Numerical Simulation of the Fractal Fractional Reaction Diffusion Equations with General Nonlinear

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In this paper we utilize the new operators using three different kernels,namely, power law, exponential decay, and the generalized Mittag-Leffler function based on the fractal-fractional differentiation. These constructed operators have two parameters: the first is considered as fractional order and the second as fractal dimension. We studied the effect of the fractal-fractional derivative order as well as the nonlinear term order $1 < q \leq 2$ on the behavior of the numerical solutions of the fractal-fractional reaction diffusion equations (FFRDE). The iterative approximations are constructed by applying the theory of fractional calculus with the help of Lagrange polynomial functions. In the case of $\beta = k = 1$, all the numerical solutions based on power kernel, exponential kernel, and the generalized Mittag-Leffler kernel are very close to each other and so one of the kernel was compared with numerical methods like finite difference methods (FDM) and an excellent agreement obtained. All calculations in this paper were done using the mathematica package.

2010 Mathematics Subject Classification. Primary 26A33, 34A08; Secondary 35A20, 35A22.

Key Words and Phrases. the fractal-fractional reaction diffusion equations ; Lagrange polynomial interpolation; the power law; the exponential law; generalized Mittag-Leffler function.

1. Introduction, Definitions and Preliminaries

Fractional calculus is a generalization of classical calculus and many researchers have paid attention to this science as they encounter many of these issues in the real world. Most of these issues do not have analytical exact solution. Which made many researchers interest and search in numerical and approximate methods to obtain solutions using these methods. There are many of these methods, such as the homotopy analysis [1, 2, 3, 4] , He's variational iteration method [5, 6] , Adomians decomposition method [7, 8, 9] , Fourier spectral methods [10] , finite difference schemes [11], collocation methods [12, 13, 14]. To find out more about the fractal calculus, refer to the following references [15, 16]. More recently, a new concept was introduced for the fractional operator, as this operator has two orders, the first representing the fractional order, and the second representing the fractal dimension.

In our work we aim to applied the idea of fractal-fractional derivative of orders β, k to a reaction-diffusion equation with q-th nonlinear. To this end [17] , we replace the derivative with respect to t by the fractal-fractional derivatives power (FFP) law, the fractal-fractional exponential(FFE) law and the fractal-fractional Mittag-Leffler (FFM) law kernels which corresponds to the [18], Caputo-Fabrizio (CF) [19] and the Atangana-Baleanu (AB) [20] fractional derivatives, respectively. This topic has attracted many researchers and has been applied to research related to the real world, such as [21]–[26]. Some recent developments in the area of numerical techniques can be found in [27]–[29].

Merkin and Needham [30] considered the reaction-diffusion travelling waves that can develop in a coupled system involving simple isothermal autocatalysis kinetics. They assumed that reactions took place in two separate and parallel regions, with, in I, the reaction being given by quadratic autocatalysis

$$F + G \to 2G(\text{rate}\, k_1 f\, g), \tag{1}$$

together with a linear decay step

$$G \to H(\text{rate}\, k_2\, g) \tag{2}$$

where f and g are the concentrations of reactant F and autocatalyst H, the $k_i (i = 1, 2)$ are the rate constants and H is some inert product of reaction. The reaction in region II was the quadratic autocatalytic step (1) only. The two regions were assumed to be coupled via a linear diffusive interchange of the autocatalytic species G. We shall consider a similar system as I, but with cubic autocatalysis

$$F + 2G \to 3G(\text{rate}\, k_3\, fg^2) \tag{3}$$

together with a linear decay step

$$G \to H(\text{rate}\, k_4 g). \tag{4}$$

For q-th autocatalytic, we have

$$F + qG \to (q+1)G(\text{rate}\, k_3 f g^q),\ 1 \le q \le 2, \tag{5}$$

together with a linear decay step

$$G \to H(\text{rate}\ \ k_4 g). \tag{6}$$

This yields to the following system

$$\frac{\partial \eta_1}{\partial \eta} = \frac{\partial^2 \eta_1}{\partial \xi^2} + \nu(\eta_2 - \eta_1) - \eta_1 \zeta_1^q, \tag{7}$$

$$\frac{\partial \zeta_1}{\partial \eta} = \frac{\partial^2 \zeta_1}{\partial \xi^2} - \kappa \zeta_1 + \eta_1 \zeta_1^q, \tag{8}$$

$$\frac{\partial \eta_2}{\partial \eta} = \frac{\partial^2 \eta_2}{\partial \xi^2} + \nu(\eta_1 - \eta_2) - \eta_2 \zeta_2^q, \tag{9}$$

$$\frac{\partial \zeta_2}{\partial \eta} = \frac{\partial^2 \zeta_2}{\partial \xi^2} + \eta_2 \zeta_2^q \tag{10}$$

where ν represents the couple between (I) and (II) and κ represents the strength of the auto-catalyst decay. For more details see [30].

Omitting the diffusion terms in the system (7)–(10), one has the following ODE system

$$\frac{\partial \eta_1}{\partial \eta} = \nu(\eta_2 - \eta_1) - \eta_1 \zeta_1^q, \tag{11}$$

$$\frac{\partial \zeta_1}{\partial \eta} = -\kappa \zeta_1 + \eta_1 \zeta_1^q, \tag{12}$$

$$\frac{\partial \eta_2}{\partial \eta} = \nu(\eta_1 - \eta_2) - \eta_2 \zeta_2^q, \tag{13}$$

$$\frac{\partial \zeta_2}{\partial \eta} = \eta_2 \zeta_2^q. \tag{14}$$

Now we will provide some basic definitions that will be needed in our work. As for the theorems and proofs related to the three fractal-fractional operators, they are found in details in [17], so we will suffice in our work by constructing the algorithms and making the numerical simulations of the system (7)–(10) with the three fractal-fractional operators.

