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On well-posedness of a magnetization-variables model for Navier-Stokes equations with damping

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Abstract

This paper aims to establish existence and uniqueness results of weak and strong solution to the three-dimensional periodic magnetization-variables formulation to Navier-Stokes equations with damping term:

$$\partial_t u - \nu\Delta u + (\mathbb{P}u\cdot\nabla)u + (\nabla u)^T u + \alpha|u|^{\beta-1}u = 0,$$

for the particular cases ($\beta = 3$ and $\nu > \frac{1}{2\alpha}$) and $\beta > 3$. Authors in precedent works addressed the question as to whether this model and similar ones possess a weak solution for $\alpha = 0$ (see [7,8]). In this vein, considering a damping term in the magnetization-variable formulation turned to be consequential as it enforces existence and uniqueness results. Energy methods, compactness methods are the main tools.

KEYWORDS:
Magnetization-variables, Damping term, Weak and Strong solutions

1 | INTRODUCTION

We consider the magnetization-variables formulation to Navier-Stokes equations:

$$\partial_t u - \nu\Delta u + (\mathbb{P}u\cdot\nabla)u + (\nabla u)^T u + \alpha|u|^{\beta-1}u = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^3 \tag{1}$$

$$u|_{t=0} = u_0(x),\ x\in\mathbb{T}^3 \quad , \tag{2}$$

where $\nu > 0$ is the viscosity, $\alpha > 0$ denotes the Darcy permeability of porous medium, $u(t,x)$ is an unknown three-dimensional vector-field which stands for velocity, $\mathbb{P}$ is the Leray projection of L^2 onto the closure of free-divergence function, and the system is subject to periodic boundary conditions with basic domain $\mathbb{T}^3 = [0, 2\pi L]^3$. In order to get it straight, we emphasize that u does not fulfill the free-divergence property while $\mathbb{P}u$ does. In fact, the Navier-Stokes equations can be reformulated as a system without a pressure term using the so-called magnetization-variable. In this formulation, previously discussed by Montgomery-Smith and Pokornỳ in [6], incompressibility is enforced explicitly via a Leray projection. The magnetization-variables formulation is more well-known in the study of the Euler equations (see e.g. [3]). In this paper, in addition to the damping term, the original system discussed in [6] is slightly modified as was done by Pooley in [7]. In [7], the author's motivation behind modifying the original system was approaching the Burgers equations, the solution of which satisfies a maximum principle (see [8,10,11]). The well-posedness of (1-2) when $\alpha = 0$ in critical Sobolev space $H^{1/2}(\mathbb{T}^3)$ was proved in [7]. It is worthwhile to note that when $\alpha = 0$, the fact that u does not fulfill the free-divergence condition prevents us from making the usual L^2 energy estimates that would give existence of weak solutions. The purpose of this paper is to establish the global in time existence of at least one weak solution to (1-2) when ($\beta = 3$ and $\nu\alpha > 1/2$). We also prove the local in time existence of strong solution. Furthermore, we prove the global in time existence of strong solution provided that the H^1-norm of initial data satisfies a smallness condition. While in the

case $\beta > 3$, we prove the global in time existence and uniqueness of weak and strong solution. Thus, introducing the damping term $\alpha|u|^{\beta-1}u$ in (1) turned to be consequential notably when $\beta > 3$, since it enforces existence and uniqueness of weak solution. The more general case $1 \leq \beta < 3$ which was investigated when considering three-dimensional incompressible convective Brinkman-Forchheimer equations or Navier-Stokes equation with damping (see e.g.[2,4,13]) cannot be tackled here. This is in fact another consequence of the lack of free-divergence condition which prevents the L^2-inner product of the term $(\nabla u)^T u$ against the solution u from vanishing as was the case of the convective term in three-dimensional Navier-Stokes equation.
Let us pause here to comment on the case $\beta = 3$. Our motivation behind considering $\beta = 3$ separately is firstly, the fact that it is critical case (not to be confused with the notion of critical spaces in which the norm of a solution is invariant under a certain scaling). Here, the term critical should be understood in the sense that when Ω is the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$, and if u satisfies equation (1) on $\mathbb{R}_+ \times \Omega$ then u_λ which is the usual parabolic rescaling of the velocity field u (i.e. $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0$) solves

$$\partial_t u_\lambda - \nu\Delta u_\lambda + (\mathbb{P}u_\lambda \cdot \nabla)u_\lambda + (\nabla u_\lambda)^T u_\lambda + \lambda^{3-\beta}\alpha|u_\lambda|^{\beta-1}u_\lambda = 0.$$

Note that when $\beta = 3$ in the above equation, one obtains the damped magnetization-variables equation (1) for the rescaled functions u_λ. Secondly, in order to ensure the existence of weak solutions when $\beta = 3$, and due to technical reasons we were constrained to add the assumption $\nu\alpha > 1/2$. From a physical perspective, this is in consistency with the fact that when both the viscosity of a fluid and the porosity of a porous medium are large enough, then the L^2-norm of the velocity remains bounded. In[7], the author was using the fact that for $\alpha = 0$, the system (1-2) exhibits conservation of momentum like the Navier-Stokes equations. Here, the solution's zero-mean property is not preserved due to the presence of the damping term. Therefore, the Poincaré inequality is not applicable, and we have to control the full H^1-norm instead. In fact, the damping term does not preserve the zero-mean property even in the case of convective Brinkman-Forchheimer equations, and coordination between weak and strong solution is necessary.

Before we state the main results, we give some notations that will be useful throughout this paper. The non-homogeneous Sobolev space is defined, for all $r \in \mathbb{R}$, by

$$H^r = \{f \in L^2(\mathbb{T}^3);\ \ u = \sum_{k\in\mathbb{Z}^3} \hat{f}_k e^{ik\cdot x},\ \ \|f\|_{H^r} < \infty\},$$

here the H^r norm is given as

$$\|f\|_{H^r} = \|f\|_{L^2} + \|\Lambda^r f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}^3} |\hat{f}_k|^2\right)^{1/2} + \left(\sum_{k\in\mathbb{Z}^3} |k|^{2r}|\hat{f}_k|^2\right)^{1/2},$$

where Λ stands for the fractional Laplacian $(-\Delta)^{1/2}$. Naturally, the homogeneous one is given by

$$\dot{H}^r = \{f \in L^2(\mathbb{T}^3);\ \ u = \sum_{k\in\mathbb{Z}^3} \hat{f}_k e^{ik\cdot x},\ \ \|f\|_{\dot{H}^r} < \infty\}$$

and endowed with the norm

$$\|f\|_{\dot{H}^r} = \|\Lambda^r f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}^3} |k|^{2r}|\hat{f}_k|^2\right)^{1/2},$$

where $\hat{f}_k$ is the k^{th}-Fourier coefficient of f. We give here the definition of weak solution for (1-2)

