Taylor Series Solution of Some Real Life Problems: ODEs & PDEs

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Abstract

Recently He et al. [31] derived an analytical solution of the system of Lane-Emden equations by using the Taylor series method and computed a closed-form solution of the system of Lane-Emden equations subject to given initial conditions. In this work, this method is further explored and extended to a class of nonlinear ODEs, PDEs, a system of Nonlinear ODEs and PDEs subject to certain Initial conditions and boundary conditions. In some cases, we could find exact solutions and if that is not possible then we compute approximate solutions. We have compared these solutions with other existing techniques and showed that the method is simple and superior to other existing iterative techniques. We have also provided Mathematica codes which user may find useful and can compute solutions as per their need.

Keywords: Taylor Series; Singular Boundary Value Problem; KDV equation; Burgers' Equation; System of Burgers Equation;

AMS Subject Classification: 35K60; 34B16

1 Introduction

For many years researchers used various real-life examples as test problems like Lane-Emden type equations, Burger equations, KDV equation, etc. to verify numerical methods. Lane-Emden type equations arise in various physical phenomena that occur in astrophysics and mathematical physics like stellar structure, thermionic currents, thermal explosions, radiative cooling, CTC, etc. In this work, we focus on such models by considering the following equation

$$\left(x^{\beta} y'(x)\right)' + x^{\beta} f(x,y) = 0, \quad 0 < x < 1 \tag{1.1}$$

where $\beta \geq 0$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ subject to both initial and boundary conditions. The status of the theoretical and numerical work on Lane-Emden type equations are well known. Many authors such as Dunninger et. al. ([15, 14]), Zhang ([61, 62]), Pandey et. al. ([43, 44, 47, 48, 49]) etc. used equation (1.1) to develop theoretical results and Russell et. al. ([51, 52]), Chawla et. al. ([8, 9, 11, 12, 10]), Jain et. al. ([33]), Pandey ([41, 42]) applied finite difference technique to find the numerical solution. Apart from these techniques many authors applied different types of numerical methods like rational Legendre approximation technique ([34]), methods based on splines polynomials ([35, 6, 50, 38]), different types of collocation approaches ([4, 45, 60, 40, 37]), methods based on Legendre function ([59, 46, 7]), Haar wavelets and other orthonormal polynomial wavelets ([56, 57, 55]), NSFD method [58] etc.

We also focus on Burgers' equation which is highly nonlinear and one dimensional analogue of Navier Stokes equation. It has a long history (Bateman [3]) and huge number of articles are available on Burgers; equations, its various generalisations to various forms in one dimension, two dimension and as system of nonlinear PDEs. Since exact solution of Burgers's equation fails for small viscosity it has posed great challenges to researchers to find its analytical solution. Fay [18] gave its solution in a particular set up. Hopf [32] and Cole [13] computed the exact solutions by transforming the Burger's equation to heat equation. Group theoretic methods for calculating the solution of Burgers' equation with appropriate boundary and initial conditions is proposed by Abd-el-Malek [16]. We list some existing methods which has been used to compute the analytical solutions of Burgers' equation: Hopf and Cole transformation ([32, 13]), Group theoretic method ([16]), Adomian decomposition method ([19]), Variation iteration method ([5, 2, 22, 23]), Tanh-function method ([53, 17]), Variational Principle ([29, 26, 30, 24, 25]), Taylor series solution ([20, 27, 31, 28]).

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He ([31]) derived an analytical solution of a system of Lane-Emden equations by using the Taylor series method and computed closed form solution of a system of Lane-Emden equations subject to given initial conditions. After that, he applied this method on fractal Bratu-type equation ([20]) arising in the electrospinning process and third order boundary value problem ([21]) arising in Architectural Engineering to derive the approximate solution which gives better accuracy than other methods. A lot of investigations are still pending and to address some of these we consider singular BVP, KDV equation, and Burgers equations. Our main aim is to extend the numerical results of He [31] and explore it further. We present several Mathematica codes for this method corresponding to IVP, BVP, SBVP, coupled IVP, coupled SBVP to find the approximate and exact solutions with the best accuracy. We test each Mathematica code by considering different real-life problems of the form (1.1) and compare our results with existing numerical results. We also extend the numerical results to Burger's equations, KDV equation, and system of nonlinear PDEs corresponding to initial conditions.

We have summarised the paper in a total of six sections. In section 2, we describe the Taylor series method. We have listed Mathematica codes in section 3. Several test examples are presented in section 4. We have derived exact solutions of Burgers' equations, KDV equation, and system of nonlinear 2 Dimensional Burgers' equation in section 5. Finally, we draw our main conclusion along with future scope in section 6.

2 Description of the method

Let us assume that the solution $y(x)$ of equation (1.1) is n times differentiable at $x = 0$ and can be written as in the following Taylor series expansion

$$y^{Taylor}(x) \approx \sum_{i=0}^{n} \frac{y^{(i)}(0)}{i!} x^i, \tag{2.1}$$

where $y^{(i)}(0)$ are unknown coefficients which are to be determined. Now, differentiating equation (1.1) n times with respect to x, we have

$$\frac{d^i}{dx^i}\Big(\big(x^\beta y'(x)\big)' + x^\beta f(x,y)\Big) = 0, \text{ for } i = 0, 1, \cdots, n \tag{2.2}$$

and setting $x = 0$. Therefore, by using initial conditions and equations (2.2) we can easily determine the unknown constants $y^{(i)}(0)$ for $i = 0, 1, \cdots, n$. Finally, the exact solution of equation (1.1) can be written as

$$y(x) = \lim_{n\to\infty} y^{Taylor}(x). \tag{2.3}$$

For better understanding, we consider the simple linear IVP

$$y' = \frac{1}{1+x}, \; y(0) = 0. \tag{2.4}$$

Differentiating equation (2.4) with respect to x three time, we have

$$y'' = -\frac{1}{(1+x)^2}, \; y''' = \frac{2}{(1+x)^3} \;\&\; y'''' = -\frac{6}{(1+x)^4}. \tag{2.5}$$