Definition 1. If $\eta(t)$ is continuous and fractal differentiable on (a, b) of order k, then the fractal-fractional derivative of $\eta(t)$ of order β in Riemann Liouville sense with power law is given by [17]:

$${}^{\text{FFP}}_{0}D_t^{\beta,k}\eta(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt^k}\int_0^t (t-\tau)^{-\beta}\eta(\tau)d\tau, \quad (0 < \beta,\, k \leq 1), \tag{15}$$

and the fractal-fractional integral of $\eta(t)$ is given by

$${}^{\text{FFP}}_{0}I_t^{\beta,k}\eta(t) = \frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\eta(\tau)d\tau. \tag{16}$$

Definition 2. If $\eta(t)$ is continuous in the (a, b) and fractal differentiable on (a, b) with order k, then the fractal-fractional derivative of $\eta(t)$ of order β in CF sense with the exponential decay kernel are given by [18]:

$${}^{FFE}_{0}D_t^{\beta,k}\eta(t) = \frac{M(\beta)}{1-\beta}\frac{d}{dt^k}\int_0^t e^{\frac{-\beta}{1-\beta}(t-\tau)}\eta(\tau)d\tau, \quad (0 < \beta,\, k \leq 1), \tag{17}$$

and the fractal-fractional integral of $\eta(t)$ is given by

$${}^{\text{FFE}}_{0}I_t^{\alpha,k}\eta(t) = \frac{(1-\beta)kt^{k-1}}{M(\beta)}\eta(t) + \frac{\beta k}{M(\beta)}\int_0^t \tau^{k-1}\eta(\tau)d\tau. \tag{18}$$

Definition 3. If $\eta(t)$ is continuous in the (a, b) and fractal differentiable on (a, b) with order k, then the fractal-fractional derivative of $\eta(t)$ of order β in AB sense with the generalized Mittag-Leffler type kernel are given by [18]:

$${}^{FFE}_{0}D_t^{\beta,k}\eta(t) = \frac{A(\beta)}{1-\beta}\frac{d}{dt^k}\int_0^t E_\beta\Big(\frac{-\beta}{1-\beta}(t-\tau)\Big)\eta(\tau)d\tau, \quad (0 < \beta,\, k \leq 1), \tag{19}$$

and the fractal-fractional integral of $\eta(t)$ is given by

$${}^{\text{FFE}}_{0}I_t^{\beta,k}\eta(t) = \frac{(1-\beta)kt^{k-1}}{A(\beta)}\eta(t) + \frac{\beta\, k}{A(\beta)\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\eta(\tau)d\tau, \tag{20}$$

where

$$\frac{d\eta(t)}{dt^k} = lim_{\tau\to t}\frac{\eta(\tau)-\eta(t)}{\tau^k - t^k}. \tag{21}$$

Our contribution to this paper is to construct the successive approximations and evaluate the numerical solutions of the FFRDE. These successive approximations allow us to study the behavior of numerical solutions based on power kernel, exponential kernel, and the generalized Mittag-Leffler kernel. Also we can study the behavior of approximate solutions in the case of nonlinearity of the FFRDE in general. To our best knowledge, this is the first study of this the FFRDE using fractal-fractional with these kernels. The structure of this paper is summarized as follows: In sections, two, three and four, the FFRDE is presented with the three kernels that proposed in this work and construct the successive approximations. In section Five, numerical solutions for the FFRDE are discussed with a study of their behavior. Chapter Six the conclusion is presented.

2. Numerical scheme for FFRDE for q-th-order autocatalysis due the power law kernel

The new model is obtained by replacing the ordinary derivative with the the fractal-fractional derivative the power law kernel as [17]

$$
\begin{aligned}
{}^{\mathrm{FFP}}_{0}D_t^{\beta}\zeta_1(t) &= \nu(\eta_2(t)-\eta_1(t)) - \eta_1(t)\zeta_1^q(t), && (22)\\
{}^{\mathrm{FFP}}_{0}D_t^{\beta}\eta_1(t) &= -\kappa\zeta_1(t) + \eta_1(t)\zeta_1^q(t), && (23)\\
{}^{\mathrm{FFP}}_{0}D_t^{\beta}\zeta_2(t) &= \nu(\eta_1(t)-\eta_2(t)) - \eta_2(t)\zeta_2^q(t), && (24)\\
{}^{\mathrm{FFP}}_{0}D_t^{\beta}\eta_2(t) &= \eta_2(t)\zeta_2^q(t). && (25)
\end{aligned}
$$

By following the procedure in [17], we can obtain the following successive approximations:

$$
\begin{aligned}
\eta_1(t)-\eta_1(0) &= \frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, && (26)\\
\zeta_1(t)-\zeta_2(0) &= \frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, && (27)\\
\eta_2(t)-\eta_3(0) &= \frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, && (28)\\
\zeta_2(t)-\zeta_2(0) &= \frac{k}{\Gamma(\beta)}\int_0^t \tau^{k-1}(t-\eta)^{\alpha-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\eta && (29)
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau) &= \Big(\nu\left(\eta_2(\tau)-\eta_1(\tau)\right) - \eta_1(\tau)\zeta_1^n(\tau)\Big), && (30)\\
\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau) &= \Big(-\kappa\,\zeta_1(t) + \eta_1(t)\zeta_1^n(t)\Big), && (31)\\
\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau) &= \Big(\nu(\eta_1(t)-\eta_2(t)) - \eta_2(t)\zeta_2^n(t)\Big), && (32)\\
\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau) &= \eta_2(t)\zeta_2^n(t). && (33)
\end{aligned}
$$

