Anisotropic nonlinear problem of infinite order with variables exponents and L^1 data

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Abstract

In this paper, we prove the existence of solutions for the strongly nonlinear equation of the type

$$Au + g(x,u) = f$$

where A is an elliptic operator of infinite order from a functional Sobolev spaces of infinite order with variables exponents to its dual. $g(x,s)$ is a lower order term satisfying essentially a sign condition on s and the second term f belongs to $L^1(\Omega)$.

keywords: Strongly nonlinear elliptic equations of infinite order, monotonicity condition, sign condition.

1 Introduction

In their work Abdou, benkirane, Chrif and EL Manouni [4] studied a class of anisotropic problems involving operators of finite and infinite higher order in the variational case. They proved the existence of solutions in generalized Sobolev spaces, also called anisotropic Sobolev spaces with variable exponents. The goal of this paper is to show the existence of solutions of the problem as the following model example:

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}D^{\alpha}(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}(x)-2}D^{\alpha}u) + g(x,u) = f \quad in \;\; \Omega, \tag{1.1}$$

where $\Omega \subset I\!R^N$ is a bounded domain, $a_\alpha \geq 0$ are real numbers, $p_\alpha(.)$ are continuous functions on $\overline{\Omega}$, such that $p_\alpha(x) > 1$ for any $x \in \overline{\Omega}$ and for any multi-indices α; the nonlinear term g has to fulfil only the sign condition $g(x,s)s \geq 0$, but we do not assume any growth conditions with respect to $|u|$. As regards the second member, we suppose that $f \in L^1(\Omega)$.

Note that in the particular case when $p_\alpha(x) = p_\alpha$ for any $x \in \overline{\Omega}$ and any multi-indices α, the solvability of (1.1) is done by M. Chrif et al. in [5], [8] and [9]. In general case when $p_\alpha(.)$ are continuous functions and f bolongs to the dual space we refer to the recent work [4]. let us recall that in the case of non-homogeneous operators :

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha (a_\alpha |D^\alpha u|^{p_\alpha(x)-2} D^\alpha u)$$

the natural setting for our approach is the use of the variable exponent anisotropic Sobolev spaces. The basic idea is to replace the Lebesgue spaces $L^{p_\alpha}(\Omega)$ by more general spaces $L^{p_\alpha(x)}(\Omega)$, called variable exponent Lebesgue spaces. In the case of finite order, the isotropic case the space $L^{p(x)}(\Omega)$ and $W_0^{m,p(x)}(\Omega)$ were thoroughly studied in the monograph by Musielak [22] and the papers by Edmunds et al. [14, 15, 16], Mih ăilescu and Rădulescu [21], Samko and Vakulov [23], Diening et al. [11] and Harjulehto et al. [17]. For more information on properties, the modeling and application of spaces of variable exponents to some fluid physical phenomenon we refer to Acerbi and Mingione [1], Alves and Souto [3], Chabrowski and Fu [7], and Diening [10].

In this paper, we assume that $f \in L^1(\Omega)$ and the main dificulty is that the anisotropic Sobolev space of infinite order $W_0^\infty(a_\alpha, p_\alpha)(\Omega)$ (see [5], [12] and [13]), are not adequate to study nonlinear problems of type (1.1) . This leads us to seek weak solutions for our problems in a more general variable anisotropic Sobolev space of infinite order, which will be introduced in the next section of this paper. To establish our main result for a general Laray-Lions types operator, we consider in section 3 the following strongly nonlinear equation :

$$Au + g(x,u) = f \quad in \ \ \Omega, \tag{1.2}$$

where A is an operator of infinite order defined as :

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \qquad |\gamma| \le |\alpha|,$$

and the coefficient functions A_α are assumed to satisfy some growth and coerciveness conditions without supposing a monotonicity condition.

2 Preliminaries

We recall in this section some definitions and basic properties of the variable exponent Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$, where Ω is a bounded subset of $I\!R^N$.
Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : min_{x\in\overline{\Omega}} h(x) > 1\},$$

for any $h \in C_+(\overline{\Omega})$. We define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)} = \{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(x)} = inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [18].

Lemma 2.1 *(see Fan and Zhao [20] and Zhao et al. [24])*

(1) The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

(2) If $p_1,\ p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).$$

and the imbedding is continuous.