Therefore, by setting $x = 0$, we have

$$y'(0) = 1, \; y''(0) = -1, \; y'''(0) = 2 \;\&\; y''''(0) = -6. \tag{2.6}$$

So, from equation (2.1) we have the first order, second order, third order and fourth order approximation are as follows

$$y^{Taylor}(x) \approx x, \; x - \frac{x^2}{2}, \; x - \frac{x^2}{2} + \frac{x^3}{3} \;\&\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}. \tag{2.7}$$

By similar analysis, for $n \to \infty$ we have the exact solution, which is

$$y(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \log(1+x). \tag{2.8}$$

We provide an algorithm of the Taylor series method which is used to develop code for different real life problems to compute the solution.

2.1 Algorithm

Step 1. Fix the value of n (number of terms of the Taylor series).
Step 2. Input the differential equation (1.1) and corresponding initial conditions.
Step 3. Differentiate equation (1.1) with respect to x upto n times and put $x = 0$.
Step 4. Identify the unknown constants $y^{(i)}(0)$ for $i = 0, 1, \cdots, n$ and solve the system of equations as in Step 3.
Step 5. Substitute all the values of $y^{(i)}(0)$ for $i = 0, 1, \cdots, n$ in equation (2.1) to get the solution.

3 Mathematica Codes

By using the algorithm 2.1 and Mathematica 11.3 software, we develop codes for this method corresponding to second-order IVP, BVP, coupled IVP, and coupled BVP to find the approximate and exact solutions with the best accuracy.

3.1 IVP

We consider the following initial value problem

$$y'' + \frac{\alpha}{x}y' + g(x, y) = 0, \ 0 < x < 1 \tag{3.1}$$

$$y(0) = c \ \& \ y'(0) = d, \tag{3.2}$$

where c, d are constants and $g(x, y)$ be arbitrary function of x and y. Below we present a Mathematica code for (3.1) and (3.2) which gives Taylor series solution upto n^{th} terms.

```
n = 3; (*Order of the  Taylor series solution*)
α = 2;
f = x (y''[x]) + α y'[x] + x (g (x, y[x])); (*Differential equation of the form (1.1)*)
y[0] = c; (*Where c is constant/Initial condition*)
y'[0] = d; (*Where d is constant/Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
A // MatrixForm
Derivativevalue = Table[D[y[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevalueatorigin = Flatten[Solve[A, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[y[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution of equation (1.1)*)
```

3.2 BVP

We consider the equation (1.1) subject to the boundary condition in the following form

$$y'' + \frac{\alpha}{x}y' + g(x, y) = 0, \ 0 < x < 1, \tag{3.3}$$

$$y'(0) = 0 \ \& \ y(1) = d, \tag{3.4}$$

where d are constants and $g(x, y)$ be arbitrary function of x and y. Since the value of $y(0)$ is not known, therefore we take $y(0) = c$. Again, we provide a Mathematica code for (3.3) and (3.4) which gives Taylor series solution upto n^{th} terms as a function of c and x. The value of c can be determined by using the boundary condition $y(1) = d$.

```
n = 2; (*Order of the Taylor series solution*)
α = 3;
f = x (y''[x]) + α (y'[x]) + x (g (x, y[x])); (*DIfferential equation of the form (1.1)*)
y[0] = α; (*Where α is constant which is to be determined/assumption*)
y'[0] = 0; (*Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
A // MatrixForm
Derivativevalue = Table[D[y[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevalueatorigin = Flatten[Solve[A, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[y[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution y(x)*)
```

3.3 Coupled IVP

We consider the coupled system of equation with the help of (1.1) subject to the initial conditions in the following form

$$u'' + \frac{\alpha}{x}u' + h_1(x,u,v) = 0,\ 0 < x < 1, \tag{3.5}$$

$$v'' + \frac{\beta}{x}v' + h_2(x,u,v) = 0,\ 0 < x < 1, \tag{3.6}$$

$$u(0) = c_1,\ v(0) = c_2,\ u'(0) = d_1\ \&\ v'(0) = d_2, \tag{3.7}$$

where c_1, c_2, d_1, d_2 are constants and $h_1(x,u,v)$, $h_2(x,u,v)$ are arbitrary functions of x, u and v. Now, we present a Mathematica code for (3.5), (3.6) and (3.7) which gives Taylor series system of solutions upto n^{th} terms as a function of x.

```
n = 2;
α = 8;
β = 4;
f = x (u''[x]) + α ( u'[x]) + x (h1 (x, u[x], v[x])); (*Differential equation of the form (1.1)*)
g = x (v''[x]) + β (v'[x]) + x (h2 (x, u[x], v[x])); (*Differential equation of the form (1.1)*)
u[0] = c1; (*Where c1 is constant/Initial condition*)
u'[0] = d1; (*Where d 1is constant/Initial condition*)
v[0] = c2; (*Where  c2 is constant/Initial condition*)
v'[0] = d2; (*Where d2 is constant/Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
For[i = 0, i ≤ n, i++, Print[Gi = D[g, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
B = Table[Gi == 0, {i, n}] /. x → 0
B // MatrixForm
A // MatrixForm
Derivativevalueu = Table[D[u[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevaluev = Table[D[v[x], {x, i + 1}], {i, n}] /. x → 0
Differentialequation = Flatten[{A, B}]
Derivativevalue = Flatten[{Derivativevalueu, Derivativevaluev}]
Derivativevalueatorigin = Flatten[Solve[Differentialequation, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[u[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution u(x)*)
Sum[(D[v[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Final Taylor series solution v(x)*)
```

