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Fixed Point Results for Suzuki Type Σ−Contractions via Simulation Functions in Modular b−Metric Spaces [†]

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Summary

This study aims to introduce Suzuki type Σ−contraction mappings with simulation functions in the frame of modular b-metric spaces. Also, some coincidence and common fixed point results are obtained for four mappings using the weakly compatibility property that these results are the extensions and improvements of the existing literature. Finally, we also present two applications on graph theory and homotopy theory, which show applicability and validity of our results.

KEYWORDS:
Modular b−metric spaces, Suzuki type contractions, Simulation functions, Common fixed points.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is one of the centers of mathematical analysis in the sense of metric space and Banach fixed point theorem (or Banach Contraction Principle), which have countless application fields such as classical functional analysis and various branches of mathematics. Banach fixed point theorem [1] has been one of the classic and most useful results of fixed point theory which asserts that every mapping ξ on a complete metric space (Q, m) satisfying for all $\varpi, q \in Q$

$$m(\xi\varpi, \xi q) \leq \lambda m(\varpi, q), \quad where \quad \lambda \in (0,1) \tag{1}$$

possesses a unique fixed point and for every $\varpi_0 \in Q$, the sequence $\left\{\xi^n \varpi_0\right\}$ is convergent to this fixed point.

In the sequel the letters $\mathbf{N}$ and $\mathbf{R}_+$ will emblematise the set of all natural numbers and the set of all positive real numbers, respectively. We also consult $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Firstly, in 2010, Chistyakov [2] acquainted a new generalized metric space, which is termed a modular metric space, hereinbelow.

Let Q be a non-void set and $\nu : (0,\infty) \times Q \times Q \to [0,\infty]$ be a function; concisely, we express:

$$\nu_\ell(\varpi, q) = \nu(\ell, \varpi, q)$$

for all $\ell > 0$ and $\varpi, q \in Q$.

Definition 1. [2] Let Q be a non-void set. A function $\nu : (0,\infty) \times Q \times Q \to [0,\infty]$ is labelled as a metric modular on Q, provided to following conditions hold: for all $\varpi, q, r \in Q$

(ν_1) for all $\ell > 0$ $\nu_\ell(\varpi, q) > 0$ if and only if $\varpi = q$,

(ν_2) for all $\ell > 0$ $\nu_\ell(\varpi, q) = \nu_\ell(q, \varpi)$,

(ν_3) for all $\ell, \mu > 0$ $\nu_{\ell+\mu}(\varpi, q) \leq \nu_\ell(\varpi, r) + \nu_\mu(r, q)$.

If instead of (ν_1), we have only the condition

[†]This is an example for title footnote.
[0]**Abbreviations:** MbMS, modular b−metric space

$(v_1{}')$ $v_\ell(\varpi,\varpi)=0$ for all $\ell>0$, then v is said to be a (metric) pseudo modular on Q.

For more details, it is refer to [2,3,4,5,6].

The notion of a b-metric space, which is more general in terms of triangle inequality from metric space, defined by Czerwik[3 4] as noted below.

Definition 2. [3] Let Q be a non−empty set and $\tau \geq 1$ be a given real number. A function $\sigma : Q \times Q \to \mathbf{R}^+$ is a b−metric on Q if, for all $\varpi, q, r \in Q$, the following aspects hold:

(σ_1) $\sigma(\varpi,q)=0 \Leftrightarrow \varpi=q$,

(σ_2) $\sigma(\varpi,q)=\sigma(q,\varpi)$,

(σ_3) $\sigma(\varpi,r)\leq \tau\left[\sigma(\varpi,q)+\sigma(q,r)\right]$.

In this case, σ is a b−metric on Q and the pair (Q,σ) is a b−metric space.

Although the ordinary metric function is continuous, the b−metric is not always. Also, obviously, for $\tau=1$, b−metric function reduces to ordinary metric function.

Very recently, Ege and Alaca[5] defined the modular b−metric space with some new concept and prove some fixed point theorems for the new space, as indicated below.

Definition 3. [5] Let Q be a non-empty set and let $\tau \geq 1$ be a real number. A map $\aleph : (0,\infty)\times Q\times Q \to [0,\infty]$ is called a modular b−metric, if the following statements hold for all $\varpi, q, r \in Q$,

$(\aleph_1)$ $\aleph_\ell(\varpi,q)>0$ for all $\ell>0$ if and only if $\varpi=q$,

$(\aleph_2)$ $\aleph_\ell(\varpi,q)=\aleph_\ell(q,\varpi)$ for all $\ell>0$,

$(\aleph_3)$ $\aleph_{\ell+\mu}(\varpi,q)\leq \tau\left[\aleph_\ell(\varpi,r)+\aleph_\ell(r,q)\right]$ for all $\ell,\mu>0$.

Then, we say that $(Q,\aleph)$,(briefly $Q_\aleph$) is a modular b−metric space, which denotes MbMS.

Note that the modular b−metric space is considered as a generalization of metric modular. In follow, we present some examples of MbMS.

- Example 1.[5] Consider the space

$$1_j=\left\{(\varpi_n)\subset \mathbf{R} : \sum_{n=1}^{\infty}|\varpi_n|^j<\infty\right\} \quad 0<j<1,$$

  $\ell \in (0,\infty)$ and $\aleph_\ell(\varpi,q)=\frac{d(\varpi,q)}{\ell}$ such that

$$d(\varpi,q)=\left(\sum_{n=1}^{\infty}|\varpi_n-q_n|^j\right)^{\frac{1}{j}}, \quad \varpi=\varpi_n,\ q=q_n\in 1_j$$

  It could be easily seen that $(Q,\aleph)$ is a modular b−metric space.

- Example 2.[6] Let $(Q,\aleph)$ be a modular b−metric space and let $k\geq 1$ be a real number.Take $\upsilon_\ell(\varpi,q)=\left(\aleph_\ell(\varpi,q)\right)^\tau$. Using the convexity of $\xi(t)=t^k$ for $t\geq 0$, by Jensen inequality, we have

$$(\alpha+\beta)^k\leq 2^{k-1}\left(\alpha^k+\beta^k\right)$$

  for non negative real numbers α,β. Thus, (Q,υ) is a modular b−metric space with the constant $\tau=2^{k-1}$.

Now, lets we bring in some basic topological terms as $\aleph$−convergent sequences, $\aleph$−Cauchy sequences, $\aleph$−complete spaces and $\aleph$−continuity of a function.

Definition 4. [5] Let $(Q,\aleph)$ be a modular b−metric space.

(i) The sequence $(\varpi_n)_{n\in\mathbf{N}}$ in $Q_\aleph$ is said to be $\aleph$−convergent to $\varpi\in Q_\aleph$, if $\aleph_\ell\left(\varpi_n,\varpi\right)\to 0$, as $n\to\infty$ for all $\ell>0$.

(ii) The sequence $(\varpi_n)_{n\in\mathbf{N}}$ in $Q_\aleph$ is said to be $\aleph$−Cauchy, if $\aleph_\ell\left(\varpi_n,\varpi_m\right)\to 0$, as $m,n\to\infty$ for all $\ell>0$.

(iii) A modular $b-$metric space $Q_{\aleph}$ is $\aleph-$complete if each $\aleph-$Cauchy sequence in $Q_{\aleph}$ is $\aleph-$convergent and its limit is in $Q_{\aleph}$.

(vi) A function $\xi : (Q, \aleph) \to (Q, \aleph)$ is called $\aleph-$continuous if $\aleph_{\ell}\left(\xi\varpi_n, \xi\varpi\right) \to 0$, whenever $\aleph_{\ell}\left(\varpi_n, \varpi\right) \to 0$.

In 2015, Khojasteh et al.[7] presented a new control function and named as the simulation function to obtain the various fixed point results that substantial in the literature.

Definition 5. [7] Let $\xi : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be a mapping and the following statements hold:

(ς_1) $\varsigma(0,0) = 0$,

(ς_2) $\varsigma(\alpha,\beta) < \beta - \alpha$ for all $\alpha, \beta > 0$,

(ς_3) if $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n > 0$ and $\lim_{n\to\infty} \sup \varsigma\left(\alpha_n, \beta_n\right) < 0$.

Then ς is described as a simulation function and we symbolize the set of all simulation functions by Υ.

Due to the axiom (ς_2), it is clear that for all $\alpha > 0$, $\varsigma(\alpha, \alpha) < 0$.
Thereafter, we present some examples of the simulation function.