Lemma 2.2 *(see Fan and Zhao [20] and Zhao et al. [24])*
If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)} dx \qquad \forall u \in L^{p(x)},$$

then

(1) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(3) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \geq \rho(u) \geq |u|_{p(x)}^{p^+}$;

(4) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0; \quad |u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Lemma 2.3 *(see Fan and Zhao [20] and Zhao et al. [24])*
If $u,\ u_n \in L^{p(x)}(\Omega)$, $n = 0, 1, 2, ...$, then the following statements are equivalent each other:

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;

(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Finally, we introduce a naturel generalization of the variable exponent Sobolev space $W_0^{m,p(x)}(\Omega)$, that will enable us to study with sufficient accuracy anisotropic problem in section 3. For this purpose, let us denote by $\vec{p}(x)$ the vectorial function

$$\vec{p}(x) = \{p_\alpha(x),\ |\alpha| \le m\},$$

where m is a positive integer such that $m \ge 1$ and $p_\alpha(.) \in C_+(\overline{\Omega})$ for all multi-indices α such that $|\alpha| \le m$.
We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order. We define $W_0^{m,\vec{p}(x)}(\Omega)$, the anisotropic variable exponent Sobolev space, as the closure of $C_0^\infty(\Omega)$ with respect the norm

$$\|u\|_{m,\vec{p}(x)} = \sum_{|\alpha|=0}^{m} |D^\alpha u|_{p_\alpha(x)}.$$

In the case when $p_\alpha(x) \in C_+(\overline{\Omega})$ are constant functions for any $|\alpha| \le m$, the resulting anisotropic space is denoted by $W_0^{m,\vec{p}}(\Omega)$. Such spaces was developed and considered by authors in [5], [8] and [9] in the study of some anisotropic strongly non linear equations. It was proved that $W_0^{m,\vec{p}}(\Omega)$ is a reflexive Banach space for any $p_\alpha > 1$ for all multi-indice $|\alpha| \le m$. This result can be easily extend to $W_0^{m,\vec{p}(x)}(\Omega)$. In fact, the following lemma follows

Lemma 2.4 *(see [4]) The space $(W_0^{m,\vec{p}(x)}(\Omega), \|.\|_{m,\vec{p}(x)})$ is a banach and reflexive space.*

In order to facilitate the manipulation of the space $W_0^{m,\vec{p}(x)}(\Omega)$, we introduce p_+^+ and p_-^- as

$$p_+^+ = max\{p_\alpha^+(x),\, |\alpha| \le m\}, \qquad p_-^- = min\{p_\alpha^-(x),\, |\alpha| \le m\}.$$

Lemma 2.5 *Let Ω be a bounded open subset of $I\!R^N$.*
If $mp_-^- > N$, then $W_0^{m,\vec{p}(x)}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{p_-^-})$.
Moreover, the embedding is compact.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W_0^{m,\vec{p}(x)}(\Omega) \subset W_0^{m,p_-^-}(\Omega)$.

Now, let $a_\alpha \ge 0$ be a real numbers for multi-indices α. The variable exponent Sobolev space of infinite order is the functional space defined by

$$W^\infty(a_\alpha, p_\alpha(x))(\Omega) = \Big\{ u \in C^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty \Big\}.$$

Since we shall deal with the Dirichlet problem in this paper, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ defined by

$$W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) = \Big\{ u \in C_0^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty \Big\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, is the question of their nontriviality (or nonemptiness), i.e. the question of the existence of a function u such that $\sigma(u) < \infty$.

Definition 2.1 *(Dubinskii[12]) The space $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ is called nontrivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^{\infty}(\Omega)$ such that $\sigma(u) < \infty$.*

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the spaces $W^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, but also on the domain Ω.
The dual space of $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega) = \Big\{ h : \ h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha, \ \sigma'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha |h_\alpha|_{p'_\alpha(x)}^{p'^+_\alpha} < \infty \Big\},$$

where $h_\alpha \in L^{p'_\alpha(x)}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$.
By the definition, the duality pairing between $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$ and its dual space $W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$ is given by the relation

$$\langle h, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x)\, D^\alpha v(x)\, dx,$$

which, as it is not difficult to verify, is correct.
In the particular case when $p_\alpha(x) = p_\alpha$ for any multi-indices α, the Sobolev space of infinite order is defined as

$$W_0^{\infty}(a_\alpha, p_\alpha)(\Omega) = \Big\{ u \in C_0^{\infty}(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha}^{r_\alpha} < \infty \Big\}.$$

$a_\alpha \geq 0$, $p_\alpha > 1$ and $r_\alpha > 1$ are real numbers for all multi-indices α and $|.|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$, (see [12], [13]).

