Synchronization methods for chaotic systems involving fractional derivative with a non-singular kernel.

Fatiha Mesdoui[a], Nabil Shawagfeh[a], Adel Ouannas[b].

[a]Department of Mathematics, University of Jordan, Amman, Jordan.

[b]Department of Mathematics, University of Larbi Tebessi, Tebessa, Algeria.

Abstract

This study considers the problem of control-synchronization for chaotic systems involving fractional derivative with a non-singular kernel. Using an extension of the Lyapunov Theorem for systems with Atangana-Baleanu-Caputo (ABC) derivative, a suitable control scheme is designed to achieve matrix projective synchronization (MP) between nonidentical ABC systems with different dimensions. The results are exemplified by the ABC version of the Lorenz system, Bloch system, and Liu system. To show the effectiveness of the proposed results, numerical simulations are performed based on the Adams-Bashforth-Mounlton numerical algorithm.

Keywords: Matrix projective synchronization. Chaos. Atangana-Baleanu-Caputo fractional derivative. Mittag-Leffler function. Lyapunov method.

1 Introduction

Fractional derivative, as an extension of the classical derivative to its non-integer derivative counterpart, has been significantly studied and successfully applied in various fields, including, bioengineering, electrical engineering, signal processing, chemical mixing, and fluid mechanics [1-5]. Several definitions of fractional differential operators have been proposed, where the kernel of these operators can be non-local and singular as proposed by the definitions of Riemann-Liouville and Liouville-Caputo [6] or local and non-singular as in the definition of Caputo-Fabrizio [7]. Based on the generalized Mittag-Leffler function, Atangana and Baleanu introduced a new definition of the fractional derivative with non-local and non-singular kernel [8]. Due to the nature of this kernel, models with ABC fractional derivative can capture the memory effects that exist in real-life problems. Many practical problem have been modeled using this derivative such as the computer worms model (SEIRA model) [9], a nanofluids model [10], the predator-prey ecosystem model [11], the coronavirus model (COVID-19 model) [12], and some other interesting results can be found in [13-15].

Chaos is a complex behavior exhibited by some natural systems. This complexity is due to the noise-like signals, random trajectories, and the sensitivity to initial conditions of such systems. The pioneering study of Pecora and Carroll [16] showed that the synchronization of two chaotic systems is possible.

Chaos synchronization refers to a phenomenon where two chaotic systems progress in strong correlations and converge towards the same behavior [17]. Recently, there has been a great deal of interest in the synchronization of fractional chaotic systems. Various methods and techniques have been introduced to achieve synchronization in chaotic systems with Caputo derivative including linear and nonlinear control, adaptive control, feedback control, active control [18-25]. Moreover, some synchronization types have been extended to fractional systems with this derivative, these include complete synchronization, anti-synchronization, projective synchronization, $Q-S$ synchronization, full state hybrid projective synchronization, generalized synchronization, $\Lambda-\phi$ synchronization and $\Theta-\Phi$ synchronization [26-33]. Among them, the matrix projective synchronization has been applied intensively in secure communications and information processing [34, 35]. Nevertheless, the synchronization of chaotic systems with ABC derivative is almost unexplored.

Motivated by the above discussion, this study aims to investigate the synchronization between chaotic systems with ABC derivative. Using an extension of the Lyapunov Theorem, sufficient conditions to achieve matrix projective synchronization for non-identical systems with different dimensions will be established.

The remainder part of this paper is organized as follows. Some basic definitions and important lemmas are given in section 2. In section 3 we propose a numerical scheme to solve the chaotic systems in ABC sense. Our main result is presented in section 4. Section 5 presents the application of our results. Finally, section 6 is devoted to the conclusion.

2 Basic concepts

In this section, we outline the Atangana-Baleanu fractional derivation and we recall to some important propositions.

Definition 2.1 *[8] The fractional derivative in the Atangana-Baleanu-Caputo operator of a function f is defined as,*

$$ {}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(\tau) E_{\alpha} \left[\frac{-\alpha}{1-\alpha} (t-\tau)^{\alpha} \right] d\tau, \tag{1} $$

where, $\alpha \in (0,1)$ is the order of the fractional derivative (1), $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$ and $t > 0$.

In this definition $E_\alpha(z)$ is the Mettag-Leffler function of one parameter

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2}$$

The associated ABC fractional integral is

$$^{ABC}I_t^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau. \tag{3}$$

The following results are the extension of the Lyapunov Theorem and Alikhanov-Aguila lemma for the ABC derivative.

Theorem 2.1 *[36] Let $X = 0$ be the trivial solution of system*

$$^{ABC}D_t^\alpha X(t) = G(X(t)), \tag{4}$$

where $X(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ and $G : \mathbb{R}^n \to \mathbb{R}^n$. Suppose there exists a positive definite Lyapunov function $V(X(t))$ such that $^{ABC}D_t^\alpha V(X(t)) < 0$, for all $t > 0$, then $X = 0$ is asymptotically stable.