- Example 3.[7,8] Let $\varsigma_i : [0,\infty) \times [0,\infty) \to R$, $\quad i = 1,2,3,4,5$ be defined by

  i. $\varsigma_1(\alpha,\beta) = \psi(\beta) - \phi(\alpha)$ for all $\alpha, \beta \in [0,\infty)$, where $\psi, \phi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(\alpha) = \phi(\alpha) = 0$ if and only if $\alpha = 0$ and $\psi(\alpha) < \alpha \le \phi(\alpha)$ for all $\alpha > 0$.

  ii. $\varsigma_2(\alpha,\beta) = \beta - \frac{J(\alpha,\beta)}{I(\alpha,\beta)}$ for all $\alpha, \beta \in [0,\infty)$, where $J, I : [0,\infty) \to (0,\infty)$ are two continuous functions with respect to each variable such that $J(\alpha,\beta) > I(\alpha,\beta)$ for all $\alpha, \beta > 0$.

  iii. $\varsigma_3(\alpha,\beta) = \beta - \Gamma(\beta) - \alpha$ for all $\alpha, \beta \in [0,\infty)$, where $\Gamma : [0,\infty) \to [0,\infty)$ is a continuous functions $\Gamma(\alpha) = 0$ if and only if $\alpha = 0$.

  iv. $\varsigma_4(\alpha,\beta) = \beta\eta(\beta) - \alpha$ for all $\alpha, \beta \in [0,\infty)$, where $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(\alpha) < \alpha$ for all $\alpha > 0$ and $\eta(0) = 0$.

  v. $\varsigma_5(\alpha,\beta) = \beta - \int_0^{\alpha} \theta(\imath)\, d\imath$ for all $\alpha, \beta \in [0,\infty)$, where $\theta : [0,\infty) \to [0,\infty)$ is a function such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} \theta(\imath) d\imath$ exists and $\int_0^{\varepsilon} \theta(\imath) d\imath > \varepsilon$.

  Then ς_i for $i = 1,2,3,4,5$ are simulation functions.

Next, Khojasteh et al.[7] defined the concept of Υ-contraction via simulation functions, as follows.

Definition 6. [7] Let (Q, m) be a metric space, ξ is a self mapping on Q and $\varsigma \in \Upsilon$. Then ς is called a Υ-contraction with respect to ς if the following condition is satisfied

$$\varsigma(m(\xi\varpi, \xi q), m(\varpi, q)) \ge 0 \text{ for all } \varpi, q \in Q.$$

If we take $\lambda \in [0,1)$ and $\varsigma(\alpha,\beta) = \lambda\beta - \alpha$ for all $\alpha, \beta \in [0,\infty)$ in the above definition, then we obtain the Banach contraction mapping, that is, Banach contraction mapping is an example of Υ-contraction mapping.

Now, we show some properties of Υ-contractions defined on a metric space.

Remark 1. By the definition of simulation function that $\varsigma(\alpha,\beta) < 0$ for all $\alpha \ge \beta > 0$. Therefore, if ξ is a Υ-contraction with respect to $\varsigma \in \Upsilon$ then

$$m(\xi\varpi, \xi q) < m(\varpi, q).$$

It implies that every Υ-contraction mapping is contractive, hence it is continuous.

In 2017, Mongkolkeha et al.[9] modified the notion of a simulation function as follow:

Definition 7. [9] Let $\varsigma^* : [0,\infty) \times (0,\infty] \to \mathbf{R}$ be a mapping such that it is called a simulation function with $\wp \ge 1$ real number, if the following statements hold:

$\left(\varsigma_1^*\right)$ $\varsigma^*(0,0) = 0$,

$\left(\varsigma_2^*\right)$ $\varsigma^*(\wp\alpha, \beta) < \beta - \wp\alpha$, for all $\alpha, \beta > 0$,

$\left(\varsigma_3^*\right)$ if $\left\{\alpha_n\right\}, \left\{\beta_n\right\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty}\sup \wp\alpha_n = \lim_{n\to\infty}\sup \beta_n > 0$ and $\alpha_n < \beta_n$ for all $n \in \mathbf{N}$, then

$$\lim_{n\to\infty}\sup \varsigma^*\left(\wp\alpha_n, \beta_n\right) < 0.$$

Suzuki [10] asserted a new preconditions, which is celebrated as the Suzuki type contraction, and proved the fixed point theorem with preconditions, as demonstrated below.

Theorem 1. [10] Let (Q, m) be a compact metric space and $\xi : Q \to Q$ be a mapping. Presume that, for all $\varpi, q \in Q$ with $\varpi \neq q$, the following statement holds:

$$\frac{1}{2}m(\varpi, \xi\varpi) < m(\varpi, q) \Rightarrow m(\xi\varpi, \xi q) < m(\varpi, q)$$

Then, ξ holds a unique fixed point in Q.

Furthermore various authors generalized Suzuki type contractions to other spaces. Kumam et al. [11] introduced the notion of Suzuki type Υ-contraction in the setting of metric spaces as follows.

Definition 8. [11] Let (Q, m) be a metric space, ξ is self mapping in Q and $\varsigma \in \Upsilon$. Then ξ is named as Suzuki type $\Upsilon-$contraction with respect to ς, provided to satisfying the below condition.

$$\frac{1}{2}m(\varpi, \xi\varpi) < m(\varpi, q) \quad \Rightarrow \quad \varsigma\left(m(\xi\varpi, \xi q), m(\varpi, q)\right) \geq 0,$$

for all $\varpi, q \in Q$ with $\varpi \neq q$.

Next, in 2018, Padcharoen et al. [12] presented the generalized Suzuki type contraction in a metric space as noted below.

Definition 9. [12] Let (Q, m) be a metric space, ξ be a self mapping on Q and $\varsigma \in \Upsilon$. Then ξ is called a generalized Suzuki type $\Upsilon-$contraction with respect to ς, if the following condition is satisfied

$$\frac{1}{2}m(\varpi, \xi\varpi) < m(\varpi, q) \quad \Rightarrow \quad \varsigma\left(m(\xi\varpi, \xi q), M(\varpi, q)\right) \geq 0,$$

for all $\varpi, q \in Q$, where

$$M(\varpi, q) = \max\left\{m(\varpi, q), m(\varpi, \xi\varpi), m(q, \xi q), \frac{m(\varpi, \xi q) + m(q, \xi\varpi)}{2}\right\}.$$

Also, Antal and Gairola [13] defined the generalized Suzuki type $\delta - \Upsilon-$contraction with respect to ς in a $b-$metric space.

Definition 10. [13] Let (Q, m) be a $b-$metric space with coefficient $\tau \geq 1$ and $\delta : Q \times Q \to \mathbf{R}$ be a function. A mapping $\xi : Q \to Q$ is named as a generalized Suzuki type $\delta - \Upsilon-$contraction with respect to ς, provided to following expression holds:

$$\frac{1}{2\tau}m(\varpi, \xi\varpi) < m(\varpi, q) \Rightarrow \varsigma\left(\tau^4\delta(\varpi, q)\, m(\xi\varpi, \xi q), M_\xi(\varpi, q)\right) \geq 0$$

for all distinct $\varpi, q \in Q$, where

$$M_\xi(\varpi, q) = \max\left\{m(\varpi, q), m(\varpi, \xi\varpi), m(q, \xi q), \frac{m(\varpi, \xi q) + m(q, \xi\varpi)}{2\tau}\right\}.$$

In 2018, A. Fulga and E. Karapınar [14] acquainted a new results on Υ-contraction of Type Σ, as follows.

Definition 11. [14] Let (Q, m) be a complete metric space and $\xi : Q \to Q$ be a mapping such that it is called as $\Upsilon-$contraction of Type Σ with respect to ς, if

$$\varsigma\left(m(\xi\varpi, \xi q), \Sigma(\varpi, q)\right) \geq 0 \text{ for all } \varpi, q \in Q$$

where

$$\Sigma(\varpi, q) = m(\varpi, q) + \left|m(\varpi, \xi\varpi) - m(q, \xi q)\right|.$$

Theorem 2. [14] Let ξ be a the $\Upsilon-$contraction of Type Σ with respect to ς defined on Q. Then ξ holds a fixed point in Q.

2. MAIN RESULTS

This section aims to establish a new contraction named as Suzuki type Σ contraction via simulation functions in modular $b-$metric space for four mappings and to present some common fixed point results related to these mappings.

Definition 12. Let $(Q, \aleph)$ be a modular $b-$metric space with coefficient $\tau \geq 1$ and let $\xi, \hbar, J$ and I be self mappings in MbMS. Then, we say that the mappings $\xi, \hbar, J$ and I are Suzuki type Σ contraction with respect to ς if the following terms are provided:

$$\frac{1}{2\tau} \min\left(\aleph_\ell\left(I\varpi, \xi\varpi\right), \aleph_\ell\left(Jq, \hbar q\right)\right) \leq \max\left(\aleph_\ell\left(I\varpi, Jq\right), \aleph_\ell\left(\xi\varpi, \hbar q\right)\right)$$

implies

$$\varsigma\left(\tau\, \aleph_\ell\left(\xi\varpi, \hbar q\right), \Sigma\left(\varpi, q\right)\right) \geq 0, \tag{2}$$

where

$$\Sigma\left(\varpi, q\right) = \frac{1}{\tau^2}\left[\aleph_\ell\left(I\varpi, Jq\right) + \left|\aleph_\ell\left(I\varpi, \xi\varpi\right) - \aleph_\ell\left(Jq, \hbar q\right)\right|\right],$$

for all distinct $\varpi, q \in Q_\aleph$ and for all $\ell > 0$.

Theorem 3. Let $Q_\aleph$ be a $\aleph-$complete MbMS with coefficient $\tau \geq 1$ and let $\xi, \hbar, J$ and I be a Suzuki type Σ contraction with respect to ς such that $\xi(Q) \subset J(Q)$ and $\hbar(Q) \subset I(Q)$. Suppose that one of the set $\xi(Q), J(Q), \hbar(Q)$ and $I(Q)$ is closed subset of $Q_\aleph$ and that the pairs $\{J, \hbar\}$ and $\{I, \xi\}$ are weakly compatible. Then, $\xi, \hbar, J$ and I possess a unique common fixed point.