3.4 Coupled BVP

Here, we consider the following coupled system of equations subject to the boundary conditions in the following form

$$u'' + \frac{\alpha}{x}u' + h_1(x,u,v) = 0,\ 0 < x < 1, \tag{3.8}$$

$$v'' + \frac{\beta}{x}v' + h_2(x,u,v) = 0,\ 0 < x < 1, \tag{3.9}$$

$$u'(0) = 0\ \&\ v'(0) = 0,\ u(1) = c_1\ \&\ v(1) = c_2, \tag{3.10}$$

where c_1, c_2 are constants and $h_1(x,u,v)$, $h_2(x,u,v)$ are arbitrary functions of x, u and v. Since the value of $u(0)$ and $v(0)$ are unknown, so we chose $u(0) = a$ and $v(0) = b$. In the following, we provide a Mathematica code for (3.8), (3.9) and (3.10) which gives Taylor series system of solutions upto n^{th} terms as a function of x, a and b.

```
n = 2;
α = 8;
β = 4;
f = x (u''[x]) + α ( u'[x]) +  x (h1 (x, u[x], v[x])); (*Differential equation 1 of the form (1.1)*)
g = x (v''[x]) + β (v'[x]) + x (h2 (x, u[x], v[x])); (*Differential equation 2 of the form (1.1)*)
u[0] = a; (*Where a is constant which is to be determined/Assumptions*)
u'[0] = 0; (*Initial condition*)
v[0] = b; (*Where b is constant which is to be determined/Assumptions*)
v'[0] = 0; (*Initial condition*)
For[i = 0, i ≤ n, i++, Print[Fi = D[f, {x, i}]]]
For[i = 0, i ≤ n, i++, Print[Gi = D[g, {x, i}]]]
A = Table[Fi == 0, {i, n}] /. x → 0
B = Table[Gi == 0, {i, n}] /. x → 0
B // MatrixForm
A // MatrixForm
Derivativevalueu = Table[D[u[x], {x, i + 1}], {i, n}] /. x → 0
Derivativevaluev = Table[D[v[x], {x, i + 1}], {i, n}] /. x → 0
Differentialequation = Flatten[{A, B}]
Derivativevalue = Flatten[{Derivativevalueu, Derivativevaluev}]
Derivativevalueatorigin = Flatten[Solve[Differentialequation, Derivativevalue]]
Derivativevalueatorigin // MatrixForm
Derivativefinaloutput = Table[Derivativevalue[[i]] /. Derivativevalueatorigin[[i]], {i, 1, n}]
Sum[(D[u[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Taylor series solution u[x]*)
Sum[(D[v[x], {x, k}] /. x → 0)/k! x^k, {k, 0, n}] /. Derivativevalueatorigin (*Taylor series solution v[x]*)
```

4 Taylor Series Solution for ODE

Here, we present few real life problems as test examples to verify our code.

4.1 IVP

We consider some initial value problems.

4.1.1 IVP 1

We consider equations (3.1) and (3.2) with $g(x,y) = y(x) - (6 + 12x + x^2 + x^3)$, $c = 0$, $d = 0$ and $\alpha = 2$, which have an exact solution $x^2 + x^3$. By using the Mathematica code as in subsection 3.1, we get the Taylor series solution upto third terms which is $y^{Taylor}(x) \approx x^2 + x^3$. The accuracy of the method is better than variational iteration method (VIM) [1] [See figure 1].

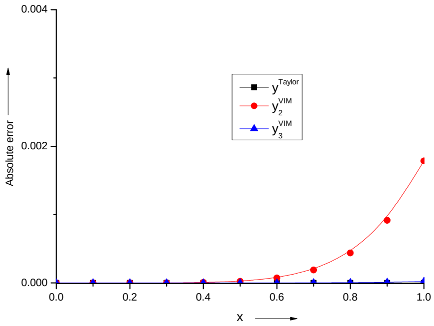

Figure 1: Comparison between Taylor series method and VIM for equations (3.1) and (3.2) with $g(x,y) = y(x) - (6 + 12x + x^2 + x^3)$, $c = 0$, $d = 0$ and $\alpha = 2$

4.1.2 IVP 2

Here we take Lane-Emden type equations (3.1) and (3.2), where $g(x,y) = (8e^y + 4e^{\frac{y}{2}})$, $c = 0$, $d = 0$ and $\alpha = 2$, which have an exact solution $-2\log(1+x^2)$. With the help of Mathematica code given in subsection 3.1, we compute the Taylor series solution which is given as follows

$$y^{Taylor} \approx -2x^2, \ -2x^2 + x^4, \ -2x^2 + x^4 - \frac{2x^6}{3}, \cdots \tag{4.1}$$

For $n \to \infty$, we get the closed form of the solution which is

$$y(x) = y^{Taylor} = -2(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots) = -2\log(1+x^2). \tag{4.2}$$

We can also verify our derived result by VIM ([1]).

4.1.3 IVP 3

We consider Lane-Emden type equations (3.1) and (3.2, where $g(x,y) = -6y(x) - 4y\log(y)$, $c = 1$, $d = 0$ and $\alpha = 2$ with an exact solution e^{x^2}. Now, by using Mathematica code given in subsection 3.1, we arrive at

$$y^{Taylor} \approx 1 + x^2, \ 1 + x^2 + \frac{x^4}{2}, \ 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}, \cdots \tag{4.3}$$

As $n \to \infty$, we get the closed form of the solution which is e^{x^2}. Also, we see that each of these approximations are same as approximations calculated by VIM ([1]).