Lemma 2.1 *[36] $\forall t > 0$ and $\alpha \in (0,1)$:*

$$^{ABC}D_t^\alpha \left(X^T(t)X(t)\right) \leq 2X^T(t) \left(^{ABC}D_t^\alpha (X(t))\right). \tag{5}$$

3 Numerical scheme

In this paper, we use an ABC version of the Adams-Bashforth-Mounlton numerical algorithm [37] to solve the fractional equation in ABC sense:

$$^{ABC}D_t^\alpha x(t) = f(x(t)). \tag{6}$$

Where $0 < \alpha \leq 1$, $t \in [0, T]$, $T > 0$ and the operator in left-hand side of this equation is the ABC operator defined by (1). Applying the $^{AB}I_t^\alpha$ integral in both side of (6), we get the Voltera integral equation

$$x(t) = x(0) + \frac{1-\alpha}{B(\alpha)} f(x(t)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x(\tau)) d\tau, \tag{7}$$

given a uniform grid $\{t_k = k.dt, k = 0, \ldots, n_T\}$, where $dt = \frac{T}{n_T}$ and n_T is positive integer, the solution of (7) at t_k can be written as

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{B(\alpha)} f(x(t_{k+1})) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} f(x(\tau)) d\tau. \qquad (8)$$

The approximation of the integral in (8), in each interval $[t_j, t_{j+1}]$ by the trapezoidal quadrature formula, leads to the ABC-predictor-corrector scheme:

$$\begin{cases} x^P_{k+1} = x(0) + \frac{1-\alpha}{B(\alpha)}\ f(x_k) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1}\, f(x_j), \\ x_{k+1} = x(0) + \frac{1-\alpha}{B(\alpha)} f(x^P_{k+1}) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \sum_{j=0}^{k} a_{j,k+1} f(x(t_j)) + a_{k+1,k+1}\, f(x^P_{k+1}), \end{cases} \qquad (9)$$

where

$$a_{j,k+1} = \frac{(dt)^\alpha}{\alpha(\alpha+1)} \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^\alpha, & j = 0, \\ (k-j+1)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, & 1 \leq j \leq k, \\ 1, & j = k+1, \end{cases}$$

and

$$b_{j,k+1} = \frac{(dt)^\alpha}{\alpha} \left[(k-j+1)^\alpha - (k-j)^\alpha\right], 0 \leq j \leq k.$$

In the numerical simulation, we can choose $B(\alpha)$ as

$$B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}.$$

4 Synchronization

In this section, we will construct the synchronization controllers to achieve synchronization between two commensurate ABC fractional-order systems. We consider the master and the slave systems,

$$^{ABC}D^\alpha_t x = f(x), \qquad (10)$$

$$^{ABC}D^\beta_t y = Ay + g(y) + U, \qquad (11)$$

where $x = (x_i(t)), (i = 1, \ldots, n)$ and $y = (y_j(t)), (j = 1, \ldots, m)$ are states of the master system (10) and the slave system (11) respectively, $f = (f_i)_{1\leq i\leq n}$ and $g = (g_i)_{1\leq i\leq m}$ are non linear continuous functions, $A = (a_{ij})_{m\times m}$, and $U = (U_j)$, $(j = 1, \ldots, m)$ is a control law to be designed.

Definition 4.1 *The master and thr slave systems (10)-(11) are said to matrix projective (MP) synchronized if there exists a constant matrix M and a suitable controller U such that the synchronization error*

$$e(t) = y(t) - Mx(t) \tag{12}$$

satisfy

$$\lim_{t\to\infty} \|e(t)\| = 0 \tag{13}$$

Theorem 4.1 *The master and thr slave systems (10)-(11) are MP synchronized under the following control law*

$$U = M\ {}^{ABC}D_t^{\beta} x - Ay - g(y) + (A-B)e \tag{14}$$

where $B = (b_{ij})_{m\times m}$ is a control matrix such that $A - B$ is negative definite.

Proof. By substituting the controllers (14) into the system (11), we get

$${}^{ABC}D_t^{\beta} y - M^{ABC}D_t^{\beta} x = (A-B)e,$$

so,

$${}^{ABC}D_t^{\beta} e(t) = (A-B)e(t). \tag{15}$$

The MP synchronization error defined in (12) converge to zero over the time if and only if the zero steady-state of the synchronization error system (15) is asymptotically stable. Now if we choose the Lyapunov function by

$$V(e(t)) = \frac{1}{2}e^T(t)e(t),$$

then, the ABC derivative of the above function with respect to time and of order β yields

$${}^{ABC}D_t^{\beta} V(e(t)) = \frac{1}{2}{}^{ABC}D_t^{\beta}\left(e^T(t)e(t)\right).$$

Using Lemma 2.1, we get

$$\begin{aligned}{}^{ABC}D_t^{\beta} V(e(t)) &\leq e^T(t)^{ABC}D_t^{\beta}(e(t)) \\ &= e^T(t)(A-B)e(t) < 0.\end{aligned}$$

From ABC fractional Lyapunov stability Theorem 2.1, we conclude that the zero steady-state of the error system (15) is asymptotically stable and therefore, the master system (10) and the slave system (11) are MP synchronized. ■

5 Applications

To show the applications of the developed scheme to non-identical chaotic systems, we propose two master-slave formulations. The first one concerns the chaotic systems with the same dimension. While the second one examines the chaotic systems with different dimensions.