Lemma 2.6 *(see [4])For all nontrivial space $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega)$, there exists a nontrivial space $W_0^{\infty}(c_\alpha, 2)(\Omega)$ such that $W_0^{\infty}(a_\alpha, p_\alpha(x))(\Omega) \subset W_0^{\infty}(c_\alpha, 2)(\Omega)$.*

3 Main result

In this section, we formulate and prove the main result of this article.
Let A be the operator of infinite order defined as

$$Au = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \qquad |\gamma| \leq |\alpha|,$$

where $A_\alpha : \Omega \times I\!R^{\lambda_\alpha} \to I\!R$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. Consider the following strongly nonlinear problem with Dirichlet conditions:

$$Au + g(x, u) = f \quad \text{in } \Omega.$$

Here, $g : \Omega \times I\!R \to I\!R$ is a measurable function and $f \in W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$.
Let us now formulate the assumptions:

(A_1) $A_\alpha(x,\xi_\alpha)$ is a Carathéodory function for all $\alpha, |\gamma| \leq |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all $m \in I\!N^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\Big|\sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\eta_\alpha\Big| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)-1}|\eta_\alpha|,$$

where $a_\alpha \geq 0$, are reals numbers and $(p_\alpha(.))_\alpha$ is a bounded sequence of functions in $C_+(\overline{\Omega})$ for all multi-indices α.

(A_3) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in I\!N^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x,\xi_\gamma)\cdot\xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)} - c_2.$$

(A_4) The space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ is nontrivial.

(G_1) The function $g : \Omega \times I\!R \mapsto I\!R$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u|<\delta} |g(x,u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G_2) We assume the "sign condition" $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in I\!R$.

Finally, we assume that

$$f \in L^1(\Omega), \tag{3.1}$$

and we shall prove the existence result without assuming any monotonicity condition.

Theorem 3.1 *Let us assume the conditions* $(A_1) - (A_4)$, (G_1) *and* (G_2). *Then for all* $f \in W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$, *there exists* $u \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ *such that*

$$\begin{cases} g(x,u) \in L^1(\Omega), \; g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega). \end{cases}$$

Proof.
In order to get our result, we will deal with the following steps:

1. We prove the existence of approximate solutions u_n.
2. We establish the a priori estimates.
3. We prove that u_n converges to an element $u \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ and we finally show that u is the solution of our problem.

`Step (1):` The approximate problem.
Consider $\varphi \in C_0^\infty(I\!R^N)$ such that $0 < \varphi(x) < 1$ and $\varphi(x) = 1$ for x close to 0.
Let f_n be a sequence of regular functions defined by

$$f_n(x) = \varphi(\frac{x}{n})T_n f(x),$$

where T_n is the usual truncation given by

$$T_n\xi = \begin{cases} \xi & \text{if } |\xi| < n \\ \frac{n\xi}{|\xi|} & \text{if } |\xi| \geq n. \end{cases}$$

It is clear that $|f_n| \leq n$ for a.e. $x \in \Omega$. Thus, it follows that $f_n \in L^\infty(\Omega)$.Using Lebesgue's dominated convergence theorem, since $f_n \to f$ a.e. $x \in \Omega$ and $|f_n| \leq |f| \in L^1(\Omega)$, we conclude that f_n strongly converges to f in $L^1(\Omega)$.
Define the operator of order $2n+2$ by

$$A_{2n+2}(u) = \sum_{|\alpha|=n+1} (-1)^{n+1} c_\alpha D^{2\alpha} u + \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leq n,$$

where c_α are constants small enough such that they fulfill the conditions of the Lemma 2.6. The operator A_{2n+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition, this follows from the result of [19]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2n+2} satisfies the growth, the coerciveness and the monotonicity conditions. Hence by Theorem 3.1 (see [4]), there exists an approximate solution u_n of the following problem:

$$(\mathrm{Pb_n}) \begin{cases} g(x, u_n) \in L^1(\Omega), \ g(x, u_n)u_n \in L^1(\Omega) \\ \langle A_{2n+2}(u_n), v\rangle + \int_\Omega g(x, u_n)v \, dx = \langle f_n, v\rangle, \quad \forall v \in W_0^{n+1,\vec{p}(x)}(\Omega) \end{cases}$$

with

$$f_n = \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha \,, \qquad f_\alpha \in L^{p'_\alpha(x)}(\Omega).$$