4.2 BVP

He ([20]) applied the Taylor series method on the Bratu type boundary value problem which is nonlinear but regular and computed the approximate solution. He also showed that this method is quite powerful than the iterative method VIM. Here, we consider a few highly non-linear singular boundary value problems of Lane-Emden type and verify the Mathematica codes given in subsection 3.2.

4.2.1 BVP 1

First we consider linear second order singular boundary value problem (3.3) and (3.4) with $g(x,y) = y(x) - \frac{5}{4} + \frac{x^2}{16}$, $d = \frac{17}{16}$ and $\alpha = 1$. The exact solution of this linear SBVP is $1 + \frac{x^2}{16}$. Now, by using the code and Mathematica 11.3 software, we have

$$y^{Taylor} \approx c + \frac{1}{16}(5 - 4c)x^2, \tag{4.4}$$

where $c = y(0)$. By using the boundary condition $y(1) = \frac{17}{16}$ we have $c = 1$. Hence Taylor series solution is

$$y^{Taylor} \approx 1 + \frac{1}{16}x^2 \tag{4.5}$$

which is equivalent to the exact solution. Also, we see that the Taylor series solution gives better accuracy than VIM approximation. Below we provide an absolute error graph (fig. 2) comparison between these two methods.

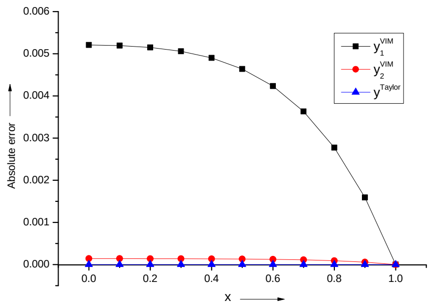

Figure 2: Comparison between absolute error of Taylor series method and VIM for SBVP (3.3) and (3.4) with $g(x,y) = y(x) - \frac{5}{4} + \frac{x^2}{16}$, $d = \frac{17}{16}$ and $\alpha = 1$

4.2.2 BVP 2

Now, we consider a model based on equilibrium isothermal gas sphere arising in astronomy of the form (3.3) and (3.4) where $g(x,y) = y^5$, $d = \frac{\sqrt{3}}{2}$ and $\alpha = 2$. The exact solution of BVP is $\frac{1}{\sqrt{1+\frac{x^2}{3}}}$. By using the Mathematica code given in subsection 3.2, we get second order and 12^{th} order Taylor series approximations, given as follows

$$y^{Taylor} \approx c - \frac{c^5x^2}{6}, \tag{4.6}$$

$$y^{Taylor} \approx c + \frac{77c^{25}x^{12}}{248832} - \frac{7c^{21}x^{10}}{6912} + \frac{35c^{17}x^8}{10368} - \frac{5c^{13}x^6}{432} + \frac{c^9x^4}{24} - \frac{c^5x^2}{6}. \tag{4.7}$$

Now, by using the boundary condition $y(1) = \frac{\sqrt{3}}{2}$ we have $c = 0.999832$. Therefore, for $n = 12$ the Taylor series approximation is

$$y^{Taylor} \approx 0.000308x^{12} - 0.001009x^{10} + 0.003366x^8 - 0.0115489x^6 + 0.04160x^4 - 0.166527x^2 + 0.999832. \tag{4.8}$$

Now, we compare our solution (4.8) with the solution computed by VIM which is

$$y_2^{VIM} \approx 7 \times 10^{-10}x^{12} - 0.000030x^{10} + 0.000577x^8 - 0.00609035x^6 + 0.039355x^4 - 0.161465x^2 + 0.993678. \tag{4.9}$$

From figure 3 we observe that Taylor series solution gives better accuracy than VIM solution ([36]).

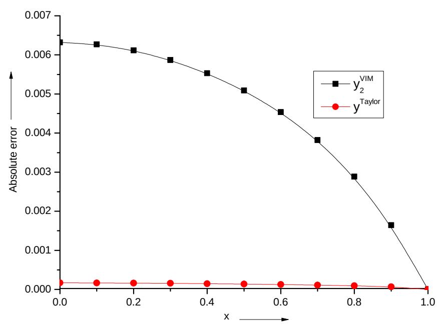

Figure 3: Comparison between absolute error of Taylor series method and VIM for SBVP (3.3) and (3.4) where $g(x,y) = y^5$, $d = \frac{\sqrt{3}}{2}$ and $\alpha = 2$

4.2.3 BVP 3

Here, we take the equation of shallow membrane cap of the form (3.3) and (3.4) where $g(x,y) = -\frac{1}{2} + \frac{1}{8y^2}$, $d = 1$ and $\alpha = 3$. It has no exact solution. Again, by using the Mathematica code given in subsection 3.2, we get Taylor series approximation upto second term as follows

$$y^{Taylor} \approx c + \frac{\left(4c^2 - 1\right) x^2}{64c^2} \tag{4.10}$$

where $c = y(0)$. By using the boundary condition $y(1) = 1$, we have $c = 0.954645$. Hence, the Taylor series approximation is

$$y^{Taylor} \approx 0.045355x^2 + 0.954645. \tag{4.11}$$

We compare our computed approximation with the VIM ([36]) approximation

$$y_1^{VIM} \approx 0.04834939252005x^2 + 0.95165060747995. \tag{4.12}$$

We have seen that Taylor series solution gives better approximation than VIM solution [See figure 4].