As master system, we consider the fractional 3D-Lorenz system in the ABC sense [38] is described as follows

$$\begin{cases} {}^{ABC}D_t^{\alpha} x_1 = \sigma(x_2 - x_1), \\ {}^{ABC}D_t^{\alpha} x_2 = x_1(\rho - x_3) - x_2, \\ {}^{ABC}D_t^{\alpha} x_3 = x_1 x_2 - \gamma x_3. \end{cases} \tag{16}$$

This system exhibits chaotic behavior when $(\sigma, \rho, \gamma) = (10, 28, \frac{8}{3})$ and $\alpha = 0.995$, the chaotic attractors are shown in Figure 1.

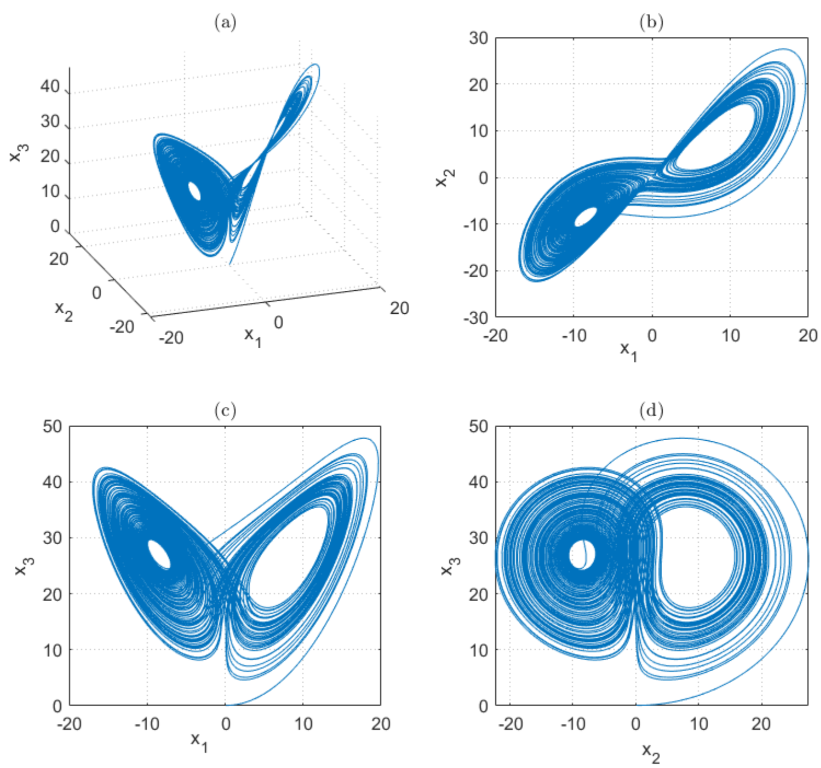

Figure 1: The chaotic attractors of the system (16) for $\alpha = 0.995$ and $(\sigma, \rho, \gamma) = (10, 28, \frac{8}{3})$.

5.1 MP synchronization of ABC-chaotic systems with the same dimension

Consider as slave system, the controlled fractional 3D-Bloch system in ABC sense [39]

$$\begin{cases} {}^{ABC}D_t^{\beta} y_1 = \psi y_2 + \lambda y_3 (y_1 \sin(\phi) - y_2 \cos(\phi)) - \frac{1}{T_2} y_1 + u_1, \\ {}^{ABC}D_t^{\beta} y_2 = -\psi y_1 - y_3 + \lambda y_3 (y_2 \sin(\phi) + y_1 \cos(\phi)) - \frac{1}{T_2} y_2 + u_2, \\ {}^{ABC}D_t^{\beta} y_3 = y_2 - \lambda y_3 \sin(\phi)(y_1^2 + y_2^2) - \frac{1}{T_1}(y_3 - 1) + u_3, \end{cases} \tag{17}$$

where u_i, $i = 1, 2, 3$ are synchronization controllers. For a specific values of parameters $\psi = 1.26$, $\lambda = 10$, $\phi = 0.7764$, $T_1 = 0.5$, $T_2 = 0.25$ and $\beta = 0.95$, the chaotic behavior of the Bloch model (i.e. system (17) with $u_i = 0$, $i = 1, 2, 3$) is shown in Figure 2.