Proof. Let $\varpi_0 \in Q_\aleph$ be an arbitrary point in $Q_\aleph$ and let choose a point $\varpi_1 \in Q_\aleph$ such that $q_0 = \xi\varpi_0 = J\varpi_1$. Since the range of J contains the range of ξ, this can be done. Similarly, we choose a point $\varpi_2 \in Q_\aleph$ such that $q_1 = \hbar\varpi_1 = I\varpi_2$ as $\hbar(Q) \subseteq I(Q)$. Continuing this manner, we construct a sequence $\{q_n\}$ in $Q_\aleph$ such that

$$q_{2n} = \xi\varpi_{2n} = J\varpi_{2n+1} \quad , \quad q_{2n+1} = \hbar\varpi_{2n+1} = I\varpi_{2n+2}.$$

Since,

$$\frac{1}{2\tau} \min\left(\aleph_\ell\left(I\varpi_{2n}, \xi\varpi_{2n}\right), \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right) \leq$$

$$\max\left(\aleph_\ell\left(I\varpi_{2n}, J\varpi_{2n+1}\right), \aleph_\ell\left(\xi\varpi_{2n}, \hbar\varpi_{2n+1}\right)\right)$$

From (2) and (ς_2), we have

$$\begin{aligned} 0 \leq\ & \varsigma\left(\tau\, \aleph_\ell\left(\xi\varpi_{2n}, \hbar\varpi_{2n+1}\right), \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right)\right) \\ <\ & \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right) - \tau\, \aleph_\ell\left(\xi\varpi_{2n}, \hbar\varpi_{2n+1}\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right) &= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(I\varpi_{2n}, J\varpi_{2n+1}\right) + \left|\aleph_\ell\left(I\varpi_{2n}, \xi\varpi_{2n}\right) - \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right|\right] \\ &= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(q_{2n-1}, q_{2n}\right) + \left|\aleph_\ell\left(q_{2n-1}, q_{2n}\right) - \aleph_\ell\left(q_{2n}, q_{2n+1}\right)\right|\right]. \end{aligned} \tag{4}$$

Consequently, by (3) and (4), we derive that

$$\aleph_\ell\left(q_{2n}, q_{2n+1}\right) \leq \tau\, \aleph_\ell\left(q_{2n}, q_{2n+1}\right) < \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right). \tag{5}$$

By letting $\eta_{2n} = \aleph_\ell\left(q_{2n-1}, q_{2n}\right)$ in (4) and (5). Thus, we have

$$\eta_{2n+1} \leq \tau\eta_{2n} < \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right) = \frac{1}{\tau^2}\left[\eta_{2n} + \left|\eta_{2n} - \eta_{2n+1}\right|\right]. \tag{6}$$

If we decide on $\eta_{2n} < \eta_{2n+1}$, then we obtain

$$\eta_{2n+1} \leq \tau\eta_{2n+1} < \frac{1}{\tau^2}\eta_{2n+1}.$$

It is a contradiction. Hence, we conclude that $\eta_{2n+1} < \eta_{2n}$ such that

$$\Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right) = \frac{1}{\tau^2}\left[2\eta_{2n} - \eta_{2n+1}\right]. \tag{7}$$

So, $\{\eta_{2n}\} = \{\aleph_\ell\left(q_{2n-1}, q_{2n}\right)\}$ is non-decreasing sequence of non negative real numbers. Thence, there exists $r \geq 0$ such that $\lim_{n\to\infty} \eta_{2n} = r$ for all $\ell > 0$. It is necessary to prove that $r = 0$. Conversely, presume that $r > 0$.

There are two situations that need to be discussed.

Case 1: If $\tau > 1$, taking the limit as $n \to \infty$ in (7) and (5), we gain

$$r \leq \tau\, r < \frac{1}{\tau^2} r,$$

which is a contradiction.

Case 2: If $\tau = 1$, then we have

$$\varsigma\left(\tau\, \aleph_{\ell}\left(q_{2n}, q_{2n+1}\right), \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right)\right) \geq 0.$$

If we put $\alpha_n = \aleph_{\ell}\left(q_{2n}, q_{2n+1}\right)$ and $\beta_n = \Sigma\left(\varpi_{2n}, \varpi_{2n+1}\right)$, then $\limsup_{n\to\infty} \tau\alpha_n = \limsup_{n\to\infty} \beta_n = r > 0$ and $\alpha_n < \beta_n$ is satisfying. Accordingly, from $\left(\varsigma_3^*\right)$, we get

$$0 \leq \limsup_{n\to\infty} \varsigma^*\left(\tau\ \alpha_n, \beta_n\right) < 0,$$

which is a contradiction. Because of that, we conclude that $r = 0$, that is, for all $\ell > 0$

$$\aleph_{\ell}\left(q_{2n-1}, q_{2n}\right) \to 0\,, \quad (n \to \infty)\,. \tag{8}$$

Now, in next step, we show that $\left\{q_n\right\}$ is a $\aleph-$Cauchy sequence. It is enough to show that $\left\{q_{2n}\right\}$ is a $\aleph-$Cauchy sequence. Assert the contrary, then given $\varepsilon > 0$ such that there exist two sequences $\left\{m_k\right\}$ and $\left\{n_k\right\}$ of positive integer satisfying $n_k > m_k \geq k$ such that n_k is smallest index for which

$$\aleph_{\ell}\left(q_{2m_k}, q_{2n_k}\right) \geq \varepsilon \quad \text{and} \quad \aleph_{\ell}\left(q_{2m_k}, q_{2n_k-2}\right) < \varepsilon, \quad \text{for all } \ell > 0. \tag{9}$$

Using (9), we obtain

$$\varepsilon \leq \aleph_{\ell}\left(q_{2m_k}, q_{2n_k}\right) \leq \tau\aleph_{\frac{\ell}{2}}\left(q_{2m_k}, q_{2n_k+1}\right) + \tau\aleph_{\frac{\ell}{2}}\left(q_{2n_k+1}, q_{2n_k}\right).$$

We take the limsup in above $k \to \infty$ and by using (8), we get

$$\lim_{n\to\infty} \sup \aleph_{\ell}\left(q_{2m_k}, q_{2n_k+1}\right) \geq \frac{\varepsilon}{\tau}\,, \quad \text{for all } \ell > 0. \tag{10}$$

From $\left(\aleph_3\right)$, we have

$$\aleph_{\ell}\left(q_{2m_k-1}, q_{2n_k}\right) \leq \tau\aleph_{\frac{\ell}{2}}\left(q_{2m_k-1}, q_{2m_k}\right) + \tau^2\aleph_{\frac{\ell}{2}}\left(q_{2m_k}, q_{2n_k-2}\right)$$

$$+\tau^3\aleph_{\frac{\ell}{4}}\left(q_{2n_k-2}, q_{2n_k-1}\right) + \tau^3\aleph_{\frac{\ell}{4}}\left(q_{2n_k-1}, q_{2n_k}\right).$$

Again, we take the limsup in above $k \to \infty$ and by use of (8) and (9), we procure

$$\limsup_{k\to\infty} \aleph_{\ell}\left(q_{2m_k-1}, q_{2n_k}\right) \leq \tau^2\varepsilon\,, \quad \text{for all } \ell > 0. \tag{11}$$

From (8) and (9), we can perceive a positive integer $\mathrm{n}_1 \in \mathbf{N}$ such that

$$\frac{1}{2\tau} \min\left(\aleph_{\ell}\left(I\varpi_{2m_k}, \xi\varpi_{2m_k}\right), \aleph_{\ell}\left(J\varpi_{2n_k+1}, \hbar\varpi_{2n_k+1}\right)\right) \leq \frac{\varepsilon}{\tau^2} \max\left[\aleph_{\ell}\left(I\varpi_{2m_k}, J\varpi_{2n_k+1}\right),\right.$$

$$\left.\aleph_{\ell}\left(\xi\varpi_{2m_k}, \hbar\varpi_{2n_k+1}\right)\right].$$

So, from the inequality (2) and $\left(\varsigma_2\right)$, we obtain

$$\begin{aligned} 0 \leq\ & \varsigma\left(\tau\ \aleph_{\ell}\left(\xi\varpi_{2m_k}, \hbar\varpi_{2n_k+1}\right), \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right)\right) \\ & < \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right) - \tau\ \aleph_{\ell}\left(\xi\varpi_{2m_k}, \hbar\varpi_{2n_k+1}\right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right) &= \frac{1}{\tau^2}\left[\aleph_{\ell}\left(I\varpi_{2m_k}, J\varpi_{2n_k+1}\right) + \left|\aleph_{\ell}\left(I\varpi_{2m_k}, \xi\varpi_{2m_k}\right) - \aleph_{\ell}\left(J\varpi_{2n_k+1}, \hbar\varpi_{2n_k+1}\right)\right|\right] \\ &= \frac{1}{\tau^2}\left[\aleph_{\ell}\left(q_{2m_k-1}, q_{2n_k}\right) + \left|\aleph_{\ell}\left(q_{2m_k-1}, q_{2m_k}\right) - \aleph_{\ell}\left(q_{2n_k}, q_{2n_k+1}\right)\right|\right]. \end{aligned} \tag{13}$$