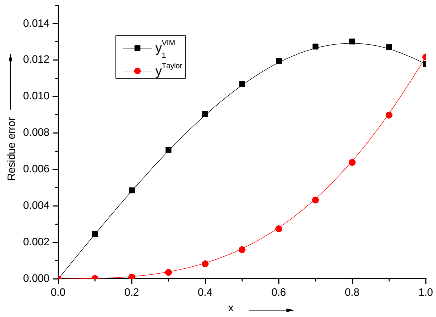

Figure 4: Comparison between absolute error of Taylor series method and VIM (3.3) and (3.4) where $g(x,y) = -\frac{1}{2} + \frac{1}{8y^2}$, $d = 1$ and $\alpha = 3$

4.3 Coupled IVP

He et al. ([31]) computed closed form solution of coupled IVP by using Taylor series method. Our main aim in this section is to verify our developed Mathematica code as in subsection 3.3 by considering different highly non-linear real life problems. To achieve our goal first we take a real life example which is described in [31]. We also found closed form solution as given in [31]. Now, we consider two real life problems.

4.3.1 Coupled IVP 1

Wazwaz et al. [39] consider the system of Lane-Emden type equations of the form (3.5), (3.7) and (3.6) with $h_1(x,u,v) = 8\left(e^u + 2e^{-\frac{v}{2}}\right)$, $h_2(x,u,v) = -8\left(e^{-v} + e^{\frac{u}{2}}\right)$, $\alpha = 5$, $\beta = 3$, $c_1 = 0$, $c_2 = 0$, $d_1 = 0$ and $d_2 = 0$. Exact solution of this coupled IVP is $(u(x), v(x)) = (-2\log(1+x^2), 2\log(1+x^2))$. They used Adomian's decomposition method (ADM) to find the approximate system of solutions. Now, by using Taylor series method, we arrive at

$$u^{Taylor} \approx \frac{x^{12}}{3} - \frac{2x^{10}}{5} + \frac{x^8}{2} - \frac{2x^6}{3} + x^4 - 2x^2, \tag{4.13}$$

$$v^{Taylor} \approx -\frac{x^{12}}{3} + \frac{2x^{10}}{5} - \frac{x^8}{2} + \frac{2x^6}{3} - x^4 + 2x^2. \tag{4.14}$$

Therefore, for $n \to \infty$ we get the closed form system of solutions $(u(x), v(x)) = (-2\log(1+x^2), 2\log(1+x^2))$ which are same as computed by ADM.

4.3.2 Coupled IVP 2

Here, we consider the system of equation ([39]) (3.5), (3.7) and (3.6) with $h_1(x,u,v) = (18u - 4u\log(v))$, $h_2(x,u,v) = (4v\log(u) - 10v)$, $\alpha = 8$, $\beta = 4$, $c_1 = 1$, $c_2 = 1$, $d_1 = 0$ and $d_2 = 0$. Exact solution of this coupled IVP is $(u(x), v(x)) = (e^{-x^2}, e^{x^2})$. With the help of mathematica code we get Taylor series solution as follows

$$u^{Taylor} \approx \frac{x^{12}}{720} - \frac{x^{10}}{120} + \frac{x^8}{24} - \frac{x^6}{6} + \frac{x^4}{2} - x^2 + 1, \tag{4.15}$$

$$v^{Taylor} \approx \frac{x^{12}}{720} + \frac{x^{10}}{120} + \frac{x^8}{24} + \frac{x^6}{6} + \frac{x^4}{2} + x^2 + 1. \tag{4.16}$$

When $n \to \infty$, we have the exact system of solutions (e^{-x^2}, e^{x^2}).

4.4 Coupled BVP

4.4.1 Coupled BVP 1

Consider the system of differential equation (3.8), (3.9) and (3.10) where $h_1(x,u,v) = (18u - 4u\log(v))$, $h_2(x,u,v) = (4v\log(u) - 10v)$, $\alpha = 8$, $\beta = 4$, $c_1 = \frac{1}{e}$ and $c_2 = e$. Exact solution of this coupled BVP is $(u(x), v(x)) = (e^{-x^2}, e^{x^2})$. Now using algorithm described in subsection 3.4 we compute the approximate Taylor series solution. For $n = 5$, we have

$$u^{Taylor} \approx -\frac{x^4}{990}\left(-20a\log^2(b) + 180a\log(b) - 495a + 36a\log(a)\right) + \frac{1}{9}x^2(2a\log(b) - 9a) + a, \tag{4.17}$$

$$v^{Taylor} \approx \frac{1}{630}x^4\left(36b\log^2(a) - 180b\log(a) + 315b - 20b\log(b)\right) - \frac{1}{5}x^2(2b\log(a) - 5b) + b, \tag{4.18}$$

where $a = u(0)$ and $b = v(0)$. For $n = 10$, we have computed the values of a and b by using boundary conditions $u(1) = \frac{1}{e}$ and $v(1) = e$ which are

$$a = 1.00299 \ \& \ b = 1.00156. \tag{4.19}$$

Therefore, for $n = 10$ the system of Taylor series solutions are

$$u^{Taylor} \approx -0.00833149x^{10} + 0.0417031x^8 - 0.166939x^6 + 0.5011x^4 - 1.00264x^2 + 1.00299, \tag{4.20}$$

$$v^{Taylor} \approx 1.00156 + 1.00037x^2 + 0.499878x^4 + 0.166539x^6 + 0.0416179x^8 + 0.00832063x^{10}. \tag{4.21}$$

Here in figure 5 we plot absolute error of Taylor series solution (4.20) and (4.21).