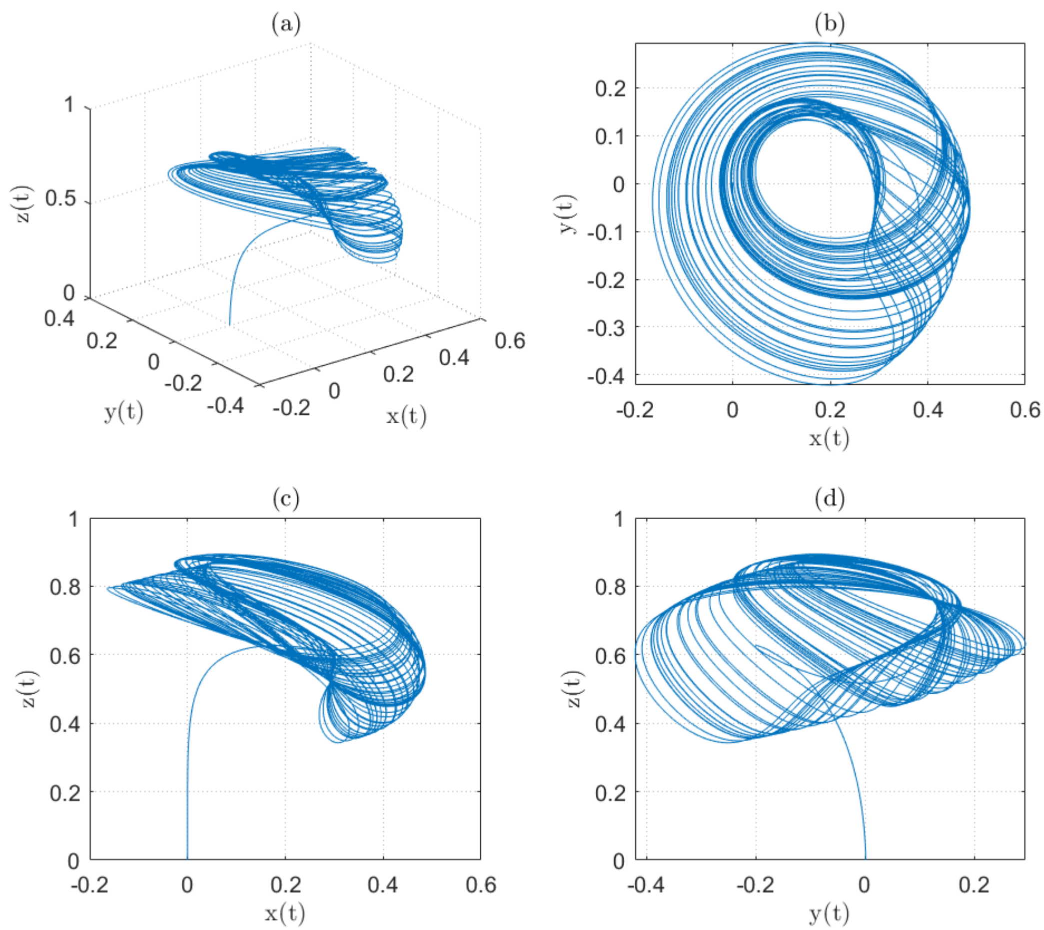

Figure 2: The chaotic attractors of the system (17) for $\beta = 0.95$, $\psi = 1.26$, $\lambda = 10$, $\phi = 0.7764$, $T_1 = 0.5$, $T_2 = 0.25$.

Comparing with the notation in (11), the linear and the nonlinear parts of the system (17) are described as follows

$$A = \begin{pmatrix} -\frac{1}{T_2} & \psi & 0 \\ -\psi & \frac{1}{T_2} & -1 \\ 0 & 1 & -\frac{1}{T_1} \end{pmatrix}, \ g(y) = \begin{pmatrix} \lambda y_3(y_1 \sin(\phi) - y_2 \cos(\phi)) \\ \lambda y_3(y_2 \sin(\phi) + y_1 \cos(\phi)) \\ -\lambda y_3 \sin(\phi)(y_1^2 + y_2^2) + \frac{1}{T_1} \end{pmatrix}.$$

The MP error synchronization system between the master system (16) and the controlled system (17) is defined as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where,

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

According to Theorem 4.1, if we choose

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -\psi & 0 & 0 \\ 0 & 1 & \frac{2}{T_1} \end{pmatrix},$$

then, the control law u_i, $i = 1, 2, 3$ are designed as

$$\begin{aligned} u_1 &= \frac{x_1}{T_1} - \psi x_2 + y_1 y_3 - \lambda \sin(\phi) y_1 y_3 + {}^{ABC}D_t^{\beta} x_1, \\ u_2 &= x_1 + \frac{x_2}{T_2} + x_3 + \psi y_1 - \lambda \cos(\phi) y_1 y_3 - \lambda \sin(\phi) y_2 y_3 + {}^{ABC}D_t^{\beta} x_2, \\ u_3 &= \frac{1}{T_1}(3x_1 + 3x_3 - 2y_3) - (1 + y_2) + \lambda \sin(\phi)(y_1^2 + y_2^2) + {}^{ABC}D_t^{\beta}(x_1 + x_3), \end{aligned}$$

since

$$A - B = \begin{pmatrix} -\frac{1}{T_2} & \psi & 0 \\ 0 & -\frac{1}{T_2} & -1 \\ 0 & 0 & -\frac{3}{T_1} \end{pmatrix}$$

is negative definite, therefore, according to the Theorem 4.1, the Lorenz system (16) and the Bloch system (17) are MP synchronized in 3-dimension.