Definition 1. A function u is said to be a weak solution to the system (1-2) on $[0,T)$ with the initial condition $u_0 \in L^2(\mathbb{T}^3)$, if

$$u \in L^\infty(0,T;L^2) \cap L^{\beta+1}(0,T;L^{\beta+1}) \cap L^2(0,T;\dot{H}^1)$$

and

$$-\int_0^t \langle u(\tau), \partial_t\varphi(\tau)\rangle d\tau + \nu\int_0^t \langle \nabla u(\tau), \nabla\varphi(\tau)\rangle d\tau + \int_0^t \langle (\mathbb{P}u(\tau)\cdot\nabla)u(\tau), \varphi(\tau)\rangle d\tau$$
$$+ \int_0^t \langle (\nabla u(\tau))^T u(\tau), \varphi(\tau)\rangle d\tau + \alpha\int_0^t \langle |u(\tau)|^{\beta-1}u(\tau), \varphi(\tau)\rangle d\tau = -\langle u(t), \varphi(t)\rangle + \langle u(0), \varphi(0)\rangle, \tag{3}$$

for almost every $0 < t < T$ and all test functions $\varphi \in C^\infty([0,T)\times\mathbb{T}^3)$.
A function u is called a global weak solution if it is a weak solution for all $T > 0$.

At this point, we are ready to state the main results. Our first result is the following theorem

Theorem 1. Having $\beta = 3$ and $\nu\alpha > 1/2$, then

- Given u_0 in $L^2(\mathbb{T}^3)$, then there exist at least a global in time weak solution u to the system (1-2) such that
$$u \in L^\infty([0,\infty); L^2(\mathbb{T}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{T}^3)) \cap L^{\beta+1}(0,\infty; L^{\beta+1}(\mathbb{T}^3)).$$
- Given u_0 in $H^1(\mathbb{T}^3)$, then there exist at least a local in time strong solution u to the system (1-2) such that
$$u \in L^\infty([0,T); H^1(\mathbb{T}^3)) \cap L^2(0,T; H^2(\mathbb{T}^3)).$$
- There exists a constant $c > 0$, such that, if $\|u_0\|_{H^1} \leq c/\nu$, then a strong solution of (1) exists for all times $t \geq 0$

Our second result concerns the case $\beta > 3$, it reads

Theorem 2. Having $\beta > 3$, then

- Given u_0 in $L^2(\mathbb{T}^3)$, then there exists a unique global in time weak solution u to (1-2) such that
$$u \in L^\infty([0,\infty); L^2(\mathbb{T}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{T}^3)) \cap L^{\beta+1}(0,\infty; L^{\beta+1}(\mathbb{T}^3)).$$
- Given u_0 in $H^1(\mathbb{T}^3)$, then there exists a unique global in time strong solution u to the the system (1-2) such that
$$u \in L^\infty([0,\infty); H^1(\mathbb{T}^3)) \cap L^2(0,\infty; H^2(\mathbb{T}^3)).$$

In order to prove the existence of strong solutions, we will make use of the following lemma, the proof of which can be found in[9]:

Lemma 1. For $\beta \geq 1$, if $u \in H^2(\mathbb{T}^3)$, then it holds that
$$\int_{\mathbb{T}^3} -\Delta u |u|^{\beta-1} u dx \geq \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{\beta-1} dx.$$
Explicitly, the left-hand side of the above yields by integrating by parts:
$$\int_{\mathbb{T}^3} -\Delta u |u|^{\beta-1} u dx = \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{\beta-1} dx + \frac{\beta-1}{4}\int_{\mathbb{T}^3} \left|\nabla |u|^2\right|^2 |u|^{\beta-1} dx.$$

To prove our results, the main difficulty stems from the nonlinear terms, and more precisely the fact that in contrast with the convective Brinkman-Forchheimer the L^2-inner product of the second nonlinear term $(\nabla u)^T u$ against the solution u does not vanish. This issue constituted a constraint to prove existence of weak solution surmountable when $\beta \geq 3$ and insurmountable when $\beta < 3$. Another issue consists of establishing an *a priori* estimate for the time derivative of the approximate solution. This issue comes from the introduction of the damping term. To beat the odds, we use functional analysis, compactness methods, and when dealing with uniqueness problem we resort to Fourier analysis to handle the second nonlinear term.
The remainder of the paper is organized as follows; the second section is assigned to provide the proof of Theorem 1, while the third one is dedicated to prove Theorem 2.

2 | WELL-POSEDNESS RESULTS FOR $\beta = 3$

2.1 | Existence of weak solution

We will use the Galerkin approximation scheme. Let a_j be an orthonormal basis in L^2 made up of eigenfunctions of the Stokes operator $-\Delta$. The set $\{a_j\}$ is an orthogonal basis in H^1. We call the function
$$u_n(t,x) := \sum_{j\leq n} c_j^n aj(x),$$
the n^{th} Galerkin approximation of the solution for Eq. (1), if it satisfies the following system of equations $\forall j = 1, ..., n$
$$\begin{aligned}\frac{d}{dt}\langle u_n(t), a_j\rangle &= -\nu\langle \nabla u_n(t), \nabla a_j\rangle - \langle (\mathbb{P}u_n(t)\cdot\nabla)u_n(t), a_j\rangle \\ &-\langle (\nabla u_n(t))^T u_n(t), a_j\rangle - \alpha\langle |u_n(t)|^2 u_n(t), a_j\rangle\end{aligned} \tag{4}$$

with the initial condition

$$u_n(0,x) = P_n u_0 = \sum_{j\le n} \langle u_0, a_j\rangle a_j.$$

For some T_n, there exists a solution $u_n \in C^\infty([0,T_n)\times\mathbb{T}^3)$ to this finite-dimensional locally-Lipschitz system of ODEs. Let us begin by proving the existence of the global in time solution when $\nu > \frac{1}{2\alpha}$. To do so, let us multiply (2.1) by c_j^n, add these equations with respect to j from 1 to n to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \nu\|u_n\|_{\dot H^1}^2 + \alpha\langle |u_n|^2 u_n, u_n\rangle_{L^2} \le |\langle(\mathbb{P}u_n\cdot\nabla)u_n, u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle(\nabla u_n)^T u_n, u_n\rangle_{L^2(\mathbb{T}^3)}|.$$

By employing the anti-symmetry rule:

$$\langle(\mathbb{P}u_n\cdot\nabla)v_1, v_2\rangle_{L^2(\mathbb{T}^3)} = -\langle(\mathbb{P}u_n\cdot\nabla)v_2, v_1\rangle_{L^2(\mathbb{T}^3)},$$

we deduce that

$$\langle(\mathbb{P}u_n\cdot\nabla)u_n, u_n\rangle_{L^2(\mathbb{T}^3)} = 0.$$

Thus, it remains to control the second nonlinear term in the right hand side, which can be done the following way

$$\begin{aligned}|\langle(\nabla u_n)^T u_n, u_n\rangle_{L^2(\mathbb{T}^3)}| &\le \|u_n\|_{L^4}^2\|\nabla u_n\|_{L^2}\\ &\le \frac{\alpha}{2}\|u_n\|_{L^4}^4 + \frac{1}{2\alpha}\|\nabla u_n\|_{L^2}^2,\end{aligned}$$

where we used Hölder and Young inequalities. It follows that

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \nu\|u_n(t)\|_{\dot H^1}^2 + \frac{\alpha}{2}\|u_n\|_{L^4}^4 \le \frac{1}{2\alpha}\|\nabla u_n\|_{L^2}^2.$$