`Step (2):` A priori estimate.
Set $v = u_n$ and using (A_3), (G_2), Lemma 2.1 and 2.2, we deduce the estimates

$$\sum_{|\alpha|=n+1} c_\alpha |D^\alpha u_n|_2^2 + \sum_{|\alpha|=0}^{n} a_\alpha |D^\alpha u_n|_{p_\alpha}^{\beta_\alpha} \leq K \tag{3.2}$$

and

$$\int_\Omega g(x, u_n)u_n \, dx \leq K \tag{3.3}$$

for some constant $K = K(f) > 0$, with

$$\beta_\alpha = \begin{cases} p_\alpha^+ & if \quad |D^\alpha u|_{p_\alpha(x)} < 1 \\ p_\alpha^- & if \quad |D^\alpha u|_{p_\alpha(x)} > 1 \end{cases} .$$

From this and since the summation in estimate (3.2) is finite, we can also write

$$\sum_{|\alpha|=n+1} c_\alpha |D^\alpha u_n|_2^2 + \sum_{|\alpha|=0}^{n} a_\alpha |D^\alpha u_n|_{p_\alpha}^{p_\alpha^+} \leq K \tag{3.4}$$

The estimate (3.4) is equivalent to

$$\sum_{|\alpha|=0}^{n+1} a_\alpha |D^\alpha u_n|_{p_\alpha(x)}^{p_\alpha^+} \leq K \tag{3.5}$$

with $a_\alpha = c_\alpha$ and $p_\alpha = 2$ for $|\alpha| = n+1$. Consequently, we have

$$\|u_n\|_{W^{n+1,\vec{p}(x)}} \leq K. \tag{3.6}$$

Then via a diagonalization process, there exists a subsequence still, denoted by u_n, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_n \to D^\alpha u$ (for more details we refer to [5], [12]).

`Step (3):` Convergence of problem (Pb_n).

There exists a solution u_n of problem (Pb_n), $n = 1, 2, \ldots$. Then by passing to the limit, we have

$$\lim_{n\to+\infty} \langle A_{2n+2}(u_n), v\rangle + \lim_{n\to+\infty} \int_\Omega g(x, u_n) v \, dx = \lim_{n\to+\infty} \langle f_n, v\rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. It is clear that

$$\lim_{n\to+\infty} \langle f_n, v\rangle = \langle f, v\rangle \quad \text{ for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

Now, we shall prove that

$$\lim_{n\to+\infty} \langle A_{2n+2}(u_n), v\rangle = \langle Au, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

In fact, let n_0 be a fix number sufficiently large ($n > n_0$) and let $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Set

$$\langle A(u) - A_{2n+2}(u_n), v\rangle = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \sum_{|\alpha|=0}^{n_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle \\
I_2 &= \sum_{|\alpha|=n_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle \\
I_3 &= -\sum_{|\alpha|=n_0+1}^{n} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle - \sum_{|\alpha|=n+1} c_\alpha \langle D^\alpha u_n, D^\alpha v\rangle,
\end{aligned}$$

or in another form,

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle.$$

with $A_\alpha(x, \xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = n+1$.

We will go to the limit as $n \to +\infty$ to prove that I_1, I_2 and I_3 tend to 0. Starting by I_1; we have $I_1 \to 0$ since $A_\alpha(x, \xi_\gamma)$ is of Carathéodory type. The term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. For what concerns I_3; in view of (A_2) and Hölder inequality (Lemma 2.1) we have

$$\Big| \sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle \Big| \leq \sum_{|\alpha|=n_0+1}^{n+1} |\langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle|$$

$$\leq \quad c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \int_\Omega |D^\alpha u_n|^{p_\alpha(x)-1} |D^\alpha v|\, dx$$
$$\leq \quad c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha | \ |D^\alpha u_n|^{p_\alpha(x)-1}|_{p'_\alpha(x)} |D^\alpha v|_{p_\alpha(x)}.$$