Equation (26)–(29) can be reformulated as

$$
\eta_1(t)-\eta_1(0) = \frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}} \tau^{k-1}(t-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{34}
$$

$$\zeta_1(t)-\zeta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}\varphi_2(\eta_1(\tau),\zeta_1(\tau),\eta_2(\tau),\zeta_2(\tau),\tau)d\tau, \tag{35}$$

$$\eta_2(t)-\eta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \tag{36}$$

$$\zeta_2(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \tag{37}$$

Using the two-step Lagrange polynomial interpolation, we obtain

$$\eta_1(t)-\eta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}Q_{1,m}(\tau)d\tau, \tag{38}$$

$$\zeta_1(t)-\zeta_1(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}Q_{2,m}(\tau)d\tau, \tag{39}$$

$$\eta_2(t)-\eta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}Q_{3,m}(\tau)d\tau, \tag{40}$$

$$\zeta_2(t)-\zeta_2(0)=\frac{k}{\Gamma(\beta)}\sum_{m=0}^{\infty}\int_{t_m}^{t_{m+1}}\tau^{k-1}(t-\tau)^{\beta-1}Q_{4,m}(\tau)d\tau, \tag{41}$$

where,

$$\begin{aligned}Q_{1,m}(\tau)=&\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_1(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}\\&\times t_{m-1}^{k-1}\varphi_1(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1})\end{aligned} \tag{42}$$

$$\begin{aligned}Q_{2,m}(\tau)=&\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_2(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}\\&\times t_{m-1}^{k-1}\varphi_2(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1})\end{aligned} \tag{43}$$

$$\begin{aligned}Q_{3,m}(\tau)=&\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_3(\eta_1(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}\\&\times t_{m-1}^{k-1}\varphi_3(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1})\end{aligned} \tag{44}$$

$$\begin{aligned}Q_{4,m}(\tau)=&\frac{\tau-t_{m-1}}{t_m-t_{m-1}}t_m^{k-1}\varphi_4(\eta_4(\tau_m),\zeta_1(\tau_m),\eta_2(\tau_m),\zeta_2(\tau_m),\tau_m)-\frac{\tau-t_m}{t_m-t_{m-1}}\\&\times t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(\tau_{m-1}),\eta_2(\tau_{m-1}),\zeta_2(\tau_{m-1}),\tau_{m-1})\end{aligned} \tag{45}$$

These integrals are evaluated directly and the numerical solutions of (22)–(25) involving the FFP derivative are given by

$$\begin{aligned}\eta_1(t_{n+1}) &= \eta_1(0)+\frac{kh^{\beta}}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_1(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\&- t_{m-1}^{k-1}\varphi_1(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{46}$$

$$\begin{aligned}\zeta_1(t_{n+1}) &= \zeta_1(0)+\frac{kh^{\beta}}{\Gamma(\beta+2)}\sum_{m=0}^{n}t_m^{k-1}\varphi_2(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\&- t_{m-1}^{k-1}\varphi_2(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{47}$$

$$\begin{aligned}\eta_2(t_{n+1}) &= \eta_2(0)+\frac{kh^{\beta}}{\Gamma(\beta+2)}\sum_{m=0}^{n} t_m^{k-1}\varphi_3(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{48}$$

$$\begin{aligned}\zeta_2(t_{n+1}) &= \zeta_2(0)+\frac{kh^{\beta}}{\Gamma(\beta+2)}\sum_{m=0}^{n} t_m^{k-1}\varphi_4(\eta_1(t_m),\zeta_1(t_m),\eta_2(t_m),\zeta_2(t_m),t_m)\Xi_1(n,m)\\ &- t_{m-1}^{k-1}\varphi_4(\eta_1(\tau_{m-1}),\zeta_1(t_{m-1}),\eta_2(t_{m-1}),\zeta_2(t_{m-1}),t_{m-1})\Xi_2(n,m)\Big),\end{aligned} \tag{49}$$

$$\Xi_1(n,m)=\Big((n+1-m)^{\beta}(n-m+2+\beta)-(n-m)^{\beta}\times(n-m+2+2\beta)\Big), \tag{50}$$

$$\Xi_2(n,m)=\Big((n+1-m)^{\beta+1}-(n-m)^{\beta}(n-m+1+\beta)\Big). \tag{51}$$

3. Numerical scheme for FFRDE for q-th-order autocatalysis due the exponential decay kernel

Considering the FFE derivative, we have from [17]

$$ {}_0^{\mathrm{FFE}}D_t^{\beta}\zeta_1(t)=\nu(\eta_2(t)-\eta_1(t))-\eta_1(t)\zeta_1^q(t), \tag{52}$$

$$ {}_0^{\mathrm{FFE}}D_t^{\beta}\eta_1(t)=-\kappa\zeta_1(t)+\eta_1(t)\zeta_1^q(t), \tag{53}$$

$$ {}_0^{\mathrm{FFE}}D_t^{\beta}\zeta_2(t)=\nu(\eta_1(t)-\eta_2(t))-\eta_2(t)\zeta_2^q(t), \tag{54}$$

$$ {}_0^{\mathrm{FFE}}D_t^{\beta}\eta_2(t)=\eta_2(t)\zeta_2^q(t). \tag{55}$$

For the successive approximations of the system (52)–(55), we follow the same procedures as in [17], we obtain

$$\begin{aligned}\eta_1(t)-\eta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{56}$$

$$\begin{aligned}\zeta_1(t)-\zeta_1(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{57}$$

$$\begin{aligned}\eta_2(t)-\eta_2(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{58}$$

$$\begin{aligned}\zeta_2(t)-\zeta_2(0) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{M(\beta)}\int_0^t k\,\tau^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau.\end{aligned} \tag{59}$$

Using $t = t_{n+1}$ the following is established

$$\begin{aligned}\eta_1(t_{n+1}) - \eta_1(t_n) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t) \\ &+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{60}$$

$$\begin{aligned}\zeta_1(t_{n+1}) - \zeta_1(t_n) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t) \\ &+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{61}$$

$$\begin{aligned}\eta_2(t_{n+1}) - \eta_2(t_n) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t) \\ &+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{62}$$

$$\begin{aligned}\zeta_2(t_{n+1}) - \zeta_2(t_n) &= \frac{kt^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t) \\ &+ \frac{\beta}{M(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \end{aligned} \tag{63}$$