Since $\nu > \frac{1}{2\alpha}$, it turns out

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + (\nu - \frac{1}{2\alpha})\|u_n(t)\|_{\dot H^1}^2 + \frac{\alpha}{2}\|u_n(t)\|_{L^4}^4 \le 0. \tag{5}$$

Integrating over $(0,\infty)$ implies that

$$\|u_n(t)\|_{L^2}^2 + (2\nu - \frac{1}{\alpha})\int_0^\infty \|u_n(t)\|_{\dot H^1}^2 dt + \alpha\int_0^\infty \|u_n(t)\|_{L^4}^4 dt \le \|u_n(0)\|_{L^2}^2 \le \|u_0\|_{L^2}^2. \tag{6}$$

From (6) and standard results from ordinary differential equations theory, it follows that we can take $T_n = T$ for all n and for every $T > 0$. Moreover, we obtain from (6) that u_n is uniformly bounded in

$$L^\infty(0,T;L^2)\cap L^4(0,T;L^4)\cap L^2(0,T;\dot H^1).$$

Notice that

$$\left\||u_n|^2u_n\right\|_{L^{4/3}}^{4/3} = \|u_n\|_{L^4}^4. \tag{7}$$

This yields that $|u_n|^2u_n$ is uniformly bounded in $L^{4/3}(0,T;L^{4/3})$. It turns out then

$$\partial_t u_n - \nu\Delta u_n + P_n[(\mathbb{P}u_n\cdot\nabla)u_n + (\nabla u_n)^T u_n + \alpha|u_n|^2u_n] = 0. \tag{8}$$

In order to use the Aubin-Lions Lemma [5], we find uniform bounds on $\partial_t u_n$ in $L^{4/3}(0,T;H^{-1})$. To do so, we assume that $\varphi\in H^1$. We take the L^2-inner product of the equation (8) with $P_n\varphi$ to obtain

$$\langle\partial_t u_n, P_n\varphi\rangle = \nu\langle\Delta u_n, P_n\varphi\rangle_{L^2} - \langle(\mathbb{P}u_n\cdot\nabla)u_n, P_n\varphi\rangle_{L^2} - \langle(\nabla u_n)^T u_n, P_n\varphi\rangle_{L^2} - \alpha\langle|u_n|^2u, P_n\varphi\rangle_{L^2}, \tag{9}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of H^{-1} with H^1. We will now estimate the norm $\|\partial_t u_n\|_{H^{-1}}$ by estimating the right-hand side of (9). The first term can be estimated the following manner

$$|\langle\Delta u_n, P_n\varphi\rangle_{L^2}| \le \|u_n\|_{\dot H^1}\|P_n\varphi\|_{H^1}.$$

The convective nonlinear term is to be dealt with, as follows

$$\begin{aligned}|\langle(\mathbb{P}u_n\cdot\nabla)u_n, P_n\varphi\rangle_{L^2}| &\le \|u_n\|_{L^3}\|\nabla u_n\|_{L^2}\|P_n\varphi\|_{L^6}\\ &\le \|u_n\|_{L^2}^{1/2}\|u_n\|_{L^6}^{1/2}\|\nabla u_n\|_{L^2}\|P_n\varphi\|_{H^1}\\ &\le \|u_n\|_{L^2}^{1/2}\|u_n\|_{H^1}^{3/2}\|P_n\varphi\|_{H^1}\end{aligned}$$

where we used the fact that $\|f\|_{L^6} \le \|f\|_{H^1}$. Similarly, we have

$$\begin{aligned} |\langle (\nabla u_n)^T u_n, P_n\varphi\rangle_{L^2}| &\le \|u_n\|_{L^3}\|\nabla u_n\|_{L^2}\|P_n\varphi\|_{L^6} \\ &\le \|u_n\|_{L^2}^{1/2}\|u_n\|_{H^1}^{3/2}\|P_n\varphi\|_{H^1}. \end{aligned}$$

To control the absorption term, we proceed as follows

$$\begin{aligned} |\langle |u_n|^2 u_n, P_n\varphi\rangle_{L^2}| &\le \left\| |u_n|^2 u_n \right\|_{L^{4/3}} \|P_n\varphi\|_{L^4} \\ &\le \|u_n\|_{L^4}^3 \|P_n\varphi\|_{H^1}, \end{aligned}$$

where we used Hölder's inequality with exponents $4/3$ and 4, Eq. (7), and the fact that $\|P_n\varphi\|_{L^4} \le \|P_n\varphi\|_{L^2}^{1/4}\|P_n\varphi\|_{H^1}^{3/4} \le \|P_n\varphi\|_{H^1}$. By (9), it turns out that

$$\|\partial_t u_n\|_{H^{-1}} = \sup_{\|P_n\varphi\|_{H^1}=1} |\langle \partial_t u_n, P_n\varphi\rangle| \le \nu\|u_n\|_{\dot H^1} + 2\|u_n\|_{L^2}^{1/2}\|u_n\|_{H^1}^{3/2} + \alpha\|u_n\|_{L^4}^3.$$

Therefore we obtain

$$\begin{aligned} \int_0^T \|\partial_t u_n(t)\|_{H^{-1}}^{4/3} dt &\le \nu^{4/3}\int_0^T \|u_n(t)\|_{\dot H^1}^{4/3} dt \\ &+2\sup_{[0,T]}\|u_n(t)\|_{L^2}^{2/3}\int_0^T \|u_n(t)\|_{H^1}^2 dt + \alpha^{4/3}\int_0^T \|u_n\|_{L^4}^4 dt \\ \le \nu^{4/3}T^{1/3}\left\{\int_0^T \|u_n(t)\|_{\dot H^1}^2 dt\right\}^{2/3} &+ 2\sup_{[0,T]}\|u_n(t)\|_{L^2}^{2/3}\int_0^T \|u_n(t)\|_{H^1}^2 dt + \alpha^{4/3}\int_0^T \|u_n(t)\|_{H^1}^2 dt \\ &\le \nu^{4/3}T^{4/3}\|u_n\|_{L_T^2(\dot H^1)}^{4/3} + 2\|u_n\|_{L_T^\infty(L^2)}^{2/3}\|u_n\|_{L_T^2(H^1)}^2 + \alpha^{4/3}\|u_n(t)\|_{L_T^4(L^4)} \end{aligned} \tag{10}$$