If we take the limsup as $k \to \infty$ in (13) and by using (11), we get

$$\begin{aligned} \limsup_{k\to\infty} \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right) &= \limsup_{k\to\infty}\left\{\frac{1}{\tau^2}\left[\aleph_{\ell}\left(q_{2m_k-1}, q_{2n_k}\right)\right]\right\} \\ &\leq \frac{1}{\tau^2}\tau^2\varepsilon = \varepsilon. \end{aligned} \tag{14}$$

Finally, if we take the limsup as $k \to \infty$ in (12) and by using (10) and (14), we have

$$
\begin{aligned}
0 \leq\ & \limsup_{k\to\infty} \left[\varsigma\left(\tau\ \aleph_\ell\left(\xi\varpi_{2m_k}, \hbar\varpi_{2n_k+1}\right), \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right)\right)\right] \\
&< \limsup_{k\to\infty} \left[\Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right) - \tau\ \aleph_\ell\left(q_{2m_k}, q_{2n_k+1}\right)\right] \\
&\leq \limsup_{k\to\infty} \Sigma\left(\varpi_{2m_k}, \varpi_{2n_k+1}\right) - \liminf_{k\to\infty} \left[\tau\ \aleph_\ell\left(q_{2m_k}, q_{2n_k+1}\right)\right] \\
&\leq \varepsilon - \tau\tfrac{\varepsilon}{\tau} = 0
\end{aligned}
$$

which is a consistency. Thereupon $\{q_{2n}\}$ is a $\aleph-$Cauchy sequence. Thus, $\{q_n\}$ is a $\aleph-$Cauchy sequence in $Q_\aleph$. As $Q_\aleph$ is $\aleph-$complete MbMS, there exists $z \in Q_\aleph$ such that

$$\lim_{n\to\infty} q_n = z. \tag{15}$$

Now, we shall prove that z is common fixed point of $\xi, \hbar, J$ and I. Firstly, we show that z is fixed point for the maps ξ and I.

It is clear that

$$
\begin{aligned}
\lim_{n\to\infty} q_{2n} &= \lim_{n\to\infty} \xi\varpi_{2n} = \lim_{n\to\infty} J\varpi_{2n+1} = z \\
\lim_{n\to\infty} q_{2n+1} &= \lim_{n\to\infty} \hbar\varpi_{2n+1} = \lim_{n\to\infty} I\varpi_{2n+2} = z.
\end{aligned}
$$

Assume that $I\left(Q_\aleph\right)$ is closed subset of $Q_\aleph$, there exists $u \in Q_\aleph$ such that $z = Su$. We claim that $\xi u = z$. Let it is not. Then, since

$$\frac{1}{2\tau}\min\left\{\aleph_\ell(Iu,\xi u), \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right\} \leq \max\left\{\aleph_\ell\left(Iu, J\varpi_{2n+1}\right), \aleph_\ell\left(\xi u, \hbar\varpi_{2n+1}\right)\right\}$$

implies

$$
\begin{aligned}
0 &\leq \varsigma\left(\tau\aleph_\ell\left(\xi u, \hbar\varpi_{2n+1}\right), \Sigma\left(u, \varpi_{2n+1}\right)\right) \\
&< \Sigma\left(u, \varpi_{2n+1}\right) - \tau\aleph_\ell\left(\xi u, \hbar\varpi_{2n+1}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma\left(u, \varpi_{2n+1}\right) &= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(Iu, J\varpi_{2n+1}\right) + \left|\aleph_\ell(Iu,\xi u) - \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right|\right] \\
&= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(z, q_{2n}\right) + \left|\aleph_\ell(z,\xi u) - \aleph_\ell\left(q_{2n}, q_{2n+1}\right)\right|\right].
\end{aligned}
$$

If we take the limit as $n \to \infty$ in above, we get

$$0 < \lim_{n\to\infty} \Sigma\left(u, \varpi_{2n+1}\right) - \lim_{n\to\infty} \tau\aleph_\ell\left(\xi u, \hbar\varpi_{2n+1}\right) = \frac{1}{\tau^2}\aleph_\ell(z,\xi u) - \tau\aleph_\ell(z,\xi u) < 0,$$

a contradiction as $\tau \geq 1$. Hence $\xi u = z$. Therefore $\xi u = Iu = z$. Because the mappings ξ and I have weakly compatibility property, we have $\xi z = \xi I u = I\xi u = Iz$.

Next we assert that $\xi z = z$. If not, then, as

$$\frac{1}{2\tau}\min\left\{\aleph_\ell(Iz,\xi z), \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right\} \leq \max\left\{\aleph_\ell\left(Iz, J\varpi_{2n+1}\right), \aleph_\ell\left(\xi z, \hbar\varpi_{2n+1}\right)\right\}$$

implies

$$
\begin{aligned}
0 &\leq \varsigma\left(\tau\aleph_\ell\left(\xi z, \hbar\varpi_{2n+1}\right), \Sigma\left(z, \varpi_{2n+1}\right)\right) \\
&< \Sigma\left(z, \varpi_{2n+1}\right) - \tau\aleph_\ell\left(\xi z, \hbar\varpi_{2n+1}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma\left(z, \varpi_{2n+1}\right) &= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(Iz, J\varpi_{2n+1}\right) + \left|\aleph_\ell(Iz,\xi z) - \aleph_\ell\left(J\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right|\right] . \\
&= \tfrac{1}{\tau^2}\left[\aleph_\ell\left(Iz, q_{2n}\right) + \left|\aleph_\ell(Iz,\xi z) - \aleph_\ell\left(q_{2n}, q_{2n+1}\right)\right|\right].
\end{aligned}
$$

If we take the limit as $n \to \infty$ in above, we get

$$0 < \lim_{n\to\infty} \Sigma\left(z, \varpi_{2n+1}\right) - \lim_{n\to\infty} \tau\aleph_\ell\left(\xi z, \hbar\varpi_{2n+1}\right) = \frac{1}{\tau^2}\aleph_\ell(Iz, z) - \tau\aleph_\ell(Iz, z) < 0$$

a contradiction with $\tau \geq 1$. Therefore $\xi z = z$.

The next step is to show that z is also fixed point for the mappings $\hbar$ and J. As $\xi\left(Q_{\aleph}\right) \subset J\left(Q_{\aleph}\right)$, there exist some υ in $Q_{\aleph}$ such that $\xi z = J\upsilon$. Then $\xi z = J\upsilon = Iz = z$. We claim that $\hbar\upsilon = z$. If $\hbar\upsilon \neq z$ then, since

$$\frac{1}{2\tau}\min\left\{\aleph_{\ell}\left(Iz,\xi z\right),\aleph_{\ell}\left(J\upsilon,\xi\varpi_{2n+1}\right)\right\} \leq \max\left\{\aleph_{\ell}\left(Iz,J\upsilon\right),\aleph_{\ell}\left(\xi z,\hbar\upsilon\right)\right\}$$

implies

$$\begin{aligned} 0 &\leq \varsigma\left(\tau\aleph_{\ell}\left(\xi z,\hbar\upsilon\right),\Sigma\left(z,\upsilon\right)\right) \\ &< \Sigma\left(z,\upsilon\right) - \tau\aleph_{\ell}\left(\xi z,\hbar\upsilon\right), \end{aligned}$$

where

$$\begin{aligned} \Sigma\left(z,\upsilon\right) &= \tfrac{1}{\tau^2}\left[\aleph_{\ell}\left(Iz,J\upsilon\right) + \left|\aleph_{\ell}\left(Iz,\xi z\right) - \aleph_{\ell}\left(J\upsilon,\hbar\upsilon\right)\right|\right] \\ &= \tfrac{1}{\tau^2}\left[\aleph_{\ell}\left(Iz,z\right) + \left|\aleph_{\ell}\left(Iz,z\right) - \aleph_{\ell}\left(z,\hbar\upsilon\right)\right|\right]. \end{aligned}$$

If we take the limit as $n \to \infty$ in above, we obtain

$$0 < \lim_{n\to\infty}\Sigma\left(z,\upsilon\right) - \lim_{n\to\infty}\tau\aleph_{\ell}\left(\xi z,\hbar\upsilon\right) = \frac{1}{\tau^2}\aleph_{\ell}\left(z,\hbar\upsilon\right) - \tau\aleph_{\ell}\left(z,\hbar\upsilon\right) < 0,$$

a contradiction with $\tau \geq 1$. Then $z = \hbar\upsilon$. Hence, $\hbar\upsilon = J\upsilon = z$. By using weak compatibility of the mappings $\hbar$ and J we get $\hbar z = \hbar J\upsilon = JJ\upsilon = Jz$.