Further, we have the following:

$$\begin{aligned}\eta_1(t_{n+1}) - \eta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) \\ &+ \frac{\beta}{M(\beta)}\int_{t_n}^{t_{n+1}} k\,\eta^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{64}$$

$$\begin{aligned}\zeta_1(t_{n+1}) - \zeta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) \\ &+ \frac{\beta}{M(\beta)}\int_{t_n}^{t_{n+1}} k\,\eta^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{65}$$

$$\begin{aligned}\eta_2(t_{n+1}) - \eta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n) \\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}) \\ &+ \frac{\beta}{M(\beta)}\int_{t_n}^{t_{n+1}} k\,\eta^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \end{aligned} \tag{66}$$

$$\begin{aligned}\zeta_2(t_{n+1})-\zeta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1})\\ &+ \frac{\beta}{M(\beta)}\int_{t_n}^{t_{n+1}} k\,\eta^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \end{aligned} \tag{67}$$

It follows from the Lagrange polynomial interpolation and integrating the following expressions:

$$\begin{aligned}\eta_1(t_{n+1})-\eta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1})+\frac{kh\beta}{2M(\beta)}\\ &\times \Big(3t_n^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)-t_{n-1}^{k-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}), \end{aligned} \tag{68}$$

$$\begin{aligned}\zeta_1(t_{n+1})-\zeta_1(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1})+\frac{kh\beta}{2M(\beta)}\\ &\times \Big(3t_n^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)-t_{n-1}^{k-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}), \end{aligned} \tag{69}$$

$$\begin{aligned}\eta_2(t_{n+1})-\eta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1})+\frac{kh\beta}{2M(\beta)}\\ &\times \Big(3t_n^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)-t_{n-1}^{k-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}), \end{aligned} \tag{70}$$

$$\begin{aligned}\zeta_2(t_{n+1})-\zeta_2(t_n) &= \frac{kt_n^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- \frac{kt_{n-1}^{k-1}(1-\beta)}{M(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1})+\frac{kh\beta}{2M(\beta)}\\ &\times \Big(3t_n^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)-t_{n-1}^{k-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}). \end{aligned} \tag{71}$$

Finally, it is appropriate to write the successive approximations of the system (52)–(55) as follows:

$$\begin{aligned}\eta_1(t_{n+1})-\eta_1(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{3h\beta}{2M}\Big)\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{h\beta}{2M(\beta)}\Big)\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}), \end{aligned} \tag{72}$$

$$\begin{aligned}\zeta_1(t_{n+1})-\zeta_1(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{3h\beta}{2M}\Big)\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{h\beta}{2M(\beta)}\Big)\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}), \end{aligned} \tag{73}$$

$$\begin{aligned}\eta_2(t_{n+1})-\eta_2(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{3h\beta}{2M}\Big)\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{h\beta}{2M(\beta)}\Big)\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}),\end{aligned} \tag{74}$$

$$\begin{aligned}\zeta_2(t_{n+1})-\zeta_2(t_n) &= kt_n^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{3h\beta}{2M}\Big)\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_n)\\ &- kt_{n-1}^{k-1}\Big(\frac{(1-\beta)}{M(\beta)}+\frac{h\beta}{2M(\beta)}\Big)\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t_{n-1}).\end{aligned} \tag{75}$$

4. Numerical scheme for FFRDE for q-th-order autocatalysis due the generalized Mittag Lefller kernel

Considering the FFM derivative, we have [18]

$${}_0^{\mathrm{FFM}}D_t^{\beta}\zeta_1(t)=\nu(\eta_2(t)-\eta_1(t))-\eta_1(t)\zeta_1^q(t), \tag{76}$$

$${}_0^{\mathrm{FFM}}D_t^{\beta}\eta_1(t)=-\kappa\zeta_1(t)+\eta_1(t)\zeta_1^q(t), \tag{77}$$

$${}_0^{\mathrm{FFM}}D_t^{\beta}\zeta_2(t)=\nu(\eta_1(t)-\eta_2(t))-\eta_2(t)\zeta_2^q(t), \tag{78}$$

$${}_0^{\mathrm{FFM}}D_t^{\beta}\eta_2(t)=\eta_2(t)\zeta_2^q(t). \tag{79}$$

Also, for this system (76)–(77), we follow the same treatment that was done in [17] to obtain the successive approximate solutions as follows:

$$\begin{aligned}\eta_1(t)-\eta_1(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{80}$$

$$\begin{aligned}\zeta_1(t)-\zeta_1(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{81}$$

$$\begin{aligned}\eta_2(t)-\eta_2(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau,\end{aligned} \tag{82}$$

$$\begin{aligned}\zeta_2(t)-\zeta_2(0) &= \frac{kt^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,t)\\ &+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^t k\,\tau^{k-1}(t-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau.\end{aligned} \tag{83}$$

At t_{n+1} we obtain the following

$$\begin{aligned}
\eta_1(t_{n+1})-\eta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (84)\\
\zeta_1(t_{n+1})-\zeta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (85)\\
\eta_2(t_{n+1})-\eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (86)\\
\zeta_2(t_{n+1})-\zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\int_0^{t_{n+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (87)
\end{aligned}$$

The integrals involving in (84)–(87) can be approximated as:

$$\begin{aligned}
\eta_1(t_{n+1})-\eta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_1(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (88)\\
\zeta_1(t_{n+1})-\zeta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_2(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (89)\\
\eta_2(t_{n+1})-\eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_3(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau, \qquad (90)\\
\zeta_2(t_{n+1})-\zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\
&+ \frac{\beta}{A(\beta)\Gamma(\beta)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} k\,\tau^{k-1}(t_{n+1}-\tau)^{\beta-1}\varphi_4(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau)d\tau. \qquad (91)
\end{aligned}$$