From (6) and (2.1), there exist functions u and v, and a subsequence of (u_n) which we relabel also u_n, such that

$$u_n \to u \ \text{ weakly-}\star \text{ in } L^\infty(0,T;L^2) \tag{11}$$

$$u_n \to u \ \text{ weakly in } L^\infty(0,T;H^1) \tag{12}$$

$$u_n \to u \ \text{ weakly in } L^4(0,T;L^4) \tag{13}$$

$$|u_n|^2 u_n \to v \text{ weakly in } L^{4/3}(0,T;L^{4/3}) \tag{14}$$

$$\partial_t u_n \to \partial_t u \text{ weakly in } L^{4/3}(0,T;H^{-1}), \tag{15}$$

as $n \to \infty$. Then, Aubin-Lions lemma[5] allows to extract a subsequence of u_n such that

$$u_n \to u \ \text{ strongly in } L^2(0,T;L^2), \tag{16}$$

as $n \to \infty$. Now, we multiply (2.1) by $\varphi \in C^1([0,T])$, with $\varphi(T) = 0$, and then we integrate these equations over the interval $(0,T)$ to obtain

$$\begin{aligned} -\int_0^T \langle u_n(t), a_j\rangle \varphi'(t)dt + \nu\int_0^T \langle \nabla u_n(t), \nabla a_j\rangle\varphi(t)dt + \int_0^T \langle (\mathbb{P}u_n(t)\cdot\nabla)u_n(t), a_j\rangle\varphi(t)dt \\ + \int_0^T \langle (\nabla u_n(t))^T u_n(t), a_j\rangle\varphi(t)dt + \alpha\int_0^T \langle |u_n(t)|^2 u_n(t), a_j\rangle\varphi(t)dt = \langle u_n(0), a_j\rangle\varphi(0). \end{aligned} \tag{17}$$

Passing to the limit in the linear terms follows from the weak convergence in $L^2(0,T;H^1)$. Weak convergence in $L^2(0,T;H^1)$ and strong convergence in $L^2(0,T;L^2)$ allow us to pass to the limit in the first and the second nonlinear terms. As for the convergence in the damping term, we notice that by taking a new subsequence, we may assume that $u_n \to u$ a.e. in $[0,T)\times\mathbb{T}^3$. This implies that

$$|u_n|^2 u_n \to |u|^2 u \ \text{ a.e. in } [0,T)\times\mathbb{T}^3.$$

Using Lemma 1.3 in [5], the above convergence and (14) yield that $v = |u|^{\beta-1}u$. At this point, we are allowed to pass to the limit as $n \to \infty$ in (2.1) to obtain

$$
\begin{aligned}
&-\int_0^t \langle u(\tau), \partial_t \varphi(\tau)\rangle d\tau + \nu \int_0^t \langle \nabla u(\tau), \nabla \varphi(\tau)\rangle d\tau + \int_0^t \langle (\mathbb{P}u(\tau)\cdot\nabla)u(\tau), \varphi(\tau)\rangle d\tau \\
&+ \int_0^t \langle (\nabla u(\tau))^T u(\tau), \varphi(\tau)\rangle d\tau + \alpha \int_0^t \langle |u(\tau)|^2 u(\tau), \varphi(\tau)\rangle d\tau = -\langle u(t), \varphi(t)\rangle + \langle u(0), \varphi(0)\rangle.
\end{aligned}
$$

This finishes the proof of existence of at least one weak solution.

2.2 | Existence of strong solution

Multiplying (8) by $-\Delta u_n$ and integrating over $\mathbb{T}^3$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + \alpha \int_{\mathbb{T}^3} -\Delta u_n |u_n|^2 u_n dx \\
&\leq |\langle (\mathbb{P}u_n\cdot\nabla)u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle (\nabla u_n)^T u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}|.
\end{aligned}
$$

By using lemma 1, it turns out that

$$
\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + \alpha \int_{\mathbb{T}^3} |\nabla u_n|^2 |u_n|^2 dx \leq |\langle (\mathbb{P}u_n\cdot\nabla)u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle (\nabla u_n)^T u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}|. \tag{18}
$$

To control the first nonlinear term in the right-hand side of the above inequality, we use Hölder and Young inequalities to get

$$
\begin{aligned}
|\langle (\mathbb{P}u_n\cdot\nabla)u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}| &\leq \|u_n\|_{L^6}\|\nabla u_n\|_{L^3}\|\Delta u_n\|_{L^2} \\
&\leq \|u_n\|_{H^1}\|\nabla u_n\|_{L^2}^{1/2}\|\nabla u_n\|_{L^6}^{1/2}\|\Delta u_n\|_{L^2} \\
&\leq \|u_n\|_{H^1}^{3/2}\|\Delta u_n\|_{L^2}^{3/2} \\
&\leq c\|u_n\|_{H^1}^6 + \frac{\nu}{4}\|\Delta u_n\|_{L^2}^2.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
|\langle (\nabla u_n)^T u_n, -\Delta u_n\rangle_{L^2(\mathbb{T}^3)}| &\leq \|u_n\|_{L^6}\|\nabla u_n\|_{L^3}\|\Delta u_n\|_{L^2} \\
&\leq c\|u_n\|_{H^1}^6 + \frac{\nu}{4}\|\Delta u_n\|_{L^2}^2.
\end{aligned}
$$

Summing up (5) and (18) and dropping some terms from the left-hand side yield

$$
\frac{1}{2}\frac{d}{dt}\|u_n\|_{H^1}^2 + \frac{\nu}{2}\|\Delta u_n\|_{L^2}^2 + \alpha \int_{\mathbb{T}^3} |\nabla u_n|^2 |u_n|^2 dx \leq c\|u_n\|_{H^1}^6. \tag{19}
$$

Now, we compare $\|u_n\|_{H^1}^2$ with the solution of the problem

$$
\frac{dX}{dt} = cX^3, \quad X(0) = \|u_0\|_{H^1}^2,
$$

and deduce that for all t such that $0 \leq ct\|u_0\|_{H^1}^2 < 1$, we have

$$
\|u_n\|_{H^1}^2 \leq \frac{\|u_0\|_{H^1}^2}{\sqrt{1 - ct\|u_0\|_{H^1}^4}}.
$$

In particular, if we choose $T = 3/4c\|u_0\|_{H^1}^4$, then for all $t \in [0, T]$ we have the uniform upper bound

$$
\sup_{t\in[0,T]} \|u_n(t)\|_{H^1}^2 \leq 2\|u_0\|_{H^1}^2. \tag{20}
$$

Now we integrate (19) over $[0, t]$, where $0 \le t \le T$, to obtain

$$\|u_n(t)\|_{H^1}^2 + \nu \int_0^t \|\Delta u_n(\tau)\|_{L^2}^2 d\tau + \alpha \int_0^t \left(\int_{\mathbb{T}^3} |\nabla u_n(\tau)|^2 |u_n(\tau)|^2 dx \right) d\tau < \infty. \tag{21}$$