To illustrate the above results numerically, the ABC-predictor-corrector scheme (9) is used to solve the MP-synchronization error system associated to the master-slave systems (16)-(17) and described by

$$\begin{cases} {}^{ABC}D_t^{\beta} e_1 = -\frac{1}{T_2} e_1 + \psi e_2, \\ {}^{ABC}D_t^{\beta} e_2 = -\frac{1}{T_2} e_2 - e_3, \\ {}^{ABC}D_t^{\beta} e_3 = -\frac{3}{T_1} e_3, \end{cases} \tag{18}$$

where the initial conditions for the Lorenz system (16) and the Bloch system (17) are ($x_1(0)$, $x_2(0)$, $x_3(0)$) = (5,-6,2) and $(y_1(0), y_2(0), y_3(0)) = (11, 3, 1)$, respectively. So the initial conditions for the error system (18) are $(e_1(0), e_2(0), e_2(0)) = (6, 9, -6)$. The synchronization errors states evolution are provided in Figure 3. This evolution indicates successful MP-synchronization between systems (16) and (17).

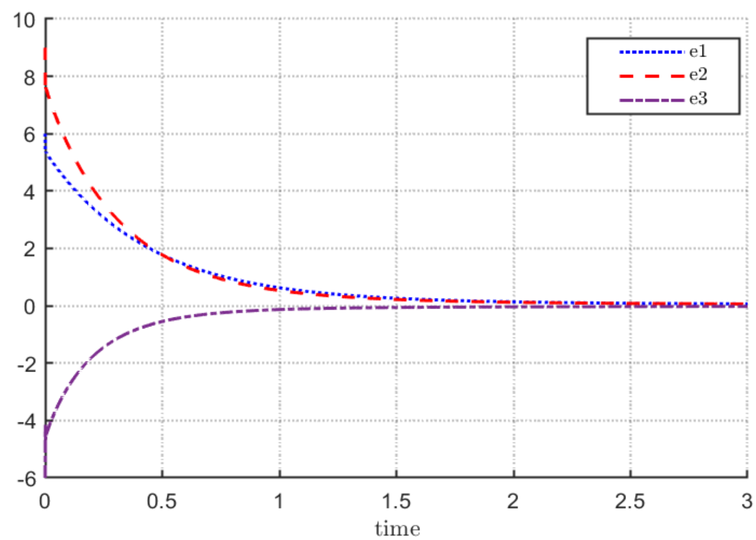

Figure 3: The time evolution of the error system (18).

5.2 Synchronization of chaotic systems with different dimensions

In this case, to have a model of a slave system with different dimensions from the master system (16), we extended the 4D-Liu system [40] to the ABC sense by replacing the integer order derivative by the ABC fractional derivative. We get the controlled ABC Liu system

$$\begin{cases} {}^{ABC}D_t^{\beta} y_1 = 10(y_2 - y_1) + y_4 + u_1, \\ {}^{ABC}D_t^{\beta} y_2 = 40y_1 + 0.5y_4 - y_1 y_3 + u_2, \\ {}^{ABC}D_t^{\beta} y_3 = -2.5y_3 + 4y_1^2 - y_4 + u_3, \\ {}^{ABC}D_t^{\beta} y_4 = -\frac{2}{3} y_2 - y_4 + u_4, \end{cases} \tag{19}$$

where u_i, $i = 1, \ldots, 4$ are synchronization controllers. Using the scheme (9), we investigate the existence of chaotic behavior of the system (19) for $\beta = 0.94$, which is shown in Figure 4.