Finally, we claim that $\hbar z = z$. If $\hbar z \neq z$, then, because

$$\frac{1}{2\tau}\min\left\{\aleph_{\ell}\left(Iz,\xi z\right),\aleph_{\ell}\left(Jz,\xi z\right)\right\} \leq \max\left\{\aleph_{\ell}\left(Iz,Jz\right),\aleph_{\ell}\left(\xi z,\hbar z\right)\right\}$$

then

$$\begin{aligned} 0 &\leq \varsigma\left(\tau\aleph_{\ell}\left(\xi z,\hbar z\right),\Sigma\left(zz\right)\right) \\ &< \Sigma\left(z,z\right) - \tau\aleph_{\ell}\left(\xi z,\hbar z\right), \end{aligned}$$

where

$$\begin{aligned} \Sigma\left(z,z\right) &= \tfrac{1}{\tau^2}\left[\aleph_{\ell}\left(Iz,Jz\right) + \left|\aleph_{\ell}\left(Iz,\xi z\right) - \aleph_{\ell}\left(Jz,\hbar z\right)\right|\right] \\ &= \tfrac{1}{\tau^2}\left[\aleph_{\ell}\left(\hbar z,z\right) + \left|\aleph_{\ell}\left(Iz,z\right) - \aleph_{\ell}\left(\hbar z,\hbar z\right)\right|\right]. \end{aligned}$$

By taking limit as $n \to \infty$ in above, we get

$$0 < \lim_{n\to\infty}\Sigma\left(z,z\right) - \lim_{n\to\infty}\tau\aleph_{\ell}\left(\xi z,\hbar z\right) = \frac{1}{\tau^2}\aleph_{\ell}\left(z,\hbar z\right) - \tau\aleph_{\ell}\left(z,\hbar z\right) < 0$$

a contradiction with $\tau \geq 1$. Hence, $\xi z = \hbar z = Iz = Jz = z$. Similar analysis is valid for the case in which $J\left(Q_{\aleph}\right)$ is closed, as well as for the cases in which $\xi\left(Q_{\aleph}\right)$ or $\hbar\left(Q_{\aleph}\right)$ is closed, as $\xi\left(Q_{\aleph}\right) \subset J\left(Q_{\aleph}\right)$ and $\hbar\left(Q_{\aleph}\right) \subset I\left(Q_{\aleph}\right)$.

Lastly, we shall show that z is a unique fixed point. If it is not, then there exists a $z \neq p$ such that $p = \xi p = \hbar p = Jp = Ip$. Because

$$0 = \frac{1}{2\tau}\min\left(\aleph_{\ell}\left(Iz,\xi z\right),\aleph_{\ell}\left(Jp,\hbar p\right)\right) \leq \max\left(\aleph_{\ell}\left(Iz,Jp\right),\aleph_{\ell}\left(\xi z,\hbar p\right)\right).$$

Then, from (2), we have

$$\begin{aligned} 0 &\leq \varsigma\left(\tau\,\aleph_{\ell}\left(\xi z,\hbar p\right),\Sigma\left(z,p\right)\right) \\ &< \Sigma\left(z,p\right) - \tau\,\aleph_{\ell}\left(\xi z,\hbar p\right), \end{aligned} \tag{16}$$

where

$$\Sigma\left(z,p\right) = \frac{1}{\tau^2}\left[\aleph_{\ell}\left(Iz,Jp\right) + \left|\aleph_{\ell}\left(Iz,\xi z\right) - \aleph_{\ell}\left(\hbar p,Jp\right)\right|\right]. \tag{17}$$

So, from (16) and (16), we conclude that

$$\aleph_{\ell}\left(z,p\right) \leq \tau\,\aleph_{\ell}\left(z,p\right) < \Sigma\left(z,p\right) = \frac{1}{\tau^2}\aleph_{\ell}\left(z,p\right),$$

is a contradiction. Then $z = p$ is a unique common fixed point of $\xi, \hbar, J$ and I. □

Now, we present some consequences, which immediately attain from our fundamental result.

Corollary 1. If we take $\Sigma\left(\varpi,q\right) = \aleph_{\ell}\left(I\varpi,Jq\right) + \left|\aleph_{\ell}\left(I\varpi,\xi\varpi\right) - \aleph_{\ell}\left(Jq,\hbar q\right)\right|$ in Theorem 3, then, we say that $\xi, \hbar, J$ and I hold a common fixed point uniquely determined in $Q_{\aleph}$.

Corollary 2. Let $Q_\aleph$ be a $\aleph-$complete modular $b-$metric space with coefficient $\tau \geq 1$ and let ξ and $\hbar$ be self mappings in MbMS. If there exists a simulation function $\varsigma \in \Upsilon$ such that

$$\frac{1}{2\tau}\min\left(\aleph_\ell(\varpi,\xi\varpi),\aleph_\ell(q,\hbar q)\right) \leq \aleph_\ell(\varpi,q)$$

implies

$$\varsigma\left(\tau\,\aleph_\ell(\xi\varpi,\hbar q),\Sigma(\varpi,q)\right) \geq 0, \tag{18}$$

where

$$\Sigma(\varpi,q) = \frac{1}{\tau^2}\left[\aleph_\ell(\varpi,q) + \left|\aleph_\ell(\varpi,\xi\varpi) - \aleph_\ell(q,\hbar q)\right|\right]$$

for all distinct $\varpi, q \in Q_\aleph$ and for all $\ell > 0$. Then, we say that ξ and $\hbar$ hold a unique common fixed point in $Q_\aleph$.

Corollary 3. If we take $\Sigma(\varpi,q) = \aleph_\ell(\varpi,q) + \left|\aleph_\ell(\varpi,\xi\varpi) - \aleph_\ell(q,\hbar q)\right|$ in Corollary 2, then, we say that ξ and $\hbar$ have a unique common fixed point in $Q_\aleph$.

If we take $\xi = \hbar$ in Corollary 2, then we obtain the following result.

Corollary 4. Let $Q_{,\aleph}$ be a $\aleph-$complete modular $b-$metric space with coefficient $\tau \geq 1$ and let ξ be self mappings. If there exists a simulation function $\varsigma \in \upsilon$ such that

$$\frac{1}{2\tau}\aleph_\ell(\varpi,\xi\varpi) \leq \aleph_\ell(\varpi,q)$$

implies

$$\varsigma\left(\tau\,\aleph_\ell(\xi\varpi,\xi q),\Sigma(\varpi,q)\right) \geq 0, \tag{19}$$

where

$$\Sigma(\varpi,q) = \frac{1}{\tau^2}\left[\aleph_\ell(\varpi,q) + \left|\aleph_\ell(\varpi,\xi\varpi) - \aleph_\ell(q,\xi q)\right|\right].$$

for all distinct $\varpi, q \in Q_\aleph$ and for all $\ell > 0$. Then, ξ holds a fixed point which is uniquely determined in $Q_\aleph$.

Corollary 5. If we take $\Sigma(\varpi,q) = \aleph_\ell(\varpi,q) + \left|\aleph_\ell(\varpi,\xi\varpi) - \aleph_\ell(q,\xi q)\right|$ in Corollary 4, then, ξ holds a unique fixed point in $Q_\aleph$.

3. APPLICATION TO GRAPH THEORY

Let $Q_\aleph$ be a $\aleph-$complete modular $b-$metric space with $\tau \geq 1$ and let define: $\Omega = \{(\varpi,\varpi) : \varpi \in Q\}$, which denotes the diagonal of the Cartesian product $Q_\aleph \times Q_\aleph$. Likewise, K be a directed graph such that the set $\mathrm{V}(\mathrm{K})$ of its vertices coincides with $Q_\aleph$, and the set $\mathrm{E}(\mathrm{K})$ of its edges contains all loops such that $\Omega \subseteq \mathrm{E}(\mathrm{K})$. The pair $(\mathrm{V}(\mathrm{K}),\mathrm{E}(\mathrm{K}))$ could be expressed as the graph K.

The graph K^{-1}, which obtained from K by reversing the direction of the edges, is conversion of a graph K such that

$$\mathrm{E}\left(\mathrm{K}^{-1}\right) = \left\{(\varpi,q) \in Q_\aleph \times Q_\aleph \right| \quad (q,\varpi) \in \mathrm{E}(\mathrm{K})\left.\right\}.$$

Define by $\tilde{\mathrm{K}}$ is the undirected graph obtained from K by ignoring the direction of the edges and it is clever to treat $\tilde{\mathrm{K}}$ as a directed graph since the set of its edges is symmetric. Under this convention, we have

$$\mathrm{E}\left(\tilde{\mathrm{K}}\right) = \mathrm{E}(\mathrm{K}) \cup \mathrm{E}\left(\mathrm{K}^{-1}\right).$$

Let H be a subgraph of a graph K such that $\mathrm{V}(\mathrm{H}) \subseteq \mathrm{V}(\mathrm{K})$ and $\mathrm{E}(\mathrm{H}) \subseteq \mathrm{E}(\mathrm{K})$. If ϖ and q be vertices in a graph K, then a path from ϖ to q of length $j \in \mathbf{N}$ is a sequence $\left(\varpi_j\right)$, which has $j+1$ distinct vertices such that $\varpi = \varpi_0, \varpi_1, ..., \varpi_j$ and $\left(\varpi_{k-1},\varpi_k\right) \in \mathrm{E}(\mathrm{K})$ for $i = 1,\ldots,j$.

Remind that if there is a path between any two vertices, then K is connected. Furthermore, K is weakly connected if $\tilde{\mathrm{K}}$ is connected. Let K_p be the component of K which consists of all edges and vertices contained in some path in K beginning at ϖ. Suppose that K is such that $\mathrm{E}(\mathrm{K})$ is symmetric; then $\mathrm{V}(\mathrm{K}) = [\varpi]_{\mathrm{K}}$ where $[\varpi]_{\mathrm{K}}$ denotes the equivalence class of relations $\mathfrak{R}$ defined on $\mathrm{V}(\mathrm{K})$ by the rule $q\mathfrak{R}r$ if there is a path in K from q to r.