These are uniform bounds on the approximate solution u_n in $L^\infty([0,T]; H^1(\mathbb{T}^3)) \cap L^2(0,T; H^2(\mathbb{T}^3))$. Furthermore, we have

$$\int_0^t \left(\int_{\mathbb{T}^3} |\nabla u_n(\tau)|^2 |u_n(\tau)|^2 dx \right) d\tau < \infty. \tag{22}$$

Here, we have the same situation as for strong solutions of the Navier-Stokes equations. However, for the damped magnetization-variables formulation, we have the additional bound (22). We turn now to the case when the H^1-norm of initial data is small as required. Specifically, we prove the global in time existence of strong solutions to the system (1-2), subject to small initial data. Taking the inner product of (8) against $-\Delta u_n$ to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu \|\Delta u_n\|_{L^2}^2 + \alpha \int_{\mathbb{T}^3} |\nabla u_n|^2 |u_n|^2 dx \le |\langle (\mathbb{P}u_n \cdot \nabla) u_n, -\Delta u_n \rangle_{L^2(\mathbb{T}^3)}| + |\langle (\nabla u_n)^T u_n, -\Delta u_n \rangle_{L^2(\mathbb{T}^3)}|.$$

To control the first term in the right hand side, we write down the following steps

$$\begin{aligned} |\langle (\mathbb{P}u_n \cdot \nabla) u_n, -\Delta u_n \rangle_{L^2(\mathbb{T}^3)}| &\le \|u_n\|_{L^6} \|\nabla u_n\|_{L^3} \|\Delta u_n\|_{L^2} \\ &\le c\|u_n\|_{H^1} \|\nabla u_n\|_{L^2}^{1/2} \|\nabla u_n\|_{L^6}^{1/2} \|\Delta u_n\|_{L^2} \\ &\le c\|u_n\|_{H^1} \|\nabla u_n\|_{L^2}^{1/2} \|\nabla u_n\|_{H^1}^{1/2} \|\Delta u_n\|_{L^2} \\ &\le \tilde{c}\|u_n\|_{H^1} \|\Delta u_n\|_{L^2}^2, \end{aligned}$$

where we used Hölder's inequality, the interpolation argument $\|f\|_{L^3} \le \|f\|_{L^2}^{1/2} \|f\|_{L^6}^{1/2}$, the embedding $H^1 \hookrightarrow L^6$ and the facts that $(\|\nabla u_n\|_{L^2} \le \|\Delta u_n\|_{L^2},\ \|\nabla u_n\|_{H^1} \le \tilde{c}\|\Delta u_n\|_{L^2})$. Similarly, we get

$$\begin{aligned} |\langle (\nabla u_n)^T u_n, -\Delta u_n \rangle_{L^2(\mathbb{T}^3)}| &\le \|u_n\|_{L^6} \|\nabla u_n\|_{L^3} \|\Delta u_n\|_{L^2} \\ &\le c\|u_n\|_{H^1} \|\Delta u_n\|_{L^2}^2. \end{aligned}$$

Consequently, we obtain

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \left(\nu - c\|u_n\|_{H^1}\right) \|\Delta u_n\|_{L^2}^2 \le 0. \tag{23}$$

Starting from an initial data such that $\|u_0\|_{H^1} \le \nu/c$, yields

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 \le 0,$$

which means that $\|\nabla u_n\|_{L^2}$ is a non-increasing function of time. Hence, it is bounded for all times $t \ge 0$ and the strong solution u does not blow up.

3 | WELL-POSEDNESS AND CONVERGENCE RESULTS FOR $\beta > 3$

3.1 | Existence of weak solution

We consider the orthonormal in L^2 basis a_j that is made up of eigenfunctions of the Stokes operator $-\Delta$. Here, the set $\{a_j\}$ is an orthogonal basis in H^2 rather than H^1. In such a way, we can handle the damping term $|u|^{\beta-1}u$ for all exponents $\beta > 3$. Since in three-dimensional torus, the embedding $H^2 \hookrightarrow L^p$ holds true for every $p \ge 1$. We call the function

$$u_n(t,x) := \sum_{j \le n} c_j^n aj(x),$$

the n^{th} Galerkin approximation of the solution for Eq. (1), if it satisfies the following system of equations $\forall j = 1, ..., n$

$$\begin{aligned} \frac{d}{dt}\langle u_n(t), a_j \rangle &= -\nu \langle \nabla u_n(t), \nabla a_j \rangle - \langle (\mathbb{P}u_n(t) \cdot \nabla) u_n(t), a_j \rangle \\ &\quad - \langle (\nabla u_n(t))^T u_n(t), a_j \rangle - \alpha \langle |u_n(t)|^{\beta-1} u_n(t), a_j \rangle \end{aligned} \tag{24}$$

with the initial condition

$$u_n(0,x) = P_n u_0 = \sum_{j\le n} \langle u_0, a_j\rangle a_j.$$

For some T_n, there exists a solution $u_n \in C^\infty([0,T_n)\times \mathbb{T}^3)$ to this finite-dimensional locally-Lipschitz system of ODEs. Let us begin by proving the existence of a global in time weak solution. To do so, let us multiply (3.1) for $\beta > 3$ by c_j^n, add these equations with respect to j from 1 to n to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \nu\|u_n\|_{\dot H^1}^2 + \alpha\langle |u_n|^{\beta-1}u_n, u_n\rangle_{L^2} \le |\langle (\nabla u_n)^T u_n, u_n\rangle_{L^2(\mathbb{T}^3)}|,$$

where we used the fact that $\langle (\mathbb{P}u_n\cdot\nabla)u_n, u_n\rangle_{L^2(\mathbb{T}^3)} = 0$. To control the nonlinear term in the right hand side of the above estimate, we write down the following steps

$$\begin{aligned}
|\langle (\nabla u_n)^T u_n, u_n\rangle_{L^2(\mathbb{T}^3)}| &\le \|u_n\|_{L^{\beta+1}}\|\nabla u_n\|_{L^2}\|u_n\|_{L^{\frac{2\beta+2}{\beta-1}}}\\
&\le \|u_n\|_{L^{\beta+1}}\|\nabla u_n\|_{L^2}\|u_n\|_{L^2}^{\gamma}\|u_n\|_{L^{\beta+1}}^{1-\gamma}\\
&\le \|u_n\|_{L^{\beta+1}}^{2-\gamma}\|u_n\|_{L^2}^{\gamma}\|\nabla u_n\|_{L^2}\\
&\le \frac{1}{2\nu}\left\{\|u_n\|_{L^{\beta+1}}^{2-\gamma}\|u_n\|_{L^2}^{\gamma}\right\}^2 + \frac{\nu}{2}\|\nabla u_n\|_{L^2}^2\\
&= \frac{1}{2\nu}\|u_n\|_{L^{\beta+1}}^{4-2\gamma}\|u_n\|_{L^2}^{2\gamma} + \frac{\nu}{2}\|\nabla u_n\|_{L^2}^2\\
&\le \tilde C(\nu,\alpha,\beta)\|u_n\|_{L^2}^2 + \frac{\alpha}{2}\|u_n\|_{L^{\beta+1}}^{\beta+1} + \frac{\nu}{2}\|\nabla u_n\|_{L^2}^2,
\end{aligned}$$