The following numerical schemes after approximating the expressions $\tau^{k-1}\varphi_i(\eta_1,\zeta_1,\eta_2,\zeta_2,\tau),\ i=1,2,3,4$ in the interval $[t_j,t_{j+1}]$ in (88)-(91) are given by

$$\begin{aligned}\eta_1(t_{n+1})-\eta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_1(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\ &+ \frac{kh^\beta}{A(\beta)\Gamma(\beta+2)}\sum_{j=0}^{n}\Big[t_j^{k-1}\varphi_1(\eta_1(t_j),\zeta_1(t_j),\eta_2(t_j),\zeta_2(t_j),(t_j))\Xi_1(n,m)\\ &- t_{j-1}^{k-1}\varphi_1(\eta_1(t_{j-1}),\zeta_1(t_{j-1}),\eta_2(t_{j-1}),\zeta_2(t_{j-1}),(t_{j-1}))\Xi_2(n,m)\Big], \qquad (92)\end{aligned}$$

$$\begin{aligned}\zeta_1(t_{n+1})-\zeta_1(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_2(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\ &+ \frac{kh^\beta}{A(\beta)\Gamma(\beta+2)}\sum_{j=0}^{n}\Big[t_j^{k-1}\varphi_2(\eta_1(t_j),\zeta_1(t_j),\eta_2(t_j),\zeta_2(t_j),(t_j))\Xi_1(n,m)\\ &- t_{j-1}^{k-1}\varphi_2(\eta_1(t_{j-1}),\zeta_1(t_{j-1}),\eta_2(t_{j-1}),\zeta_2(t_{j-1}),(t_{j-1}))\Xi_2(n,m)\Big], \qquad (93)\end{aligned}$$

$$\begin{aligned}\eta_2(t_{n+1})-\eta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_3(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\ &+ \frac{kh^\beta}{A(\beta)\Gamma(\beta+2)}\sum_{j=0}^{n}\Big[t_j^{k-1}\varphi_3(\eta_1(t_j),\zeta_1(t_j),\eta_2(t_j),\zeta_2(t_j),(t_j))\Xi_1(n,m)\\ &- t_{j-1}^{k-1}\varphi_3(\eta_1(t_{j-1}),\zeta_1(t_{j-1}),\eta_2(t_{j-1}),\zeta_2(t_{j-1}),(t_{j-1}))\Xi_2(n,m)\Big], \qquad (94)\end{aligned}$$

$$\begin{aligned}\zeta_2(t_{n+1})-\zeta_2(0) &= \frac{kt_n^{k-1}(1-\beta)}{A(\beta)}\varphi_4(\eta_1(t_n),\zeta_1(t_n),\eta_2(t_n),\zeta_2(t_n),t_n)\\ &+ \frac{kh^\beta}{A(\beta)\Gamma(\alpha+2)}\sum_{j=0}^{n}\Big[t_j^{k-1}\varphi_4(\eta_1(t_j),\zeta_1(t_j),\eta_2(t_j),\zeta_2(t_j),(t_j))\Xi_1(n,m)\\ &- t_{j-1}^{k-1}\varphi_4(\eta_1(t_{j-1}),\zeta_1(t_{j-1}),\eta_2(t_{j-1}),\zeta_2(t_{j-1}),(t_{j-1}))\Xi_2(n,m)\Big]. \qquad (95)\end{aligned}$$

5. Numerical results

In this section, we study in details the effect of the non-linear term in general, as well as the effect of the fractal-fractional order on the numerical solutions that we obtained by using successive approximations in the above sections. First we begin by satisfying the effective of the numerical solutions of the proposed system when $\beta = 1, k = 1$.

We compare only for the power kernel with a known numerical method which is the finite differences method. This is because all numerical solutions based on the three fractal-fractional operators that presented in this paper are very close each other when $\beta = 1, k = 1$. Figure 1 (a)-(c) illustrates the comparison between numerical solutions (46)–(49) and numerical solutions computed by using the finite differences method with $\beta = 1, k = 1$. The parameters that used are $\gamma = 0.4, \kappa = 0.004, h = 0.02$. From this figure we note that an excellent agreement. And the accurate is increasing as we take small h. From, Figure 1(a) and 1(c), we can see, that the profiles for η_1 and η_2 are very similar, but the profiles of ζ_1 and ζ_2 are more distinct with $\zeta_2 > \zeta_2$. For Figure 1(b), the profiles of ζ_1 and ζ_2 are very close

than in figures 1(a) and 1(c), also for ζ_1 and ζ_2. Figures 2–3 show that the behavior of the approximate solutions based on FFP, FFE and FFM, when the degree of the non-linear term is cubic and for different values of k, β. For the parameters γ, κ , we fixed them in all computations. The remain parameters are the same as in Figure 1. Similarly, in Figures 4–5, the approximate solutions are plotted in the case of a non-linear with quadratic degree and for different values of k, β. Finally in 6–7, the approximate solutions are shown in the case of non-linear with fraction order and for different values for k, β. From 4–5, we can see in the case of fraction non-linear, the profiles of η_1 and η_2 are very close to each other than the profiles of ζ_1 and ζ_2. For the figures 2–3 which the nonlinear is cubic, all the profiles are distinct. Similarly with figures 6–7 when the nonlinear is quadratic.