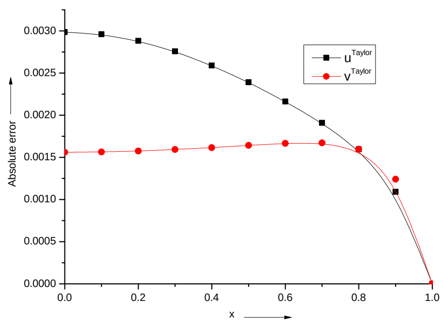

Figure 5: Absolute error of Taylor series method

4.4.2 Coupled BVP 2

We consider the system of differential equation (3.8), (3.9) and (3.10) with $h_1(x,u,v) = 8\left(e^{u-1} + 2e^{-\frac{v-1}{2}}\right)$, $h_2(x,u,v) = -8\left(e^{-(v-1)} + e^{\frac{u-1}{2}}\right)$, $\alpha = 5$, $\beta = 3$, $c_1 = 1 - 2\log(2)$ and $c_2 = 1 + 2\log(2)$. Exact solution of this coupled BVP is $(u(x), v(x)) = (1 - 2\log(1 + x^2), 1 + 2\log(1 + x^2))$. Now, using algorithm developed in subsection 3.4, for $n = 5$, we have the following Taylor series approximation

$$u^{Taylor} \approx \frac{1}{12}x^4\left(3e^{\frac{a}{2}-\frac{b}{2}} + 4e^{a-\frac{b}{2}-\frac{1}{2}} + 2e^{2a-2} + 3e^{\frac{3}{2}-\frac{3b}{2}}\right) + \frac{2}{3}x^2\left(e^{a-1} + 2e^{\frac{1}{2}-\frac{b}{2}}\right) + a, \tag{4.22}$$

$$v^{Taylor} \approx \frac{1}{9}x^4\left(-3e^{\frac{a}{2}-b+\frac{1}{2}} - 2e^{\frac{a}{2}-\frac{b}{2}} - e^{\frac{3a}{2}-\frac{3}{2}} - 3e^{2-2b}\right) - e^{-b-\frac{1}{2}}x^2\left(e^{\frac{a}{2}+b} + e^{3/2}\right) + b, \tag{4.23}$$

where $a = u(0)$ and $b = v(0)$. Now from $u(1) = 1-2\log(2)$ and $v(1) = 1+2\log(2)$ we have two nonlinear system of equations. Therefore, by using Newton Raphson method, we get the values of a and b which are given by

$$a = 1 \ \& \ b = 1. \tag{4.24}$$

Hence, for $n = 5$ Taylor series solution of this coupled system are

$$u^{Taylor} \approx x^4 - 2x^2 + 1, \tag{4.25}$$

$$v^{Taylor} \approx -x^4 + 2x^2 + 1, \tag{4.26}$$

and for $n \to \infty$ we get exact system of solutions.

5 Taylor Series Solution for PDE

In this section we derive analytical solution for three sets of problems. First is two Nonlinear PDEs with initial condition and the second one is system of nonlinear PDEs subject to given initial conditions.

5.1 Burgers' Equations

We consider the following class of Burgers' equations

$$\text{PDE:} \qquad u_t + uu_x = \nu u_{xx}, \ 0 \le x \le 1, \ t > 0, \tag{5.1}$$

$$\text{Initial Condition:} \qquad u(x, 0) = 2x, \tag{5.2}$$

$$\text{Exact Solution ([5]):} \qquad u(x, t) = \frac{2x}{1 + 2t}. \tag{5.3}$$

Using (5.1) and (5.2) we get

$$u(0, 0) = 0, \ u_x(0, 0) = 2, \ u_{xx}(0, 0) = 0, \ u_t(0, 0) = 0. \tag{5.4}$$

Differentiating (5.1) with respect to x and t, respectively, we get

$$u_{xt} + (u_x)^2 + uu_{xx} = \nu u_{xxx}, \tag{5.5}$$

$$u_{tt} + u_t u_x + uu_{tx} = \nu u_{txx}. \tag{5.6}$$

Differentiating (5.5) with respect to x and t, and (5.6) with respect to t, we get the following three equations,

$$u_{xxt} + 2u_x u_{xx} + u_x u_{xx} + uu_{xxx} = \nu u_{xxxx}, \tag{5.7}$$

$$u_{xtt} + 2u_x u_{tx} + u_t u_{xx} + uu_{txx} = \nu u_{txxx}, \tag{5.8}$$

$$u_{ttt} + u_{tt} u_x + u_t u_{tx} + u_t u_{tx} + uu_{ttx} = \nu u_{ttxx}. \tag{5.9}$$

Differentiating (5.7) with respect to x, we get

$$u_{xxxt} + 2u_x u_{xxx} + 3u_{xx}^2 + 2u_x u_{xxx} + uu_{xxxx} = \nu u_{xxxxx}. \tag{5.10}$$

Therefore we get

$$u_{xt}(0, 0) = -4, \ u_{tt}(0, 0) = 0, \ u_{xxx}(0, 0) = 0, \ u_{xxt}(0, 0) = 0, \ u_{xtt}(0, 0) = 16, \ u_{ttt}(0, 0) = 0. \tag{5.11}$$

Taylor series expansion of $u(x, t)$ around the point $(0, 0)$ can be written as

$$\begin{aligned} u(x, t) = u(0, 0) &+ \frac{1}{1!}\left(u_x(0, 0)x + u_t(0, 0)t\right) + \frac{1}{2!}\left(u_{xx}(0, 0)x^2 + 2u_{xt}(0, 0)xt + u_{tt}(0, 0)t^2\right) \\ &+ \frac{1}{3!}\left(3x^2 t u_{xxt}(0, 0) + 3xt^2 u_{xtt}(0, 0) + x^3 u_{xxx}(0, 0) + t^3 u_{ttt}(0, 0)\right) + \cdots . \end{aligned} \tag{5.12}$$

Substituting the values of $u(0,0)$, $u_x(0,0)$ and all other values from (5.4) and (5.11), in the (5.12) we get

$$u(x,t) = 0 + 2x + 0 \times t + \frac{1}{2}\left(0 \times x^2 + 2(-4)xt + 0 \times t^2\right) + \frac{1}{6}\left(3x^2 t \times 0 + 3xt^2 \times 16 + x^3 \times 0 + t^3 \times 0\right) + \cdots \quad (5.13)$$

Hence, we get

$$u(x,t) = 2x - 4xt + 8xt^2 + \cdots = 2x\left(1 - 2t + 4t^2 - \cdots\right) = \frac{2x}{1+2t}. \quad (5.14)$$

Which is same as computed in [5, 19].