Now, in view Lemma 2.3, one get

$$\begin{aligned} | \ |D^\alpha u_n|^{p_\alpha(x)-1}|_{p'_\alpha(x)} &\leq \Big(\int_\Omega |D^\alpha u_n|^{(p_\alpha(x)-1)p'_\alpha(x)}\ dx \Big)^{\nu_\alpha} \\ &\leq \Big(\int_\Omega |D^\alpha u_n|^{p_\alpha(x)}\ dx \Big)^{\nu_\alpha} \\ &\leq |D^\alpha u_n|^{\nu_\alpha\ \beta_\alpha}_{p_\alpha(x)} \\ &\leq |D^\alpha u_n|^{p_\alpha^+ -1}_{p_\alpha(x)}, \end{aligned}$$

where ν_α and β_α are real numbers for all multi-indices $|\alpha| \leq n$, defined as

$$\nu_\alpha = \begin{cases} \frac{1}{{p'_\alpha}^+} & if \quad | \ |D^\alpha u_n|^{p_\alpha(x)-1}|_{p'_\alpha(x)} < 1 \\ \frac{1}{{p'_\alpha}^-} & if \quad | \ |D^\alpha u_n|^{p_\alpha(x)-1}|_{p'_\alpha(x)} > 1 \end{cases} .$$

$$\beta_\alpha = \begin{cases} p_\alpha^+ & if \quad |D^\alpha u_n|_{p_\alpha(x)} < 1 \\ p_\alpha^- & if \quad |D^\alpha u_n|_{p_\alpha(x)} > 1 \end{cases} .$$

It's very easy to verify that for all multi-indices $|\alpha| \leq n$, on has

$$\nu_\alpha\ \beta_\alpha \leq p_\alpha^+ - 1.$$

Therefore, for all $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ (see [6, p. 56]) such that

$$\begin{aligned} \Big| \sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v \rangle \Big| &\leq \varepsilon c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha |D^\alpha u_n|^{p_\alpha^+}_{p_\alpha(x)} + c_0 k(\varepsilon) \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha |D^\alpha v|^{p_\alpha^+}_{p_\alpha(x)} \\ &\leq \varepsilon c_0 K + c_0 k(\varepsilon) \sum_{|\alpha|=n_0+1}^{\infty} a_\alpha |D^\alpha v|^{p_\alpha^+}_{p_\alpha(x)}, \end{aligned}$$

where K is the constant given in the estimate (3.5). Since the sequence $(p_\alpha(x))$ is bounded and $\sum_{|\alpha|=n_0+1}^{\infty} a_\alpha |D^\alpha v|^{p_\alpha^+}_{p_\alpha(x)}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Hence $\langle A_{2n+2}(u_n), v \rangle \to \langle A(u), v \rangle$ as $n \to +\infty$ for all $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. It remains to show, for our purposes, that

$$\lim_{n \to +\infty} \int_\Omega g(x, u_n) v\, dx = \int_\Omega g(x, u) v\, dx,$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. Indeed, we have $u_n \to u$ uniformly in Ω, hence $g(x, u_n) \to g(x, u)$ for a.e. $x \in \Omega$. In view of (3.3), we deduce by Fatou's lemma that

$$\int_\Omega g(x, u) u\, dx \leq \lim_{n \to +\infty} \int_\Omega g(x, u_n) u_n\, dx \leq K.$$

This implies that $g(x,u)u \in L^1(\Omega)$. On the other hand, let $\delta > 0$, since $|g(x,t)|\delta \leq |g(x,t)t|$ and then $|g(x,t)| \leq \delta^{-1}|g(x,t)t|$ for $|t| \geq \delta$, we have

$$\begin{aligned} |g(x,u_n)| &\leq \sup_{|t|\leq\delta} |g(x,t)| + \delta^{-1}|g(x,u_n).u_n| \\ &\leq h_\delta(x) + \delta^{-1}|g(x,u_n)u_n|. \end{aligned}$$

It follows that

$$\int_E |g(x,u_n)|\,dx \leq \int_E h_\delta(x)\,dx + \delta^{-1}K,$$

for some measurable subset E of Ω and for some $\varepsilon > 0$. Here, K is the constant of (3.3) which is independent of n. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x,u_n)|\,dx < \varepsilon$. Then, using Vitali's, we get theorem $g(x,u_n) \to g(x,u)$ in $L^1(\Omega)$. Hence it follows that $g(x,u) \in L^1(\Omega)$.