where we used Hölder's inequality, the Lebesgue Interpolation argument $\|u_n\|_{L^{\frac{2\beta+2}{\beta-1}}} \le \|u_n\|_{L^2}^{\gamma}\|u_n\|_{L^{\beta+1}}^{1-\gamma}$, where $\gamma = \frac{\beta-3}{\beta-1}$, and Young's inequality ($ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $\frac{1}{p}+\frac{1}{q} = 1$, $1 \le p \le q \le \infty$) twice. It follows that

$$\frac{d}{dt}\|u_n(t)\|_{L^2}^2 + \nu\|u_n(t)\|_{\dot H^1}^2 + \alpha\|u_n(t)\|_{L^{\beta+1}}^{\beta+1} \le C\|u_n(t)\|_{L^2}^2, \tag{25}$$

where C is a positive constant that depends on ν, α and β. By applying the Grönwall's inequality, we get

$$\|u_n(t)\|_{L^2}^2 \le \|u_n(0)\|_{L^2}^2 \exp\{Ct\} \le \|u_0\|_{L^2}^2\exp\{Ct\}. \tag{26}$$

Integrating estimate (25) over $[0,T_n]$ yields

$$\begin{aligned}
\sup_{t\in[0,T_n]}\|u_n(t)\|_{L^2}^2 + \nu\int_0^{T_n}\|u_n(t)\|_{\dot H^1}^2\,dt + \alpha\int_0^{T_n}\|u_n(t)\|_{L^{\beta+1}}^{\beta+1}\,dt&\\
\le \|u_0\|_{L^2}^2 + C^\star\|u_0\|_{L^2}^2\exp\left\{CT_n\right\}.&
\end{aligned}\tag{27}$$

From (3.1) and standard ODEs argument, it follows that we can take $T_n = T$ for all n and for every $T > 0$. Moreover, we deduce from (3.1) that u_n is uniformly bounded in

$$L^\infty(0,T;L^2)\cap L^{\beta+1}(0,T;L^{\beta+1})\cap L^2(0,T;\dot H^1).$$

Notice that

$$\left\||u_n|^{\beta-1}u_n\right\|_{L^{(\beta+1)^*}}^{(\beta+1)^*} = \|u_n\|_{L^{\beta+1}}^{\beta+1}, \tag{28}$$

where $(\beta+1)^* = (\beta+1)/\beta$. It turns out that $|u_n|^{\beta-1}u_n$ is uniformly bounded in $L^{(\beta+1)^*}(0,T;L^{(\beta+1)^*})$. It follows that

$$\partial_t u_n - \nu\Delta u_n + P_n[(\mathbb{P}u_n\cdot\nabla)u_n + (\nabla u_n)^T u_n + \alpha|u_n|^{\beta-1}u_n] = 0. \tag{29}$$

We need to find uniform bounds on $\partial_t u_n$ in $L^{(\beta+1)^*}(0,T;H^{-2})$. To do so, let us consider $\psi \in H^2(\mathbb{T}^3)$. We take the L^2-inner product of the equation (29) against $P_n\psi$ to obtain

$$\langle \partial_t u_n, P_n\psi\rangle = \nu\langle \Delta u_n, P_n\psi\rangle_{L^2} - \langle (\mathbb{P}u_n\cdot\nabla)u_n, P_n\psi\rangle_{L^2} - \langle (\nabla u_n)^T u_n, P_n\psi\rangle_{L^2} - \alpha\langle |u_n|^{\beta-1}u_n, P_n\psi\rangle_{L^2}, \tag{30}$$

Controlling the linear terms as well as the terms $(\mathbb{P}u_n\cdot\nabla)u_n$ and $(\nabla u_n)^T u_n$ can be done the same as previously (i.e. as for $\beta = 3$). So, let us turn our attention to our particular concern which is the damping term, it holds that

$$\begin{aligned}
|\langle |u_n|^{\beta-1}u_n, P_n\psi\rangle_{L^2}| &\le \left\||u_n|^{\beta-1}u_n\right\|_{L^{(\beta+1)^*}}\|P_n\psi\|_{L^{\beta+1}}\\
&\le c\|u_n\|_{L^{\beta+1}}^{\beta}\|P_n\psi\|_{H^2},
\end{aligned}$$

where we used Hölder's inequality with exponents $(\beta+1)^*$ and $\beta+1$, identity (28), and the embedding $H^2 \hookrightarrow L^{\beta+1}$. By using (29), we deduce that $\partial_t u_n$ remains bounded in $L^{(\beta+1)^*}(0,T;H^{-2})$. Thus, Aubin-Lions lemma[5] allows to extract a subsequence u_n that converges strongly to u in $L^2(0,T;L^2)$.
Now, we multiply (3.1) by $\psi \in C^1([0,T])$, with $\psi(T)=0$, and then we integrate these equations over the time interval $(0,T)$ to obtain

$$-\int_0^T \langle u_n(t), a_j\rangle \psi'(t)dt + \nu \int_0^T \langle \nabla u_n(t), \nabla a_j\rangle \psi(t)dt + \int_0^T \langle (\mathbb{P}u_n(t)\cdot\nabla)u_n(t), a_j\rangle \psi(t)dt$$
$$+\int_0^T \langle (\nabla u_n(t))^T u_n(t), a_j\rangle \psi(t)dt + \alpha \int_0^T \langle |u_n(t)|^{\beta-1}u_n(t), a_j\rangle \psi(t)dt = \langle u_n(0), a_j\rangle \psi(0). \tag{31}$$

Passing to the limit in the above yields

$$-\int_0^t \langle u(\tau), \partial_t\varphi(\tau)\rangle d\tau + \nu \int_0^t \langle \nabla u(\tau), \nabla\varphi(\tau)\rangle d\tau + \int_0^t \langle (\mathbb{P}u(\tau)\cdot\nabla)u(\tau), \varphi(\tau)\rangle d\tau$$
$$+\int_0^t \langle (\nabla u(\tau))^T u(\tau), \varphi(\tau)\rangle d\tau + \alpha \int_0^t \langle |u(\tau)|^{\beta-1}u(\tau), \varphi(\tau)\rangle d\tau = -\langle u(t), \varphi(t)\rangle + \langle u(0), \varphi(0)\rangle.$$

It should be pointed out that the main ingredients allowing us to pass to the limit were as in the previous case (i.e. $\beta=3$), the strong convergence in $L^2(0,T;L^2)$ and the fact that $|u_n|^{\beta-1}u_n \to |u|^{\beta-1}u$, as $n\to\infty$ a.e. in $[0,T]\times\mathbb{T}^3$. At this point, the global in time existence of weak solution is established for $\beta>3$.