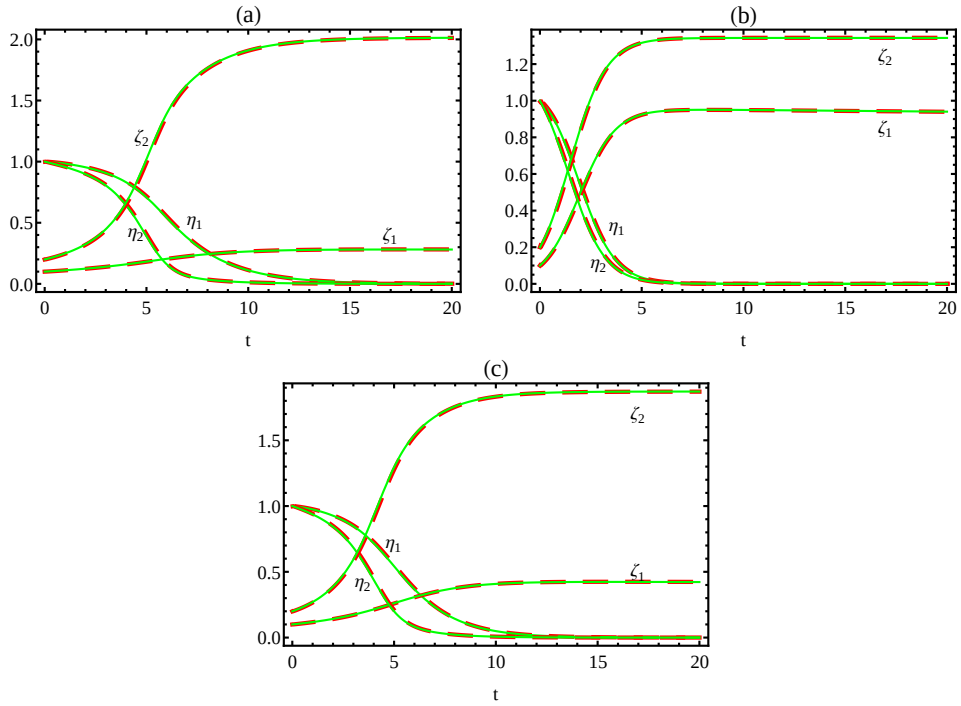

FIGURE 1. **Comparison between the numerical solutions (46)–(49) and numerical based on finite difference methods for** $\beta = 1$, $k = 1$, $\gamma = 0.4$, $\kappa = 0.001$, $h = 0.01$. $(a)\, q = 2$; $(b)\, q = 1$; $(c)\, q = 1.8$; **(Green solid color: Numerical solutions (46)–(49); Red dashed color: FDM)**

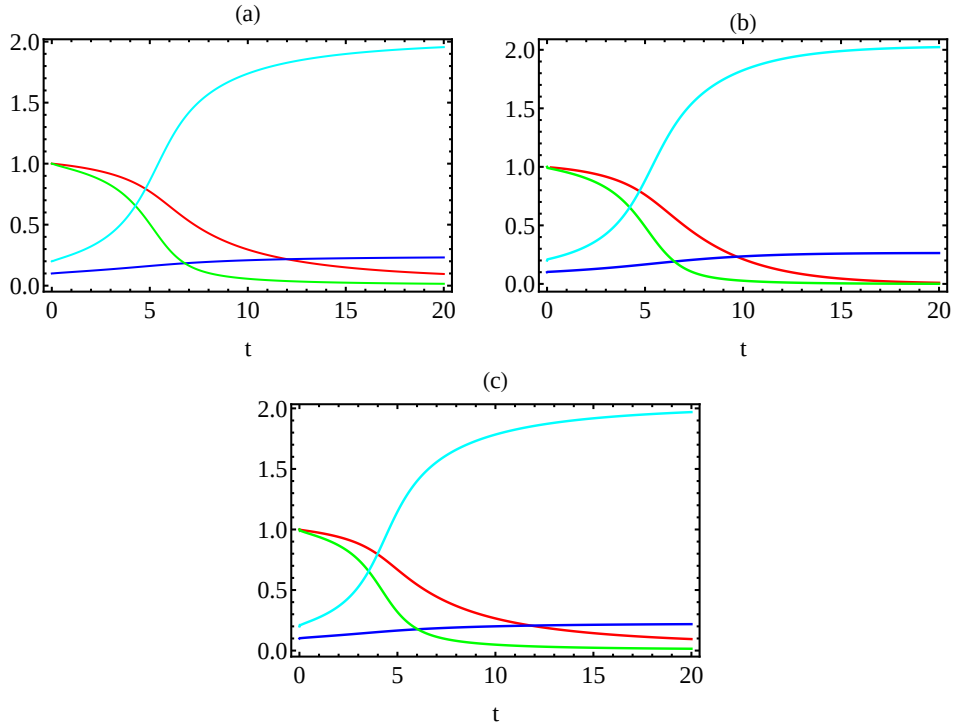

FIGURE 2. **Graph of the numerical solutions with** $q = 2$ **for** $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ **(a) FFP; (b) FFE; (c) FFM; (Red color:** η_1**; Blue color:** ζ_1**; Green color:** η_2**; Cyan color:** ζ_2**).**