5.2 KDV equation

We consider KDV equation

$$u_t - 6uu_x + u_{xxx} = 0, \; 0 < x < 1, \; t > 0, \quad (5.15)$$

$$u(x,0) = -\frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right). \quad (5.16)$$

The exact solution of KDV equation is $-\frac{k^2}{2}\operatorname{sech}^2\left(\frac{k}{2}(x - k^2 t)\right)$.

By using equations (5.15) and (5.16), we have

$$u(0,0) = -\frac{k^2}{2}, \; u_x(0,0) = 0, \; u_{xx}(0,0) = \frac{k^4}{4}, \; u_{xxx}(0,0) = 0 \;\&\; u_t(0,0) = 0. \quad (5.17)$$

Differentiating equation (5.15) with respect to x and t we have

$$u_{xt} - 6uu_{xx} - 6(u_x)^2 + u_{xxxx} = 0, \quad (5.18)$$

$$u_{tt} - 6u_t u_x - 6uu_{tx} + u_{txxx} = 0. \quad (5.19)$$

To find the value of u_{txxx}, we differentiate equation (5.18) with respect to x and we get

$$u_{xxt} - 6u_x u_{xx} - 6uu_{xxx} - 12u_x u_{xx} + u_{xxxxx} = 0, \quad (5.20)$$

$$u_{xxxt} - 18(u_{xx})^2 - 24u_x u_{xxx} - 6uu_{xxxx} + u_{xxxxxx} = 0. \quad (5.21)$$

By using eqautions (5.19), (5.20) and (5.21), we have

$$u_{xxxx}(0,0) = -\frac{k^6}{2}, \; u_{xxxxx}(0,0) = 0, \; u_{xxxxxx}(0,0) = \frac{17k^8}{8}, \; u_{xt}(0,0) = -\frac{k^6}{4} \;\&\; u_{tt}(0,0) = \frac{k^8}{4}. \quad (5.22)$$

Therefore, first order Taylor series solution is

$$u^{Taylor}(x,t) \approx -\frac{k^2}{2}\left(1 - \left(\frac{k}{2}(x - k^2 t)\right)^2\right). \quad (5.23)$$

The KDV equation (5.15) and (5.16) have numerically solved by VIM in [54]. First approximation of VIM solution is given by

$$u_1^{VIM}(x,t) \approx -\frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right) - \frac{k^5}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right)\tanh^2\left(\frac{kx}{2}\right)t. \quad (5.24)$$

From figure 6 we see that Taylor series solution provide better accuracy than VIM approximation.

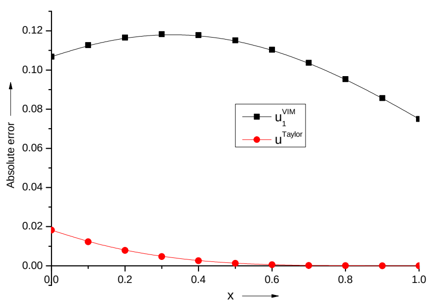

Figure 6: Absolute error of $u(x,t)$ for $t = 1$ for KDV equation.

5.3 System of Nonlinear PDEs

We consider the following system of nonlinear PDEs with given initial conditions

$$u_t + uu_x + vu_y = \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \ 0 \le x \le 1, \ t > 0, \tag{5.25}$$

$$v_t + uv_x + vv_y = \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \ 0 \le x \le 1, \ t > 0, \tag{5.26}$$

$$\text{Initial Conditions:} \quad u(x,y,0) = x + y, \quad v(x,y,0) = x - y, \tag{5.27}$$

$$\text{Exact Solution ([5]):} \quad u(x,y,t) = \frac{x + y - 2xt}{1 - 2t^2}, \tag{5.28}$$

$$v(x,y,t) = \frac{x - y - 2yt}{1 - 2t^2}. \tag{5.29}$$

Here R_e is known as Reynold's number which is related to viscous property of fluid.

We deduce the following:

$$u_x(x,y,0) = 1; \ u_y(x,y,0) = 1; \ u_{xx}(x,y,0) = 0; \ u_{yy}(x,y,0) = 0, \tag{5.30}$$

$$v_x(x,y,0) = 1; \ v_y(x,y,0) = -1; \ v_{xx}(x,y,0) = 0; \ v_{yy}(x,y,0) = 0, \tag{5.31}$$

$$u(0,0,0) = 0; \ v(0,0,0) = 0; \ u_t(0,0,0) = 0. \tag{5.32}$$

Differentiating (5.25) with respect to x, t and x twice we arrive at

$$u_{xt} = -u_x^2 - uu_{xx} - v_x u_y - vu_{xy} + \frac{1}{Re}\left(u_{xxx} + u_{xyy}\right), \tag{5.33}$$

$$u_{tt} = -u_t u_x - uu_{tx} - v_t u_y - vu_{ty} + \frac{1}{Re}\left(u_{txx} + u_{tyy}\right), \tag{5.34}$$