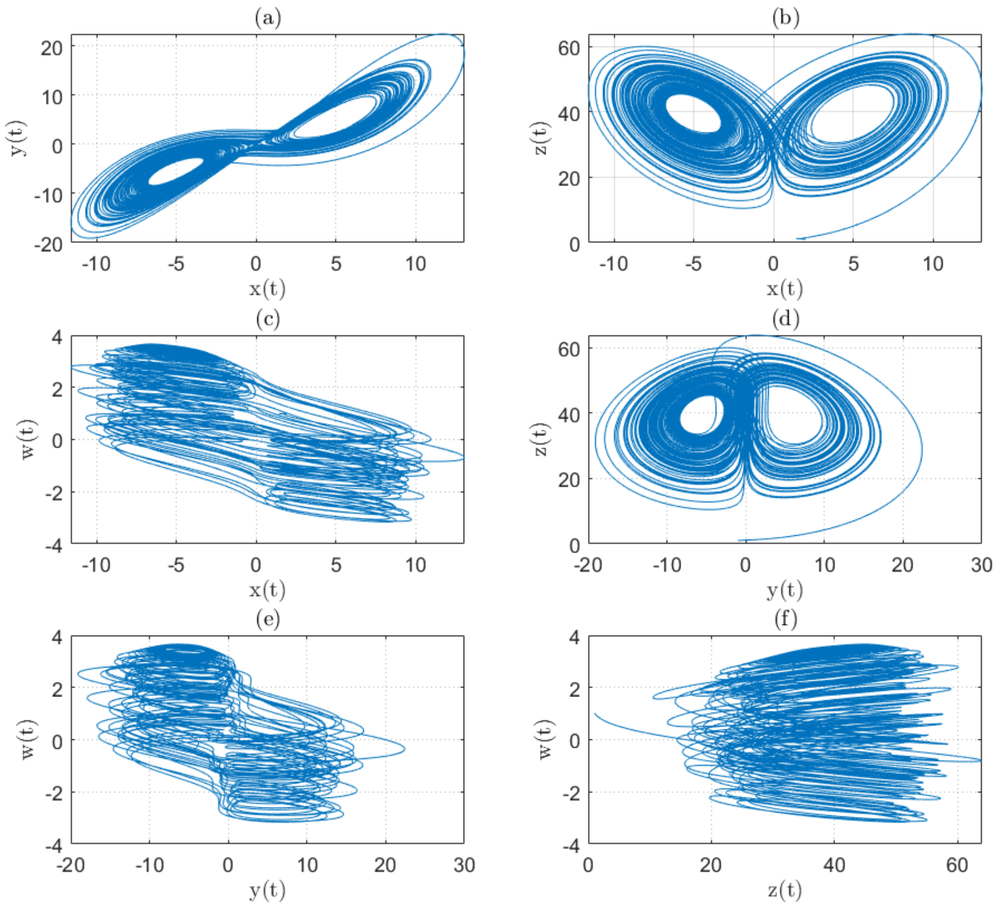

Figure 4: The chaotic attractors of the system (19) for $\beta = 0.94$.

Comparing with (11), the linear and the nonlinear parts of the system (19) are described as follows

$$A = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 40 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{5}{2} & -1 \\ 0 & -\frac{2}{3} & 0 & -1 \end{pmatrix}, \quad g(y) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ 4y_1^2 \\ 0 \end{pmatrix}.$$

The MP error synchronization system between the master system (16) and the controlled system (19) is defined as

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} - M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where,

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix}.$$

According to the Theorem 4.1, if we choose

$$B = \begin{pmatrix} 0 & 10 & 0 & 1 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then, the controllers u_i, $i = 1, \ldots, 4$ are designed as

$$\begin{aligned} u_1 &= 10x_1 - 10y_2 - y_4 + {}^{ABC}D_t^{\beta} x_1, \\ u_2 &= 40x_1 + 4x_2 - 2y_2 + y_1y_3 - 0.5y_4 + {}^{ABC}D_t^{\beta}(2x_2), \\ u_3 &= -4y_1^2 + 7.5x_3 + y_4 + {}^{ABC}D_t^{\beta}(3x_3), \\ u_4 &= 8x_1 + \frac{4}{3}x_2 - y_4 + {}^{ABC}D_t^{\beta}(4x_1). \end{aligned}$$

Obviously, the matrix

$$A - B = \begin{pmatrix} -10 & 0 & 0 & 0 \\ 40 & -2 & 0 & 0 \\ 0 & 0 & -\frac{5}{2} & 0 \\ 0 & -\frac{2}{3} & 0 & -2 \end{pmatrix}$$

is negative definite so the Lorenz system (16) and the Liu system (19) are MP-synchronized in 4-dimension.

Synchronization error system associated to the master-slave systems (16) and (19) is described by

$$\begin{cases} {}^{ABC}D_t^{\beta} e_1 = -10e_1, \\ {}^{ABC}D_t^{\beta} e_2 = 40e_1 - 2e_2, \\ {}^{ABC}D_t^{\beta} e_3 = -\frac{5}{2}e_3, \\ {}^{ABC}D_t^{\beta} e_4 = -\frac{2}{3}e_2 - 2e_4, \end{cases} \tag{20}$$

where the initial conditions for the Lorenz system (16) and the Liu system (19) are $(x_1(0), x_2(0), x_3(0)) = (1, -2, 7)$and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (8, -1, 1, 3)$, respectevely. So the initial conditions for the error system (20) are $(e_1(0), e_2(0), e_2(0), e_4(0)) = (7, 3, -20, -1)$. The time evolution of the synchronization error system states are provided in Figure 5. Which indicates successful MP-synchronization between the 3D-Lorenz system (16) and the 4D-Liu system (19).

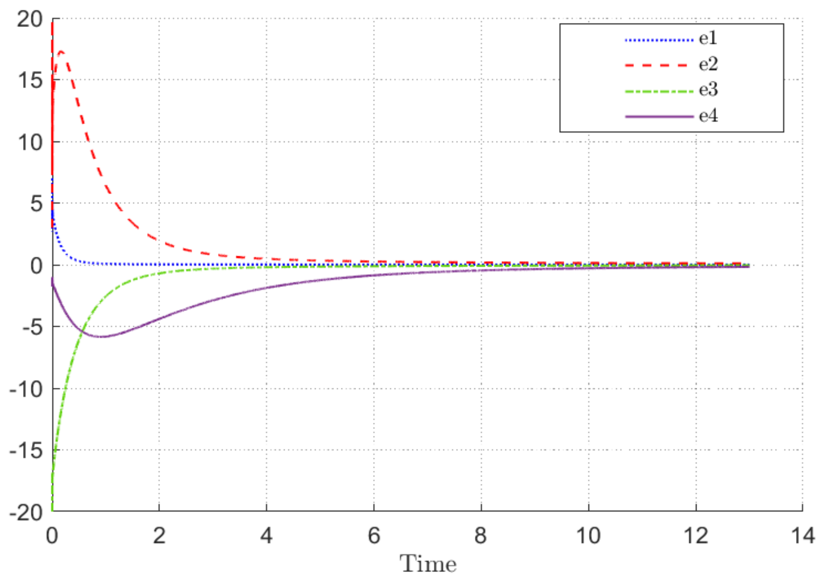

Figure 5: The time evolution of the error system (20).