3.2 | Existence of strong solution

We take the inner product of (29) against $-\Delta u_n$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + \alpha\int_{\mathbb{T}^3} -\Delta u_n |u_n|^{\beta-1}u_n dx \le |\langle(\mathbb{P}u_n\cdot\nabla)u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle(\nabla u_n)^T u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}|.$$

By using lemma 1, it turns out that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + \alpha\int_{\mathbb{T}^3} |\nabla u_n|^2 |u_n|^{\beta-1} dx \le |\langle(\mathbb{P}u_n\cdot\nabla)u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle(\nabla u_n)^T u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}|. \tag{32}$$

To control the nonlinear term in the right-hand side of the above inequality, we use Hölder and Young inequalities to get

$$\begin{aligned}|\langle(\mathbb{P}u_n\cdot\nabla)u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}| + |\langle(\nabla u_n)^T u_n, \Delta u_n\rangle_{L^2(\mathbb{T}^3)}| &\le 2\int_{\mathbb{T}^3} |u_n|\,|\nabla u_n|\,|\Delta u_n| dx \\ &\le \frac{2}{\nu}\int_{\mathbb{T}^3} |u_n|^2|\nabla u_n|^2 dx + \frac{\nu}{2}\int_{\mathbb{T}^3} |\Delta u_n|^2 dx.\end{aligned}$$

It turns out

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + 2\alpha\int_{\mathbb{T}^3} |\nabla u_n|^2|u_n|^4 dx \le \frac{4}{\nu}\int_{\mathbb{T}^3} |u_n|^2|\nabla u_n|^2 dx. \tag{33}$$

Let us make the following observation

$$\begin{aligned}
\int_{\mathbb{T}^3} |u_n|^2|\nabla u_n|^2 dx &= \int_{\mathbb{T}^3} \left(|u_n|^2|\nabla u_n|^{\frac{4}{\beta-1}}\right)\left(|\nabla u_n|^{2\frac{\beta-3}{\beta-1}}\right) dx \\
&\leq \left(\int_{\mathbb{T}^3} |u_n|^{\beta-1}|\nabla u_n|^2 dx\right)^{\frac{2}{\beta-1}} \left(\int_{\mathbb{T}^3} |\nabla u_n|^2 dx\right)^{\frac{\beta-3}{\beta-1}} \\
&\leq \frac{\alpha\nu}{4}\int_{\mathbb{T}^3} |u_n|^{\beta-1}|\nabla u_n|^2 dx + c(\nu,\alpha,\beta)\int_{\mathbb{T}^3} |\nabla u_n|^2 dx, \qquad (34)
\end{aligned}$$

where we used Hölder's and Young's inequalities. By plugging the estimate (34) into (35), we obtain

$$\frac{d}{dt}\|\nabla u_n\|_{L^2}^2 + \nu\|\Delta u_n\|_{L^2}^2 + \alpha\int_{\mathbb{T}^3} |u_n|^{\beta-1}|\nabla u_n|^2 dx \leq \frac{c}{\nu}\|\nabla u_n\|_{L^2}^2. \qquad (35)$$

In particular, an application of Grönwall's inequality yields

$$\|\nabla u_n(t)\|_{L^2}^2 \leq \frac{c}{\nu}\|\nabla u_n(0)\|_{L^2}^2 \exp\left\{\frac{ct}{\nu}\right\}.$$

Summing up (25) and (35) to get

$$\begin{aligned}
\frac{d}{dt}\|u_n\|_{H^1}^2 + \nu\left(\|\nabla u_n\|_{L^2}^2 + \|\Delta u_n\|_{L^2}^2\right) + \alpha\left(\|u_n\|_{L^6}^6 + \int_{\mathbb{T}^3} |u_n|^{\beta-1}|\nabla u_n|^2 dx\right) \\
\leq C\|u_n\|_{L^2}^2 + \frac{c}{\nu}\|\nabla u_n\|_{L^2}^2 \\
\leq \|u_0\|_{L^2}^2 + C^\star\|u_0\|_{L^2}^2 \exp\{Ct\} + \frac{c}{\nu}\|\nabla u_0\|_{L^2}^2 \exp\left\{\frac{ct}{\nu}\right\} < \infty. \qquad (36)
\end{aligned}$$

It follows in particular, that u_n is uniformly bounded in $L^\infty(0,T;H^1)$. Then one infers from (3.2) that $\int_0^T \|\Delta u_n\|_{L^2}^2 dt < \infty$. Therefore, we deduce the existence of strong solution on the time interval $[0,T]$ for all $T > 0$.

3.3 | Continuous dependence on initial data and uniqueness

We show the continuous dependence of the weak solutions on the initial data, in particular their uniqueness when $\beta > 3$. Let u and v be two solutions to (1) associated to the initial data u_0 and v_0. More explicitly, $u = u(t,x)$ and $v = v(t,x)$ satisfy the following two systems

$$\begin{cases} \partial_t u - \nu\Delta u + (\mathbb{P}u\cdot\nabla)u + (\nabla u)^T u + \alpha|u|^{\beta-1}u = 0, \\ u|_{t=0} = u_0(x), \end{cases}$$

and

$$\begin{cases} \partial_t v - \nu\Delta v + (\mathbb{P}v\cdot\nabla)v + (\nabla v)^T v + \alpha|v|^{\beta-1}v = 0, \\ v|_{t=0} = v_0(x). \end{cases}$$

Let $w := u - v$, subtracting the equation satisfied by v from the one satisfied by u, and taking the inner product against w to obtain

$$\begin{aligned}
&\langle\partial_t w, w\rangle - \nu\langle\Delta w, w\rangle_{L^2} + \alpha\int_{\mathbb{T}^3}\left(|u|^{\beta-1}u - |v|^{\beta-1}v\right) w dx \\
&+\langle(\mathbb{P}u\cdot\nabla)w + (\nabla u)^T w, w\rangle_{L^2} + \langle(\mathbb{P}w\cdot\nabla)v + (\nabla w)^T v, w\rangle_{L^2} = 0. \qquad (37)
\end{aligned}$$

We need to use the following monotonicity inequality (see e.g.[1]):

$$c|u-v|^2\left(|u|+|v|\right)^{\beta-1} \leq \left(|u|^{\beta-1}u - |v|^{\beta-1}v\right)(u-v)$$

By employing the fact that $\mathbb{P}u$ fulfills the free-divergence property, we get

$$\langle(\mathbb{P}u\cdot\nabla)w, w\rangle_{L^2} = 0.$$

Let us control the second nonlinear term appearing in estimate (3.3) which involves $(\nabla u)^T w$. To do so, we write down the following steps

$$|\langle(\nabla u)^T w, w\rangle_{L^2}| \leq \sum_{k\in\mathbb{Z}^3} \left|\mathcal{F}\left((\nabla u)^T w\right)(k)\right| \left|\hat{w}_k\right|.$$