$$u_{yt} = -u_y u_x - uu_{yx} - v_y u_y - vu_{yy} + \frac{1}{Re}\left(u_{yxx} + u_{yyy}\right), \tag{5.35}$$

$$u_{xxt} = -2u_x u_{xx} - u_x u_{xx} - uu_{xxx} - v_{xx} u_y - v_x u_{xxy} + \frac{1}{Re}\left(u_{xxxx} + u_{xxyy}\right), \tag{5.36}$$

$$u_{yyt} = -u_{yy} u_x - u_y u_{yx} - u_y u_{yx} - uu_{yyx} - v_{yy} u_y - v_y u_{yy} - v_y u_{yy} - vu_{yyy} + \frac{1}{Re}\left(u_{yyxx} + u_{yyyy}\right), \tag{5.37}$$

$$u_{xtt} = -2u_x u_{tx} - u_t u_{xx} - uu_{txx} - v_{tx} u_y - v_x u_{ty} - v_t u_{xy} - vu_{txy} + \frac{1}{Re}\left(u_{txxx} + u_{txyy}\right), \tag{5.38}$$

$$u_{xyt} = -u_{xy} u_x - u_y u_{xx} - u_x u_{yx} - uu_{xyx} - v_{xy} u_y - v_y u_{xy} - v_x u_{yy} - vu_{xyy} + \frac{1}{Re}\left(u_{xyxx} + u_{xyyy}\right), \tag{5.39}$$

$$u_{ytt} = -u_{yt} u_x - u_t u_{yx} - u_y u_{tx} - uu_{ytx} - v_{yt} u_y - v_t u_{yy} - v_y u_{ty} - vu_{yty} + \frac{1}{Re}\left(u_{ytxx} + u_{ytyy}\right). \tag{5.40}$$

$$\tag{5.41}$$

Hence, we deduce the following:

$$u_{xt}(0,0,0) = -2, \ u_{tt}(0,0,0) = 0, \ u_{yt}(0,0,0) = 0, \tag{5.42}$$

$$u_{xxt}(0,0,0) = 0, \ u_{yyt}(0,0,0) = 0, \tag{5.43}$$

$$u_{xyt}(0,0,0) = 0, \ u_{xtt}(0,0,0) = 4, \ u_{ytt}(0,0,0) = 4, \tag{5.44}$$

$$u_{xxxt}(0,0,0) = 0, \ u_{yyyt}(0,0,0) = 0, \ u_{yyxt}(0,0,0) = 0. \tag{5.45}$$

Similarly, for $v(x,y,t)$ we can calculate:

$$v_{xt}(0,0,0) = -2, \ v_{tt}(0,0,0) = 0, \ v_{yt}(0,0,0) = 0, \tag{5.46}$$

$$v_{xxt}(0,0,0) = 0, \ v_{yyt}(0,0,0) = 0, \tag{5.47}$$

$$v_{xyt}(0,0,0) = 0, \ v_{xtt}(0,0,0) = 4, \ v_{ytt}(0,0,0) = -4, \tag{5.48}$$

$$v_{xxxt}(0,0,0) = 0, \ v_{yyyt}(0,0,0) = 0, \ v_{yyxt}(0,0,0) = 0. \tag{5.49}$$

For brevity we are not providing calculations further. Taylor series expansion of $u(x,y,t)$ and $v(x,y,t)$ around the point $(0,0,0)$ can be written as

$$\begin{aligned} u(x,y,t) &= u(0,0,0) + \frac{1}{1!}\left(u_x(0,0,0)x + u_y(0,0,0)y + u_t(0,0,0)t\right) \\ &+ \frac{1}{2!}\big(u_{xx}(0,0,0)x^2 + u_{yy}(0,0,0)y^2 + u_{tt}(0,0,0)t^2 + 2u_{xy}(0,0,0)xy + 2u_{xt}(0,0,0)xt + 2u_{yt}(0,0,0)yt\big) + \cdots, \end{aligned} \tag{5.50}$$

$$v(x,y,t) = v(0,0,0) + \frac{1}{1!}\left(v_x(0,0,0)x + v_y(0,0,0)y + v_t(0,0,0)t\right)$$
$$+\frac{1}{2!}\left(v_{xx}(0,0,0)x^2 + v_{yy}(0,0,0)y^2 + v_{tt}(0,0,0)t^2 + 2v_{xy}(0,0,0)xy + 2v_{xt}(0,0,0)xt + 2v_{yt}(0,0,0)yt\right) + \cdots . \tag{5.51}$$

Substituting the values of u, v and their derivatives at $(0,0,0)$ in (5.50) and (5.51), we arrive at

$$u(x,y,t) = x + y - 2xt + 2xt^2 + 2yt^2 + \cdots = \frac{x+y-2xt}{1-2t^2}, \tag{5.52}$$
$$v(x,y,t) = x - y - 2yt + 2xt^2 - 2yt^2 + \cdots = \frac{x-y-2yt}{1-2t^2}. \tag{5.53}$$

Remark 5.1 *These derivations are based on assumption that mixed derivatives u_{xy} and u_{yx} and all other higher order derivatives all are same.*

6 Conclusion

In this paper, we successfully extend the work of He et al. [31] to different real-life problems. We successfully developed a few Mathematica codes to solve a class of singular nonlinear ODEs subject to initial conditions and boundary conditions. We also develop codes for the system of nonlinear singular ODEs and solve them too successfully. We also extend this approach to PDEs. This approach can further be extended to a different class of problems that do not have exact solutions. Finally, we conclude that this simple technique is very useful for engineering science, chemical, and physical science.

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