Definition 13. Let (Q,d) be a metric space, and $\xi : Q \to Q$ be a self-mapping on Q. Then ξ is called a Banach K$-$contraction if the followings hold:

(i) ξ preserves edges of K, that is, for all $\varpi, q \in Q$

$$(\varpi,q) \in \mathrm{E}(\mathrm{K}) \quad \Rightarrow \quad (\xi\varpi,\xi q) \in \mathrm{E}(\mathrm{K}),$$

(ii) ξ decreases weights of edges of K: there exists $\lambda \in (0,1)$ such that

$$d(\xi\varpi, \xi q) \leq \lambda d(\varpi, q)$$

for all $(\varpi, q) \in \mathrm{E}(\mathrm{K})$.

Definition 14. The triple $\left(Q_{\aleph}, \aleph, \mathrm{K}\right)$ is regular if

(i) For any sequence $\left(\varpi_k\right)$ in $Q_{\aleph}$ with $\varpi_k \to p$ and $\left(\varpi_k, \varpi_{k+1}\right) \in \mathrm{E}(\mathrm{K})$ for all $\varpi \in \mathbf{N}$, then $\left(\varpi_k, \varpi\right) \in \mathrm{E}(\mathrm{K})$ for all $k \in \mathbf{N}$.

(ii) For any sequence $\left(\varpi_k\right)$ in $Q_{\aleph}$ with $\varpi_k \to p$ and $\left(\varpi_{k+1}, \varpi_k\right) \in \mathrm{E}(\mathrm{K})$ for all $\varpi \in \mathbf{N}$, then $\left(\varpi, \varpi_k\right) \in \mathrm{E}(\mathrm{K})$ for all $k \in \mathbf{N}$.

Let $Q_{\aleph}$ be a modular $b-$metric space endowed with a graph K and $\xi, \hbar : Q_{\aleph} \to Q_{\aleph}$. We set:

$$Q_{\aleph}^{\xi} = \left\{\varpi \in Q_{\aleph} \,|(\varpi, \xi\varpi) \in \mathrm{E}(\mathrm{K})\right\} \quad \text{and} \quad Q_{\aleph}^{\hbar} = \left\{\varpi \in Q_{\aleph} \,|(\varpi, \hbar\varpi) \in \mathrm{E}(\mathrm{K})\right\}.$$

Now, we will give a new contraction and a fixed point theorem by using the graph structure.

Definition 15. Let $(Q, \aleph)$ be a modular $b-$metric space endowed with the graph K. Assume that the following conditions hold:

(i) $\xi, \hbar$ preserves edges of K, that is, for all $\varpi \in Q_{\aleph}$

$$(\varpi, \xi\varpi) \in \mathrm{E}(\mathrm{K}) \quad \Rightarrow \quad (\xi\varpi, \hbar\xi\varpi) \in \mathrm{E}(\mathrm{K})$$

and

$$(\varpi, \hbar\varpi) \in \mathrm{E}(\mathrm{K}) \quad \Rightarrow \quad (\hbar\varpi, \xi\hbar\varpi) \in \mathrm{E}(\mathrm{K}),$$

(ii) there exists a simulation function $\varsigma \in \Upsilon$ and $\tau \geq 1$ such that

$$\frac{1}{2\tau} \min\left(\aleph_{\ell}(\varpi, \xi\varpi), \aleph_{\ell}(q, \hbar q)\right) \leq \aleph_{\ell}(\varpi, \varpi)$$

implies

$$\varsigma\left(\tau\, \aleph_{\ell}(\xi\varpi, \hbar q), \Sigma(\varpi, q)\right) \geq 0, \tag{20}$$

where

$$\Sigma(\varpi, q) = \frac{1}{\tau^2}\left[\aleph_{\ell}(\varpi, q) + \left|\aleph_{\ell}(\varpi, \xi\varpi) - \aleph_{\ell}(q, \hbar q)\right|\right]$$

for all $\varpi, q \in \mathrm{E}(\mathrm{K})$ and for all $\ell > 0$.

Then the pair $(\xi, \hbar)$ is named a Suzuki Type Σ_{K} graphic contraction via simulation function.

Theorem 4. Let $Q_{\aleph}$ be a $\aleph-$complete modular $b-$metric space endowed with the graph K and $\xi, \hbar : Q_{\aleph} \to Q_{\aleph}$ be self-mappings. Assume that the following conditions hold:

(i) there exists $\varpi_0 \in Q_{\aleph}^{\xi}$,

(ii) the pair $(\xi, \hbar)$ is a Suzuki Type Σ_{K} graphic contraction via simulation function,

(iii) ξ or $\hbar$ are continuous, or

(iv) the triple $\left(Q_{\aleph}, \aleph, \mathrm{K}\right)$ is regular,

(v) K is weakly connected.

Then, ξ or $\hbar$ hold a unique common fixed point in $Q_{\aleph}$.

Proof. Let $\varpi_0 \in Q_{\aleph}^{\xi}$. Thus $\left(\varpi_0, \xi\varpi_0\right) \in \mathrm{E}(\mathrm{K})$. From (i), we get

$$\left(\varpi_0, \xi\varpi_0\right) \in \mathrm{E}(\mathrm{K}) \quad \Rightarrow \quad \left(\xi\varpi_0, \hbar\xi\varpi_0\right) \in \mathrm{E}(\mathrm{K}).$$

If we indicate $\varpi_1 = \xi\varpi_0$, then we have $\left(\varpi_1, \hbar\varpi_1\right) \in \mathrm{E}(\mathrm{K})$. Again, from (i), we get

$$\left(\varpi_1, \hbar\varpi_1\right) \in \mathrm{E}(\mathrm{K}) \quad \Rightarrow \quad \left(\hbar\varpi_1, \xi\hbar\varpi_1\right) \in \mathrm{E}(\mathrm{K}).$$

Indicating $\varpi_2 = \hbar\varpi_1$, this also gives $\left(\varpi_2, \xi\varpi_2\right) \in \mathrm{E}(\mathrm{K})$.

Continuing this way, we determine a sequence $\left\{\varpi_n\right\}_{n\in\mathbf{N}}$ by

$$\varpi_{2n+2} = \hbar\varpi_{2n+1} \quad \text{and} \quad \varpi_{2n+1} = \xi\varpi_{2n}$$

such that $\left(\varpi_{2n}, \varpi_{2n+1}\right) \in \mathrm{E}(\mathrm{K})$.

Then, from Theorem 3, we have that $\left\{\varpi_n\right\}$ is a $\aleph-$Cauchy sequence in $Q_\aleph$. Because $Q_\aleph$ is a $\aleph-$complete, there exists $z \in Q_\aleph$ such that

$$\lim_{n\to\infty} \varpi_n = z. \tag{21}$$

Now, we demonstrate that z is a common fixed point of ξ or $\hbar$. Suppose that the condition (iii) holds. It is easy to show that z is a common fixed point. Then, we presume that the condition (iv) holds. We allow $\varpi = z \in {Q_\aleph}^{\xi}$ and $q = \varpi_{2n+1} \in {Q_\aleph}^{\hbar}$ in condition (iv), then we have

$$\left(z, \varpi_{2n+1}\right) \in \mathrm{E}(\mathrm{K}) \quad \text{for all n} \geq 0.$$

It is true that:

$$\frac{1}{2\tau}\min\left(\aleph_\ell\left(z, \xi z\right), \aleph_\ell\left(\varpi_{2n+1}, \hbar\varpi_{2n+1}\right)\right) \leq \aleph_\ell\left(z, \varpi_{2n+1}\right).$$

Again, similar with the proof of Theorem 3, we obtain z is a common fixed point of ξ or $\hbar$.

Finally, we show that z is a unique common fixed point. On the contrary, we suppose that υ is the another common fixed point with $z \neq \upsilon$ such that $\upsilon = T\upsilon = S\upsilon$. Then, there exists $\mu \in Q_\aleph$ such that $(z, \mu) \in \mathrm{E}(\mathrm{K})$ and $(\mu, \upsilon) \in \mathrm{E}(\mathrm{K})$. Using (v), we get that $(u, \upsilon) \in \mathrm{E}(\mathrm{K})$. Also,

$$0 = \frac{1}{2\tau}\min\left\{\aleph_\ell\left(z, \xi z\right), \aleph_\ell\left(\upsilon, \hbar\upsilon\right)\right\} \leq \aleph_\ell\left(z, \upsilon\right).$$

So, using the similar procedure as in the proof of Theorem 3, we determine that z is a unique common fixed point of ξ or $\hbar$ in $Q_\aleph$. □

4. APPLICATION TO HOMOTOPY THEORY

Initially, we establish a corollary, which is an analysis of Theorem 3.

Corollary 6. Let $Q_\aleph$ be a $\aleph-$complete modular $b-$metric space with $\tau \geq 1$ and let ξ be a self mapping. If, for all $\varpi, q \in Q_\aleph$ and for all $\ell > 0$, the following statement hold:

$$\tau\,\aleph_\ell\left(\xi\varpi, \xi q\right) \leq k\left(\Sigma\left(\varpi, q\right)\right), \quad \text{for all } k \in [0, 1), \tag{22}$$

where

$$\Sigma\left(\varpi, q\right) = \frac{1}{2\tau}\left[\aleph_\ell\left(\varpi, q\right) + \left|\aleph_\ell\left(\varpi, \xi\varpi\right) - \aleph_\ell\left(q, \xi q\right)\right|\right].$$

Then, ξ holds a unique fixed point in $Q_\aleph$.