Our strategy consists of transferring the gradient that is applied on u into w. It should be pointed out that in the case of convective Brinkman-Forchheimer, damped NS equations, and MHD model in porous media ([2,4,13,12]), this problematic does not arise as the solution is divergent-free. Indeed, we have

$$\begin{aligned}
\left|\mathcal{F}\left((\nabla u)^T w\right)(k)\right| &= |(\mathcal{F}(\nabla u) * \mathcal{F}(w))(k)| \\
&= \left| i \sum_p \hat{w}_p (k-p)\hat{u}_{(k-p)}\right| \\
&\leq \sum_p |\hat{w}_p||k-p||\hat{u}_{(k-p)}| \\
&\leq \sum_p |\hat{w}_p|\left(|k|+|p|\right)|\hat{u}_{(k-p)}| \\
&\leq |k| \sum_p |\hat{w}_p||\hat{u}_{(k-p)}| \\
&+ \sum_p |p||\hat{w}_p||\hat{u}_{(k-p)}| \\
&\leq |k|\mathcal{F}(\vartheta_1) * \mathcal{F}(\vartheta_2) + \mathcal{F}(\vartheta_3) * \mathcal{F}(\vartheta_2),
\end{aligned}$$

where $\vartheta_1 = \mathcal{F}^{-1}(|\hat{w}_k|)$, $\vartheta_2 = \mathcal{F}^{-1}(|\hat{u}_k|)$, and $\vartheta_3 = \mathcal{F}^{-1}(|k|\ |\hat{w}_k|)$. It turns out

$$\begin{aligned}
|\langle(\nabla u)^T w, w\rangle_{L^2}| &\leq \sum_{k\in\mathbb{Z}^3} \mathcal{F}\left(\vartheta_1 \cdot \vartheta_2\right)(k)\ |k|\ |\hat{w}_k| \\
&+ \sum_{k\in\mathbb{Z}^3} \mathcal{F}\left(\vartheta_3 \cdot \vartheta_2\right)(k)\left|\hat{w}_k\right| \\
&\leq |\langle \vartheta_1\cdot\vartheta_2, \vartheta_3\rangle_{L^2}| + |\langle \vartheta_3\cdot\vartheta_2, \vartheta_1\rangle_{L^2}| \\
&\leq 2\|\vartheta_3\|_{L^2}\|\vartheta_1\cdot\vartheta_2\|_{L^2} \\
&\leq 2\|\nabla w\|_{L^2}\||u|\ |w|\|_{L^2} \\
&\leq 2\|\nabla w\|_{L^2}\||u|\ |w|^{\frac{2}{\beta-1}}|w|^{1-\frac{2}{\beta-1}}\|_{L^2} \\
&\leq 2\|\nabla w\|_{L^2}\||u|\ |w|^{\frac{2}{\beta-1}}\|_{L^{\beta-1}}\||w|^{1-\frac{2}{\beta-1}}\|_{L^{2\frac{\beta-1}{\beta-3}}} \\
&\leq \frac{\nu}{6}\|\nabla w\|^2_{L^2} + \frac{6}{\nu}\||u|\ |w|^{\frac{2}{\beta-1}}\|^2_{L^{\beta-1}}\||w|^{1-\frac{2}{\beta-1}}\|^2_{L^{2\frac{\beta-1}{\beta-3}}} \\
&= \frac{\nu}{6}\|\nabla w\|^2_{L^2} + \frac{6}{\nu}\||u|^{\frac{\beta-1}{2}}\ |w|\|^{\frac{4}{\beta-1}}_{L^2}\|w\|^{2\frac{\beta-3}{\beta-1}}_{L^2} \\
&\leq \frac{\nu}{6}\|\nabla w\|^2_{L^2} + \frac{c\alpha}{6}\||u|^{\frac{\beta-1}{2}}\ |w|\|^2_{L^2} + C(\nu,\alpha,\beta)\|w\|^2_{L^2} \\
&\leq \frac{\nu}{6}\|\nabla w\|^2_{L^2} + \frac{c\alpha}{6}\int_{\mathbb{T}^3} |u-v|^2\left(|u|+|v|\right)^{\beta-1}dx + C\|w\|^2_{L^2},
\end{aligned}$$

where we employed Hölder inequality and Young inequality several times. By using the anti-symmetry rule, we get

$$
\begin{aligned}
|\langle(\mathbb{P}w\cdot\nabla)v, w\rangle_{L^2}| &= |\langle(\mathbb{P}w\cdot\nabla)w, v\rangle_{L^2}| \\
&\leq \|\nabla w\|_{L^2}\| |v|\ |w| \|_{L^2} \\
&\leq \|\nabla w\|_{L^2}\| |v|\ |w|^{\frac{2}{\beta-1}} |w|^{1-\frac{2}{\beta-1}} \|_{L^2} \\
&\leq \|\nabla w\|_{L^2}\| |v|\ |w|^{\frac{2}{\beta-1}} \|_{L^{\beta-1}} \| |w|^{1-\frac{2}{\beta-1}} \|_{L^{2\frac{\beta-1}{\beta-3}}} \\
&\leq \frac{\nu}{6}\|\nabla w\|_{L^2}^2 + \frac{3}{2\nu}\| |v|\ |w|^{\frac{2}{\beta-1}} \|_{L^{\beta-1}}^2 \| |w|^{1-\frac{2}{\beta-1}} \|_{L^{2\frac{\beta-1}{\beta-3}}}^2 \\
&= \frac{\nu}{6}\|\nabla w\|_{L^2}^2 + \frac{3}{2\nu}\| |v|^{\frac{\beta-1}{2}}\ |w| \|_{L^2}^{\frac{4}{\beta-1}} \|w\|_{L^2}^{2\frac{\beta-3}{\beta-1}} \\
&\leq \frac{\nu}{6}\|\nabla w\|_{L^2}^2 + \frac{c\alpha}{6}\| |v|^{\frac{\beta-1}{2}}\ |w| \|_{L^2}^2 + \tilde{C}(\nu,\alpha,\beta)\|w\|_{L^2}^2 \\
&\leq \frac{\nu}{6}\|\nabla w\|_{L^2}^2 + \frac{c\alpha}{6}\int_{\mathbb{T}^3} |u-v|^2\,(|u|+|v|)^{\beta-1}\,dx + \tilde{C}\|w\|_{L^2}^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|\langle(\nabla w)^T v, w\rangle_{L^2}| &\leq \|\nabla w\|_{L^2}\| |v|\ |w| \|_{L^2} \\
&\leq \frac{\nu}{6}\|\nabla w\|_{L^2}^2 + \frac{c\alpha}{6}\int_{\mathbb{T}^3} |u-v|^2\,(|u|+|v|)^{\beta-1}\,dx + \tilde{C}\|w\|_{L^2}^2.
\end{aligned}
$$

In particular, we get by applying the Gronwall's inequality

$$
\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 \exp\left\{C^{\star}t\right\}. \tag{38}
$$

Hence, the continuous dependence of the weak solution on the initial data follows, in particular we get uniqueness when $u_0 = v_0$. We have already proved that weak solution is unique. As strong solutions are also weak, one deduces uniqueness of strong solution. This finishes the proof of theorem 2.

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