trajectory by finding a function for the motion in the y – axis as a function of the motion in the x – axis by solving the x – component for t, and substituting the result into y. We write the simplified equation as:

$$y(x) = \left[\frac{m}{c}\left(|\dot{r}_0| \sin\theta + \frac{mg}{c}\right)\left(1 - exp\left(\ln\left(-\frac{cx - m|\dot{r}_0| \cos\theta}{m|\dot{r}_0| \cos\theta}\right)\right)\right) - \frac{m^2 g}{c^2}\left(\ln\left(-\frac{cx - m|\dot{r}_0| \cos\theta}{m|\dot{r}_0| \cos\theta}\right)\right)\right]$$

For illustrative purposes I have added a computer simulated graph by using the equation above:

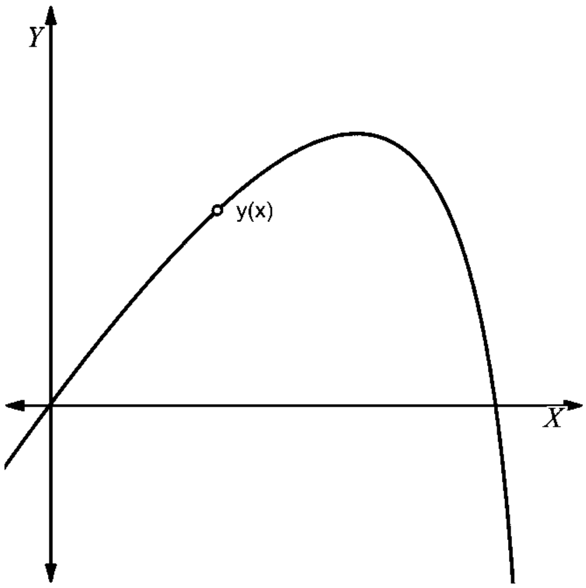

Figure 1 Projectile Path with Drag

We can clearly see the effect when a drag force is applied by the asymmetric property of the graph. Conversely, in the case where no forces of constraint are present, we expect a perfectly symmetric parabolic function by which a projectile's path of motion can be modelled.

APPLYING THE LAGRANGIAN TO A MECHANICAL SYSTEM BOUNDED IN THREE – DIMENSIONAL EUCLIDEAN SPACE

By now we have only solved the Euler – Lagrange equations for a mechanical system in $\mathbb{R}^2$ – space, however a more realistic approach would be to extend our generalisation beyond a two – dimensional system to one that contains three degrees of freedom.

It seems quite trivial that we can expect to solve three Euler – Lagrange equations in order to obtain the equations of motion in a three – dimensional system, however the method becomes somewhat more difficult in that finding an equation that accurately models the path taken by a projectile in $\mathbb{R}^3$ – space demands complex equations. We will find a work – around to this problem by solving for the Euler – Lagrange equations as before and then translating the velocity vector onto the $xy - plane$.

We begin by defining a new mechanical system in $\mathbb{R}^3$ – space where the coordinate system with origin O is centred such that it contains point p, where p is the mechanical body in question. We assign an initial velocity vector $\vec{v}_0$ to p such that it has a positive angle of trajectory θ and an azimuth deviation φ from the positive x – axis.

We say that $\vec{v}_0$ forms a shadow onto the $xz - plane$ which we will denote by $\vec{v}_{H0}$ where $\vec{v}_{H0} = \vec{v}_0 \cos\theta$. Furthermore, we define the Lagrangian of this system as:

$$L = T - V$$

Such that:

$$L = \frac{1}{2}m\left(\dot{r}_x{}^2 + \dot{r}_y{}^2 + \dot{r}_z{}^2\right) - mgr_y$$

We then write the Euler – Lagrange equation as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

Where $q_j = \{r_x, r_y, r_z\}$.

Solving the Euler – Lagrange equation when $j = x$:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r_x}}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right) = \frac{\partial}{\partial r_x}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right)$$

$$\ddot{r_x} = 0$$

Solving the Euler – Lagrange equation when $j = y$:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r_y}}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right) = \frac{\partial}{\partial r_y}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right)$$

$$\ddot{r_y} = -g\hat{\jmath}$$

Finally, solving the Euler – Lagrange equation when $j = z$:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r_z}}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right) = \frac{\partial}{\partial r_z}\left(\frac{1}{2}m\left(\dot{r_x}^2 + \dot{r_y}^2 + \dot{r_z}^2\right) - mgr_y\right)$$

$$\ddot{r_z} = 0$$

We find that $\ddot{r}(t) = -g\hat{\jmath}$.

Furthermore, we find the velocity function by integrating the acceleration function with respect to time:

$$\dot{r}(t) = \int \ddot{r}(t)\, dt$$

$$\dot{r}(t) = (\vec{v}_{H0}\cos\varphi)\hat{\imath} + (|\dot{r}_0|\sin\theta - gt)\hat{\jmath} + (\vec{v}_{H0}\sin\varphi)\hat{k}$$

Which we then write as:

$$\dot{r}(t) = (|\dot{r}_0|\cos\theta\cos\varphi)\hat{\imath} + (|\dot{r}_0|\sin\theta - gt)\hat{\jmath} + (|\dot{r}_0|\cos\theta\sin\varphi)\hat{k}$$

We then find the position function by integrating $\dot{r}(t)$ with respect to time:

$$r(t) = \int \dot{r}(t)\ dt$$

$$r(t) = (|\dot{r}_0|\cos\theta\cos\varphi\, t)\hat{\imath} + \left(|\dot{r}_0|\sin\theta t - \frac{1}{2}gt^2\right)\hat{\jmath} + (|\dot{r}_0|\cos\theta\sin\varphi\, t)\hat{k}$$

Which clearly provides a time dependent function, and we have obtained the equations of motion in three – dimensional Euclidean space in a similar way as before.