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

Consequently,

$$\begin{cases} g(x,u) \in L^1(\Omega), g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) \end{cases}$$

This completes the proof.

Remark 3.1 *Note that the existence result is given with no monotonicity condition on the operator.*

References

[1] E. Acerbi, G. Mingione, Gradient estimates for the p(x)-Laplacian system, J. Reine Angew. Math. 584 (2005), 117 -148.

[2] R. Adams, *Sobolev Spaces.* New York: Academic Press 1975.

[3] C.O. Alves, M.A. Souto, Existence of solutions for a class of problems in $I\!R^N$ involving the p(x)-Laplacian, in: T. Cazenave, D. Costa, O. Lopes, R. Manàsevich, P. Rabinowitz, B. Ruf, C. Tomei (Eds.), Contributions to Nonlinear Analysis, A Tribute to D.G. de Figueiredo on the Occasion of his 70th Birthday, in: Progr. Nonlinear Differential Equations Appl., vol. 66, Birkhäuser, Basel, (2006), 17 -32.

[4] M. H. Abdou, A. Benkirane M. Chrif and S. El Manouni, Strongly anisotropic elliptic problems of infinite order with variable exponents. Complex Var. Elliptic Equ. 2014, vol. 59, no. 10, 1403-1417.

[5] A. Benkirane M. Chrif and S. El Manouni, Existence Results for Strongly Nonlinear Equations of Infinite Order, Z. Anal. Anwend. (J. Anal. Appl.) 26, (2007), 303–312.

[6] H. Brezis, *Analyse fonctionelle. Théorie, Méthodes et Applications,* Masson, Paris, 1992.

[7] J. Chabrowski, Y. Fu, Existence of solutions for p(x)-Laplacian problems on a bounded domain, J. Math. Anal. Appl. 306 (2005), 604 -618.

[8] M. Chrif and S. El Manouni, On a strongly anisotropic equation with L1-data, Appl. Anal. 87(7) (2088), 865–871.

[9] M. Chrif and S. El Manouni, Anisotropic equations in weighted Sobolev spaces of higher order, Ricerche mat. 58, (2009), 1–14.

[10] L. Diening, Theoretical and numerical results for electrorheological fluids, PhD thesis, University of Frieburg, Germany, 2002.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Ružička, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.

[12] Ju. A. Dubinskii, *Sobolev Spaces of Infinite Order and Differential Equations*, Teubner-Texte Math. Band 87. Leipzig: Teubner, 1986.

[13] Ju. A. Dubinskii, *Sobolev spaces for infinite order and the behavior of solutions of some boundary value problems with unbounded increase of the order of the equation*, Math. USSR-Sb. 27(2) (1975), 143–162.

[14] D.E. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}$ norms, Proc. Roy. Soc. London Ser. A 455 (1999), 219 -225.

[15] D.E. Edmunds, J. Rakosnik, Density of smooth functions in $W^{k,p(x)}(\Omega)$, Proc. Roy. Soc. London Ser. A 437 (1992), 229–236.

[16] D.E. Edmunds, J. Rákosník, Sobolev embedding with variable exponent, Studia Math. 143 (2000), 267 -293.

[17] P. Harjulehto, P. Hästö, Ú. V. Lê and M. Nuortio, Overview of differential equations with non-standard growth, Nonlinear Anal. 72 (2010), 4551–4574.

[18] O. Kovacik, J. Rakosnik, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41 (1991), 592–618.

[19] J. L. Lions, *Quelque Méthodes de Résolution des Problémes aux Limites Non Linéaires.* Paris: Dunod; Gauthier-Villars 1969.

[20] X.L. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$, J. Gansu Educ. College 12 (1) (1998) 1–6.

[21] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. Roy. Soc. London Ser. A 462 (2006) 2625 -2641.

[22] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.

[23] S. Samko, B. Vakulov,Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, J.Math. Anal. Appl. 310 (2005), 229 -246.

[24] D. Zhao, W.J. Qiang, X.L. Fan, On generalized Orlicz spaces $L^{p(x)}$, J. Gansu Sci. 9 (2) (1996), 1–7.