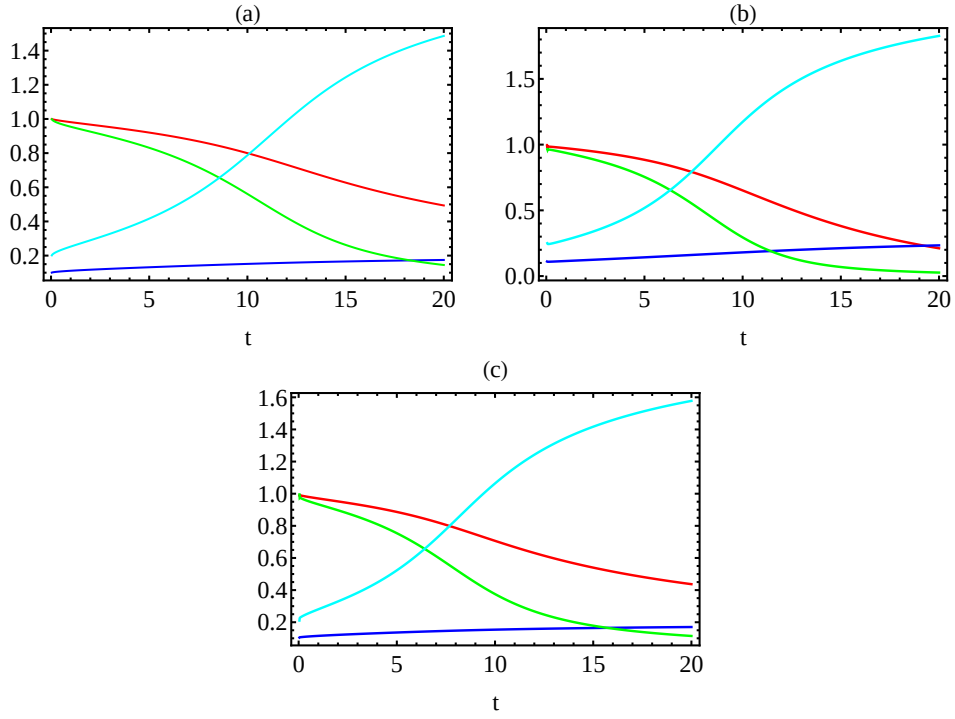

FIGURE 3. **Graph of the numerical solutions with $q = 2$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).**

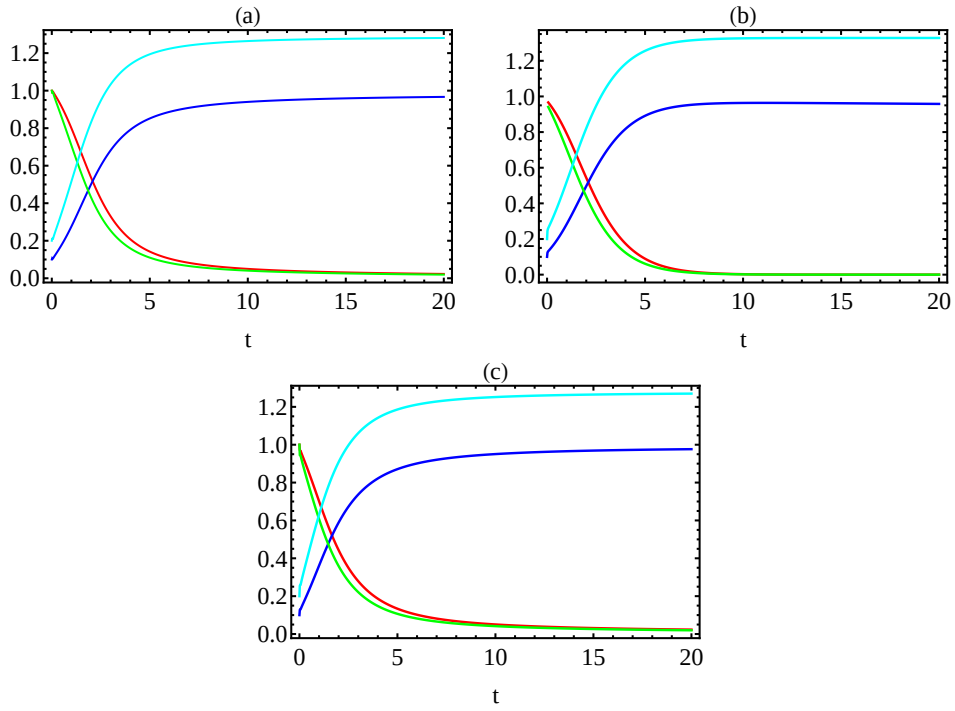

FIGURE 4. **Graph of the numerical solutions with $q = 1$ for $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).**

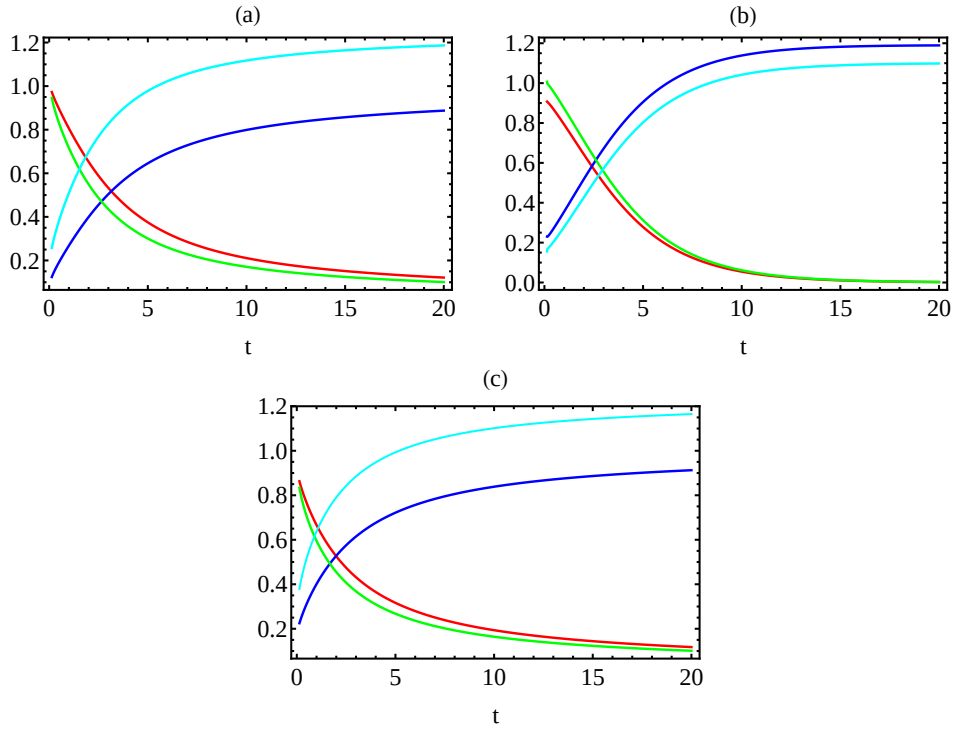

FIGURE 5. **Graph of the numerical solutions with $q = 1$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).**