Now we will introduce a different way of representing the velocity vector:

$$\dot{\boldsymbol{r}} = \langle \dot{r}_x, \dot{r}_y, \dot{r}_z \rangle$$

$$\dot{\boldsymbol{r}} = \langle |\dot{r}_0|\cos\theta\cos\varphi\,, |\dot{r}_0|\sin\theta\,, |\dot{r}_0|\cos\theta\sin\varphi \rangle$$

We are then able to simplify the three – dimensional system by translating $\dot{\boldsymbol{r}}$ onto the xy – plane by the translation matrix T:

$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -|\dot{r}_0|\cos\theta\sin\varphi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We find the transformation of $\dot{\boldsymbol{r}}$ as $T\dot{\boldsymbol{r}}$ where $\dot{\boldsymbol{r}}$ is written in homogeneous coordinates:

$$T_{xy}\dot{\boldsymbol{r}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -|\dot{r}_0|\cos\theta\sin\varphi \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} |\dot{r}_0|\cos\theta\cos\varphi \\ |\dot{r}_0|\sin\theta \\ |\dot{r}_0|\cos\theta\sin\varphi \\ 1 \end{bmatrix}$$

Which gives us the translated velocity vector $\dot{\boldsymbol{r}}'$:

$$\dot{\boldsymbol{r}}' = (|\dot{r}_0|\cos\theta\cos\varphi)\hat{\imath} + (|\dot{r}_0|\sin\theta)\hat{\jmath}$$

By integrating with respect to time and finding the respective time functions, we find $y(x)$:

$$y(x) = \left(\frac{\tan\theta}{\cos\varphi}\right)x - \left(\frac{g}{2|\dot{r}_0|^2\cos^2\theta\cos^2\varphi}\right)x^2$$

We are able to illustrate the 2 – Dimensional deviation from the actual trajectory:

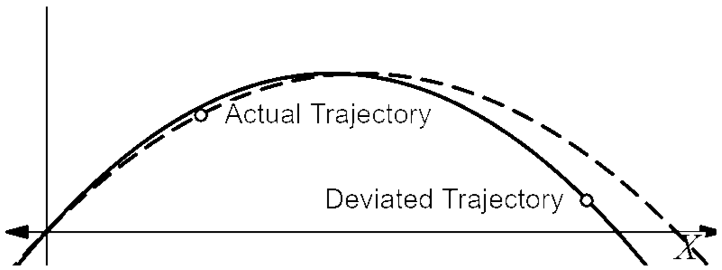

Figure 2 Deviated Trajectory Due to Translation in the XY - Plane

We repeat the same process to project the motion onto the yz – plane by taking the product of the transformation matrix T_{yz} and $\dot{\boldsymbol{r}}$:

$$T_{yz}\dot{\boldsymbol{r}} = \begin{bmatrix} 1 & 0 & 0 & -|\dot{r}_0|\cos\theta\cos\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} |\dot{r}_0|\cos\theta\cos\varphi \\ |\dot{r}_0|\sin\theta \\ |\dot{r}_0|\cos\theta\sin\varphi \\ 1 \end{bmatrix}$$

This gives us the velocity vector with respect to the yz – plane as:

$$\dot{\boldsymbol{r}} = |\dot{r}_0|\sin\theta\,\hat{j} + |\dot{r}_0|\cos\theta\sin\varphi\,\hat{k}$$

Furthermore, we find y(z) by integration with respect to time and elementary algebra:

$$y(z) = \left(\frac{\tan\theta}{\tan\varphi}\right)z - \left(\frac{g}{2|\dot{r}_0|^2\sin^2\varphi\cos^2\theta}\right)z^2$$

The projected motion is illustrated by:

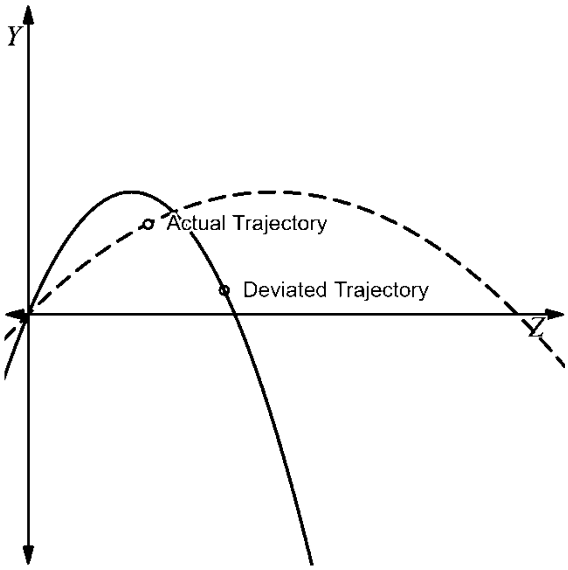

Figure 3 Deviated Trajectory Due to Translation in the YZ - Plane

We can also show that the projection onto the xz – plane yields a linear function x(z) such that $x(z) = (\tan\varphi)z$, which is a trivial and expected result. By using linear transformations, we can simplify an n – dimensional system such that we get the (n – 1) – dimensional projection.

We also find that this projection deviates from the original system, which we expect.