6 Conclusion

In the present paper, we investigated the problem of chaos synchronization in chaotic systems with the Atangana-Baleanu-Caputo derivative. First, we proposed the master-slave formulation for the matrix projective synchronization and we introduced a novel control scheme to achieve synchronization between chaotic systems with the same or different dimensions. Numerical simulations are performed to verify the effectiveness of the approach developed herein. The scheme presented in this paper can be applied to various classes of chaotic systems with derivative in ABC sense.

In our future research, we will extend different types of synchronization [27-28], to fractional chaotic systems with discrete Mittag-Leffler Kernels. Moreover, we plan to study the control and synchronization of spatiotemporal models with ABC time fractional derivative [43, 44].

References

[1] R. L., Magin, Fractional calculus in bioengineering. Begell House Publishers. (2006)

[2] G. E., Carlson, C. A., Halijak, Approximation of fractional capacitors $(\frac{1}{s})^{\frac{1}{n}}$ by a regular Newton process. IEEE Transactions on Circuit Theory (1964), vol 11. pp 210–213.

[3] E. J. S., Pires, J. A. T., Machado, P. B., de Moura, Fractional order dynamics in a GA planner. Signal Process, (2003), vol 83 pp 2377–2386.

[4] D., Kusnezov, A., Bulgac, G. D., Dang, Quantum Lévy processes and fractional kinetics. Physical Review Letters (1999), vol 82.pp 1136–1139.

[5] V. V., Kulish, J. L., Lage, Application of fractional calculus to fluid mechanics.Journal of Fluids Engineering, (2002), vol 124. pp 803–806.

[6] I. Podlubny, Fractional Differential Equations. Academic Press, New York (1999)

[7] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel. Progr Fract DifferentAppl. (2015) vol 1. pp. 73–85.

[8] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel, Theory Appl. Heat Trans. Model. Therm. Sci. (2016) vol. 20. pp. 763–769.

[9] S. Uçar, Analysis of a basic SEIRA model with Atangana-Baleanu derivative. AIMS Mathematics (2020), vol 5(2). pp. 1411.

[10] M., Saqib, I., Khan, S., Shafie, Application of Atangana–Baleanu fractional derivative to MHD channel flow of CMC-based-CNT's nanofluid through a porous medium. Chaos, Solitons & Fractals (2018) vol 116. pp79–85.

[11] B. Ghanbari,H. Günerhan, H. M, Srivastava, An application of the Atangana-Baleanu fractional derivative in mathematical biology: A three-species predator-prey model. Chaos, Solitons & Fractals (2020) vol 138. pp 109910.

[12] M. A., Khan, A., Atangana, E., Alzahrani, Fatmawati, The dynamics of COVID-19 with quarantined and isolation. Advances in Difference Equations (2020) vol 2020(1). pp425.

[13] I., Koca, Analysis of rubella disease model with non-local and non-singular fractional derivatives. An International Journal of Optimization and Control. Theories & Applications (IJOCTA) (2017) vol 8(1). pp 17–25.

[14] N. H., Sweilam, S. M., Al-Mekhlafi, T., Assiri, A., Atangana, Optimal control for cancer treatment mathematical model using Atangana–Baleanu–Caputo fractional derivative. Advances in Difference Equations (2020) vol 2020(1). pp 334.

[15] S., Uçar, E., Uçar, N., Özdemir, Z., Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative. Chaos, Solitons & Fractals (2019) vol 118(C). pp 300–306.

[16] L. M., Pecora, T. L., Carrol, T.L, Synchronization in chaotic systems. Phys. Rev A.

(1990), vol. 64. pp. 8.

[17] A. Luo, A theory for synchronization of dynamical systems. Communication in Nonlinear Sciences and Numerical Simulation (2009), vol. 14 (10). pp. 1901-1951.

[18] Z., Odibat, N., Corson,M. A. A., Alaoui, C. Bertelle, Synchronization of chaotic fractional-order systems via linear control. International Journal of Bifurcation and Chaos, (2010), vol 20, pp 81–97.

[19] X. R., Chen, C. X., Liu, Chaos Synchronization of fractional order unified chaotic system via nonlinear control. International Journal of Modern Physics B, (2012), vol 25 pp 407–415.

[20] M. Srivastava, S. P., Ansari, S. K., Agrawal, S. Das, S, A. Y. T., Leung, Antisynchronization between identical and non-identical fractional-order chaotic systems using active control method. Nonlinear Dynamics (2014), vol 76, pp 905–914.