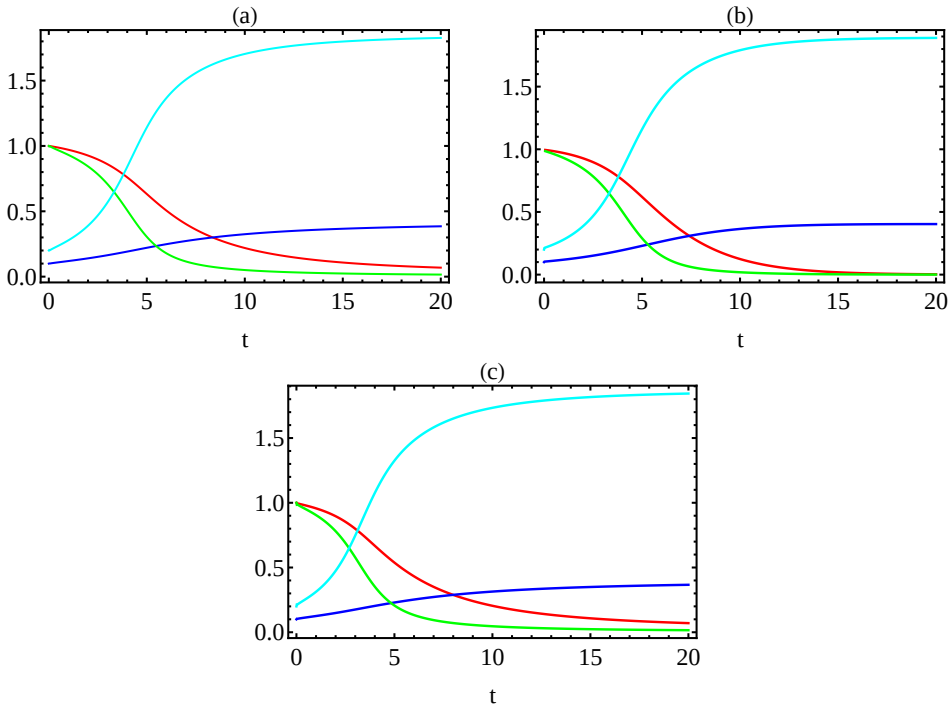

FIGURE 6. **Graph of the numerical solutions with $q = 1, 8$ for $\beta = 0.8, k = 1, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).**

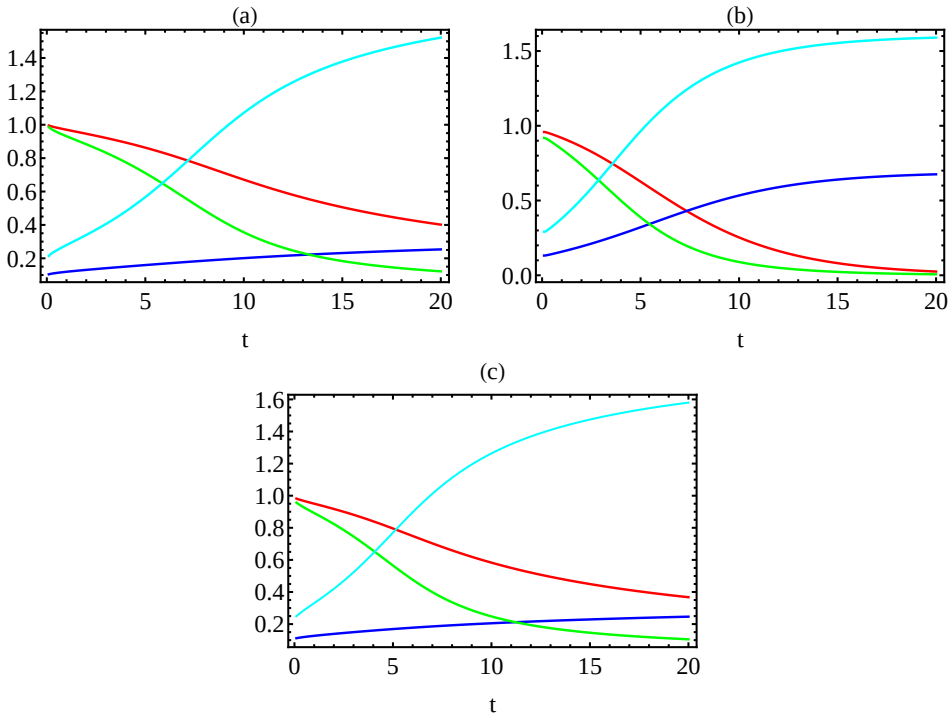

FIGURE 7. **Graph of the numerical solutions with $q = 1.8$ for $\beta = 0.7, k = 0.8, \gamma = 0.4, \kappa = 0.001, h = 0.01$ (a) FFP; (b) FFE; (c) FFM; (Red color: η_1; Blue color: ζ_1; Green color: η_2; Cyan color: ζ_2).**

6. Conclusion

In this paper, numerical solutions of the of the fractal-fractional reaction diffusion equations with general nonlinear have been studied. We introduced the FFRDE in three instances of fractional derivatives based on power, exponential, and Mittag-Leffler kernels. After that, we used the fundamental calculus, and with the help of Lagrange polynomial functions. We obtained the iterative and approximate formulas in the three cases. We studied the effect of the non-linear term order, in the case of cubic,quadratic, and fractional for different values of the fractal-fractional derivative order. The accuracy of the numerical solutions in the classic case of the FFRDE was tested in the case of power kernel, where all the numerical solutions in the classic case of integer order coincide to each other, and the comparison result has excellent agreement . In all calculations we used the Mathematica Program Package.

Acknowledgments

The authors would like to express their Gratitudes to the ministry of education and the deanship of scientific research Najran University Kingdom of Saudi Arabia for their financial and Technical support under code number (NU/ESCI/17/025)

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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