Proof. Let $\varpi_0 \in Q_\aleph$ be an arbitrary point and we generate a sequence $\left\{\varpi_n\right\}$ by $\varpi_n = \xi\varpi_{n-1} = \xi^n\varpi_0$ for all $\varpi \in Q_\aleph$. Then, from 22, we have

$$\aleph_\ell\left(\varpi_n, \varpi_{n+1}\right) = \aleph_\ell\left(\xi\varpi_{n-1}, \xi\varpi_n\right) \leq \tau\aleph_\ell\left(\xi\varpi_{n-1}, \xi\varpi_n\right) \leq k\left(\Sigma\left(\varpi_{n-1}, \varpi_n\right)\right) \tag{23}$$

where

$$\Sigma\left(\varpi_{n-1}, \varpi_n\right) = \tfrac{1}{\tau^2}\left[\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) + \left|\aleph_\ell\left(\varpi_{n-1}, \xi\varpi_{n-1}\right) - \aleph_\ell\left(\varpi_n, \xi\varpi_n\right)\right|\right]$$

$$= \tfrac{1}{2\tau}\left[\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) + \left|\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) - \aleph_\ell\left(\varpi_n, \varpi_{n+1}\right)\right|\right].$$

We suppose that $\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) < \aleph_\ell\left(\varpi_n, \varpi_{n+1}\right)$. Then, we get

$$\aleph_\ell\left(\varpi_n, \varpi_{n+1}\right) \leq \frac{k}{2\tau}\aleph_\ell\left(\varpi_n, \varpi_{n+1}\right).$$

Because $\frac{k}{2\tau} \in (0,1)$, we obtain an inconsistency. Thus, we have the other case, that is, $\aleph_\ell\left(\varpi_n, \varpi_{n+1}\right) < \aleph_\ell\left(\varpi_{n-1}, \varpi_n\right)$. So, from (23), we get

$$\begin{aligned}\aleph_\ell\left(\varpi_n, \varpi_{n+1}\right) &\le \tfrac{k}{2\tau}\left[2\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) - \aleph_\ell\left(\varpi_n, \varpi_{n+1}\right)\right] \\ &\le \tfrac{k}{\tau}\aleph_\ell\left(\varpi_{n-1}, \varpi_n\right) \\ &\le \left(\tfrac{k}{\tau}\right)^2\aleph_\ell\left(\varpi_{n-2}, \varpi_{n-1}\right) \\ &\vdots \\ &\le \left(\tfrac{k}{\tau}\right)^n\aleph_\ell\left(\varpi_0, \varpi_1\right).\end{aligned}$$

If we take the limit as $n \to \infty$ in above, we conclude that, for all $\ell > 0$,

$$\lim_{n\to\infty} \aleph_\ell\left(\varpi_n, \varpi_{n+1}\right) = 0. \tag{24}$$

Now, we show that $\left\{\varpi_n\right\}$ is a $\aleph-$Cauchy sequence.

For $m > n$ and $m, n \in \mathbf{N}$, we have

$$\aleph_\ell\left(\varpi_n, \varpi_m\right) \le \tau\aleph_{\frac{\ell}{2}}\left(\varpi_n, \varpi_{n+1}\right) + \tau^2\aleph_{\frac{\ell}{4}}\left(\varpi_{n+1}, \varpi_{n+2}\right) + \tau^3\aleph_{\frac{\ell}{8}}\left(\varpi_{n+2}, \varpi_{n+3}\right) + ...$$

So, without loss a generality, we obtain that

$$\begin{aligned}\aleph_\ell\left(\varpi_n, \varpi_m\right) &\le \tau\left(\tfrac{k}{\tau}\right)^n\aleph_\ell\left(\varpi_0, \varpi_1\right) + \tau^2\left(\tfrac{k}{\tau}\right)^{n+1}\aleph_\ell\left(\varpi_0, \varpi_1\right) + \tau^3\left(\tfrac{k}{\tau}\right)^{n+2}\aleph_\ell\left(\varpi_0, \varpi_1\right) + ... \\ &\le \tau\left(\tfrac{k}{\tau}\right)^n\aleph_\ell\left(\varpi_0, \varpi_1\right)\left[1 + k + k^2 + ...\right].\end{aligned}$$

Again, by taking limit as $n, m \to \infty$ in above, then we get that $\left\{\varpi_n\right\}$ is a $\aleph-$Cauchy sequence. Because $Q_\aleph$ is a $\aleph-$complete modular $b-$metric space, there exists $z \in Q_\aleph$ such that

$$\lim_{n\to\infty} \varpi_n = z. \tag{25}$$

Now, we shall prove that z is a fixed point of ξ, that is, $z = \xi z$. We suppose that $z \neq \xi z$. Then, from (22), we have

$$\begin{aligned}\tau\aleph_\ell\left(\varpi_n, \xi z\right) &= \tau\aleph_\ell\left(\xi\varpi_{n-1}, \xi z\right) \\ &\le k\left(\Sigma\left(\varpi_{n-1}, z\right)\right) \\ &= k\left[\aleph_\ell\left(\varpi_{n-1}, z\right) + \left|\aleph_\ell\left(\varpi_{n-1}, \xi\varpi_{n-1}\right) - \aleph_\ell(z, \xi z)\right|\right]\end{aligned}.$$

So, for $n \to \infty$, we get

$$\aleph_\ell(z, \xi z) \le \frac{k}{\tau}\aleph_\ell(z, \xi z)$$

which is a contradiction. Then, z is a fixed point of ξ.

Finally, for uniqueness, we suppose that υ is a another fixed point, that is, $\xi\upsilon = \upsilon$ such that $z \neq \upsilon$. Again, from 22, we obtain

$$\begin{aligned}\tau\aleph_\ell(z, \upsilon) = \tau\aleph_\ell(\xi z, \xi\upsilon) &\le k(\Sigma(z, \upsilon)) \\ &= k\left[\aleph_\ell(z, \upsilon) + \left|\aleph_\ell(z, \xi z) - \aleph_\ell(\upsilon, \xi\upsilon)\right|\right].\end{aligned}$$

So, we decide on that

$$\aleph_\ell(z, \upsilon) \le \frac{k}{\tau}\aleph_\ell(z, \upsilon),$$

is a contradiction. Then $z = \upsilon$ is a unique fixed point of ξ. □

Now, we have the main result in this section.

Theorem 5. Let $Q_\aleph$ be a $\aleph-$complete modular $b-$metric space with $\tau \geq 1$ and P, R be an open and closed subset of $Q_\aleph$, respectively. Let the operator $\mathcal{H} : R \times [0,1] \to Q_\aleph$ be satisfying the following conditions.

(a) $\varpi \neq \mathcal{H}(\varpi, \iota)$ for every $\varpi \in R \backslash P$ and $\iota \in [0,1)$.

(b) For all $\varpi, q \in R$ and $\iota, k \in [0,1)$, we have

$$\tau\aleph_{\ell}\left(\mathcal{H}\left(\varpi,\iota\right),\mathcal{H}\left(q,\iota\right)\right)\le k\left(\Sigma\left(\varpi,q\right)\right)$$

where,

$$\Sigma\left(\varpi,q\right)=\frac{1}{2\tau}\left[\aleph_{\ell}\left(\varpi,q\right)+\left|\aleph_{\ell}\left(\varpi,\mathcal{H}\left(\varpi,\iota\right)\right)-\aleph_{\ell}\left(q,\mathcal{H}\left(q,\iota\right)\right)\right|\right].$$

(c) There is a continuous function $\psi : [0,1] \to \mathbf{R}$ such that

$$\tau\aleph_{\ell}\left(\mathcal{H}\left(\varpi,\iota\right),\mathcal{H}\left(\varpi,\iota^{*}\right)\right)\le\left|\psi\left(\iota\right)-\psi\left(\iota^{*}\right)\right|$$

for all $\iota, \iota^{*} \in [0,1)$ and $\forall \varpi \in R$.

Then, $\mathcal{H}(\cdot,0)$ holds a fixed point $\Leftrightarrow$ $\mathcal{H}(\cdot,1)$ holds a fixed point.

Proof. Define the following set

$$\Lambda=\left\{\iota\in[0,1] : \varpi=\mathcal{H}\left(\varpi,\iota\right)\ \text{for some}\ \varpi\in P\right\}.$$

($\Rightarrow$:) Presume that $\mathcal{H}(\cdot,0)$ holds a fixed point. Then Λ is nonempty, that is, $0 \in \Lambda$. We will show that Λ is both open and closed in $[0,1]$ and hence, by connectedness, we have that $\Lambda = [0,1]$. As a result, $\mathcal{H}(\cdot,1)$ holds a fixed point in P.