[21] S. K., Agrawal, M. Srivastava, S., Das, Synchronization of fractional order chaotic systems using active control method. Chaos, Solitons & Fractals (2012) vol 45(6). pp 737–752.

[22] S. Das, M. Srivastava, A. Y. T., Leung, Hybrid phase synchronization between identical and nonidentical three-dimensional chaotic systems using the active control method. Nonlinear Dynamics (2013) vol 73(4). pp 2261–2272.

[23] Z. Odibat, Adaptive feedback control and synchronization of non-identical chaotic fractional order systems. Nonlinear Dynamics, (2010) vol 60, pp 479–487.

[24] S. K., Agrawal, S. Das, Function projective synchronization between four dimensional chaotic systems with uncertain parameters using modified adaptive control method. Journal of Process Control (2014) vol 24(5). pp 517–530.

[25] S. K., Agrawal, S. Das, Modified adaptive control method for synchronization of some fractional chaotic systems with unknown parameters. Nonlinear Dynamics, (2013), vol 73, pp 907–919.

[26] A. Razminia, D. Baleanu, Complete synchronization of commensurate fractional order chaotic systems using sliding mode control. Mechatronics, (2013), vol 23, pp 873–879.

[27] M. M., Al-sawalha, A. K., Alomari, S. M., Goh, M. S. M., Nooran, Active antisynchronization of two identical and different fractional-order chaotic systems. International Journal of Nonlinear Science (2011), vol 11. pp 267–274.

[28] G. Si., Z. Sun, Y. Zhang, W. Chen, Projective synchronization of different fractional-order chaotic systems with non-identical orders. Nonlinear Analytics: Real World Applications (2012) vol 13, pp 1761–1771.

[29] C. Yi, C. Liping, W. Ranchao, D. Juan, Q-S synchronization of the fractional order unified system. Pramana, (2013), vol 80, pp 449–461.

[30] H. Feng, Y. Yang, S. P., Yang, S. P, A new method for full state hybrid projective synchronization of different fractional order chaotic systems. Applied Mechanics and Materials, (2013), 385–38, 919–922.

[31] X. D., Zhang, P. D., Zhao, A. H., Li, Construction of a new fractional chaotic system and generalized synchronization. Communications in Theoretical Physics, (2010) vol 53, pp 1105–1110.

[32] A. Ouannas, M. M., Al-sawalha. On $\Lambda - \varphi$ generalized synchronization of chaotic dynamical systems in continuous-time. European Physical Journal Special Topics (2016) vol 225, pp 187–196.

[33] A. Ouannas,A.,M. M., Al-sawalha, T. Ziar, Fractional chaos synchronization schemes for different dimensional systems with non-identical fractional-orders via two scaling matrices. Optik , (2016) vol 127, pp 8410–8418.

[34] A., Ouannas, R., Abu-Saris, On Matrix Projective Synchronization and Inverse Matrix Projective Synchronization for Different and Identical Dimensional Discrete-Time Chaotic Systems. Journal of Chaos (2016) vol 2016 pp. 7.

[35] W., Yan, Q., Ding, A New Matrix Projective Synchronization and Its Application in Secure Communication. IEEE Access (2019) vol 7. pp 112977–112984.

[36] M. Taneco-Hernández, C. Vargas-De-León, Stability and Lyapunov functions for systems with Atangana–Baleanu Caputo derivative: An HIV/AIDS epidemic model, Chaos, Solitons & Fractals (2020) vol 132 pp 109586.

[37] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. (2002), vol. 29 (1–4). pp. 3–22.

[38] A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order, Chaos, Solitons & Fractals, vol 89 (2016) pp 447-454.

[39] J.F. Gómez-Aguilar, Chaos in a nonlinear bloch system with atangana–baleanu fractional derivatives Numer Methods Partial Differ Equations, (2018), vol 34. pp. 1716-1738

[40] Q. Han, C.X. Liu, L. Sun, D.R Zhu, A fractional order hyperchaotic system derived from a Liu system and its circuit realization. Chinese Physics B, (2013), vol 22. pp 6–020502.

[41] A. Ouannas, Z. Odibat, N. Shawagfeh, A new Q-S synchronization results for discrete chaotic systems, Diff. Eq. Dyn. Syst.(2016), vol. 27(4), pp. 413–422.

[42] A. Ouannas, Z. Odibat, N. Shawagfeh, A. Alsaedi, B. Ahmad, Universal chaos synchronization control laws for general quadratic discrete systems. Appl. Math. Model. (2017), vol. 45, pp. 636-641.

[43] F. Mesdoui, A. Ouannas A, N. Shawagfeh, G. Grassi, V. T. Pham, Synchronization Methods for the Degn-Harrison Reaction-Diffusion Systems. IEEE Access, (2020) vol 8 pp. 91829-91836.

[44] F. Mesdoui, N. Shawagfeh, A. Ouannas, Global synchronization of fractional-order and integer-order N component reaction diffusion systems: Application to biochemical models. Math Meth Appl Sci (2020) pp 1– 10.