We first show that Λ is closed in $[0,1]$. Let $\left\{\iota_n\right\}_{n=1}^{\infty} \subseteq \Lambda$ with $\iota_n \to \iota \in [0,1]$ as $n \to \infty$. t is necessary to show that $\iota \in \Lambda$. Because $\iota_n \in \Lambda$ for $n = 1,2,3,\ldots$, there exists $\varpi_n \in P$ with $\varpi_n = \mathcal{H}\left(\varpi_n,\iota_n\right)$. Also for $n,m \in \{1,2,3,\ldots\}$, we have

$$\begin{aligned}\aleph_{\ell}\left(\varpi_n,\varpi_m\right)&=\aleph_{\ell}\left(\mathcal{H}\left(\varpi_n,\iota_n\right),\mathcal{H}\left(\varpi_m,\iota_m\right)\right)\\&\le\tau\aleph_{\frac{\ell}{2}}\left(\mathcal{H}\left(\varpi_n,\iota_n\right),\mathcal{H}\left(\varpi_n,\iota_m\right)\right)+\tau\aleph_{\frac{\ell}{2}}\left(\mathcal{H}\left(\varpi_n,\iota_m\right),\mathcal{H}\left(\varpi_m,\iota_m\right)\right),\end{aligned}$$

where

$$\begin{aligned}\tau\aleph_{\ell}\left(\mathcal{H}\left(\varpi_n,\iota_m\right),\mathcal{H}\left(\varpi_m,\iota_m\right)\right)&\le k\left(\Sigma\left(\varpi_n,\varpi_m\right)\right)\\&=\tfrac{k}{2\tau}\left[\aleph_{\ell}\left(\varpi_n,\varpi_m\right)+\left|\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi_n,\iota_m\right)\right)-\aleph_{\ell}\left(\varpi_m,\mathcal{H}\left(\varpi_m,\iota_m\right)\right)\right|\right]\\&=\tfrac{k}{2\tau}\left[\aleph_{\ell}\left(\varpi_n,\varpi_m\right)+\aleph_{\ell}\left(\mathcal{H}\left(\varpi_n,\iota_n\right),\mathcal{H}\left(\varpi_n,\iota_m\right)\right)\right].\end{aligned}$$

So, we obtain that

$$\aleph_{\ell}\left(\varpi_n,\varpi_m\right)\le\left|\psi\left(\iota_n\right)-\psi\left(\iota_m\right)\right|+\tfrac{k}{2\tau}\left[\aleph_{\ell}\left(\varpi_n,\varpi_m\right)+\tfrac{\left|\psi(\iota_n)-\psi(\iota_m)\right|}{\tau}\right]$$

$$\aleph_{\ell}\left(\varpi_n,\varpi_m\right)\le\left(\tfrac{2\tau^2+k}{\tau(2\tau-k)}\right)\left|\psi\left(\iota_n\right)-\psi\left(\iota_m\right)\right|.$$

Then, if we use the convergence of $\left\{\iota_n\right\}_{n\in\mathbf{N}}$ with $n,m\to\infty$, we get $\lim_{n,m\to\infty}\aleph_{\ell}\left(\varpi_n,\varpi_m\right)=0$. It means $\left\{\varpi_n\right\}$ is $\aleph-$Cauchy sequence in $Q_{\aleph}$. As $Q_{\aleph}$ is $\aleph-$complete, there exists $\varpi^{*} \in R$ such that

$$\lim_{n\to\infty}\aleph_{\ell}\left(\varpi^{*},\varpi_n\right)=0.$$

Since

$$\begin{aligned}\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi^{*},\iota\right)\right)&=\aleph_{\ell}\left(\mathcal{H}\left(\varpi_n,\iota_n\right),\mathcal{H}\left(\varpi^{*},\iota\right)\right)\\&\le\tau\aleph_{\frac{\ell}{2}}\left(\mathcal{H}\left(\varpi_n,\iota_n\right),\mathcal{H}\left(\varpi_n,\iota\right)\right)+\tau\aleph_{\frac{\ell}{2}}\left(\mathcal{H}\left(\varpi_n,\iota\right),\mathcal{H}\left(\varpi^{*},\iota\right)\right),\end{aligned}$$

where

$$\begin{aligned}\tau\aleph_{\frac{\ell}{2}}\left(\mathcal{H}\left(\varpi_n,\iota\right),\mathcal{H}\left(\varpi^{*},\iota\right)\right)&\le k\Sigma\left(\varpi_n,\varpi^{*}\right)\\&=\tfrac{k}{2\tau}\left[\aleph_{\ell}\left(\varpi_n,\varpi^{*}\right)+\left|\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi_n,\iota\right)\right)-\aleph_{\ell}\left(\varpi^{*},\mathcal{H}\left(\varpi^{*},\iota\right)\right)\right|\right].\end{aligned}$$

Then, we get

$$\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi^{*},\iota\right)\right)\le\left|\psi\left(\iota_n\right)-\psi\left(\iota\right)\right|+\frac{k}{2\tau}\aleph_{\ell}\left(\varpi_n,\varpi^{*}\right).$$

Letting $n\to\infty$ the above, we obtain $\lim_{n\to\infty}\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi^{*},\iota\right)\right)=0$ and hence

$$\aleph_{\ell}\left(\varpi^{*},\mathcal{H}\left(\varpi^{*},\iota\right)\right)=\lim_{n\to\infty}\aleph_{\ell}\left(\varpi_n,\mathcal{H}\left(\varpi_n,\iota\right)\right)=0.$$

It implies that $\varpi^* = \mathcal{H}(\varpi^*, \iota)$. Since (a) is hold, we have $\varpi^* \in P$. Thus $\iota \in \Lambda$ and Λ is closed in $[0,1]$.

Next we show that Λ is open in $[0,1]$. Let $\iota_0 \in \Lambda$. Then there exists $\varpi_0 \in P$ with $\varpi_0 = \mathcal{H}(\varpi_0, \iota_0)$. Because P is open, then there exists $r > 0$ such that $B_{\aleph}(\varpi_0, r) \subseteq P$ in $Q_{\aleph}$. Considering $\varepsilon = \frac{r(2\tau - k)\tau}{2\tau^2 + k} > 0$ with $k \in [0,1)$ and $\tau \geq 1$, then there exists $\vartheta(\varepsilon) > 0$ such that $\left|\psi(\iota) - \psi(\iota_0)\right| < \varepsilon$ for all $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$ because ψ is continuous on ι_0.

Let $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$, for $p \in \overline{B_{\aleph}(\varpi_0, r)} = \{\varpi \in Q_{\aleph} : \aleph_{\ell}(\varpi, \varpi_0) \leq r\}$, we obtain

$$\begin{aligned}\aleph_{\ell}(\mathcal{H}(\varpi,\iota),\varpi_0) &= \aleph_{\ell}(\mathcal{H}(\varpi,\iota),\mathcal{H}(\varpi_0,\iota_0)) \\ &\leq \tau\aleph_{\frac{\ell}{2}}(\mathcal{H}(\varpi,\iota),\mathcal{H}(\varpi,\iota_0)) + \tau\aleph_{\frac{\ell}{2}}(\mathcal{H}(\varpi,\iota_0),\mathcal{H}(\varpi_0,\iota_0)),\end{aligned}$$

where

$$\begin{aligned}\tau\aleph_{\ell}(\mathcal{H}(\varpi,\iota_0),\mathcal{H}(\varpi_0,\iota_0)) &\leq k\Sigma(\varpi,\varpi_0) \\ &\leq \frac{k}{2\tau}\left[\aleph_{\ell}(\varpi,\varpi_0) + \left|\aleph_{\ell}(\varpi,\mathcal{H}(\varpi,\iota_0)) - \aleph_{\ell}(\varpi_0,\mathcal{H}(\varpi_0,\iota_0))\right|\right] \\ &= \frac{k}{2\tau}\left[\aleph_{\ell}(\varpi,\varpi_0) + \aleph_{\ell}(\mathcal{H}(\varpi,\iota),\mathcal{H}(\varpi,\iota_0))\right].\end{aligned}$$

Finally, we combine the above inequalities, we get

$$\begin{aligned}\aleph_{\ell}(\mathcal{H}(\varpi,\iota),\varpi_0) &\leq \left|\psi(\iota) - \psi(\iota_0)\right| + \frac{k}{2\tau}\left[\aleph_{\ell}(\varpi,\varpi_0) + \frac{\left|\psi(\iota)-\psi(\iota_0)\right|}{\tau}\right] \\ &= \left(1 + \frac{k}{2\tau^2}\right)\left|\psi(\iota) - \psi(\iota_0)\right| + \frac{k}{2\tau}\aleph_{\ell}(\varpi,\varpi_0) \\ &\leq \left(1 + \frac{k}{2\tau^2}\right)\varepsilon + \frac{k}{2\tau}r \\ &\leq r\end{aligned}$$

and $\mathcal{H}(\varpi, \iota) \in \overline{B_{\aleph}(\varpi_0, r)}$. Therefore

$$\mathcal{H}(\cdot,\iota) : \overline{B_{\aleph}(\varpi_0, r)} \to \overline{B_{\aleph}(\varpi_0, r)}$$

for every fixed $\iota \in (\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon))$. We can now apply to Corollary 6 to deduce that $\mathcal{H}(\cdot, \iota)$ holds fixed point in R. But it must be in P since (a) is true. So $(\iota_0 - \vartheta(\varepsilon), \iota_0 + \vartheta(\varepsilon)) \subseteq \Lambda$ and thus we conclude that Λ is open in $[0,1]$. □

5. CONCLUSIONS

Consequently, in this study we extended and improved the result of Karapinar and Fulga[14] in the setting of modular $b-$metric space by using Suzuki type contraction for four mappings. As well as, we proved that Banach contraction endowed with Type Σ holds in the same space and showed that it can be applied to homotopy theory. Also, we advanced the main results to graph structure. On the other hand, if we choose $\tau = 1$ in modular b-metric space, then the above results valid for modular metric space, too.

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