Optimal three-ball inequality, quantitative uniqueness for the bi-Laplace equations*

Xiaoyu Fu †and Zhonghua Liao‡

Abstract

In this paper, we prove an optimal three-ball inequality for y satisfying an equation of the form

$$\Delta^2 y = V_0 y + V_1 \cdot \nabla y + V_2 \Delta y + V_3 \cdot \nabla \Delta y$$

in some open, connected set Ω of $\mathbb{R}^n$ with $V_0,\ V_2 \in L^\infty(\Omega;\mathbb{C})$ and $V_1,\ V_3 \in L^\infty(\Omega;\mathbb{C}^n)$. The derivation of such estimate relies on a delicate Carleman estimate for the bi-Laplace equation and some Caccioppoli inequalities to estimate the lower-terms. Based on three-ball inequality, we then derive the vanishing order of y is less than $C\Big(|V_0|_\infty^{\frac{1}{3}} + |V_1|_\infty^{\frac{1}{2}} + |V_2|_\infty^{\frac{2}{3}} + |V_3|_\infty^{2}\Big)$, where $|\cdot|_\infty$ means the L^∞ norm, which is a quantitative version of the strong unique continuation property for y. Furthermore, under some priori assumptions on V_j and y, we prove that the nontrivial solution y satisfies the decay property $e^{-CR^2 \log R}$ around the point at infinity. In particular, if $V_1 = V_3 = (0, \cdots, 0)$, this decaying rate can be improved to $e^{-CR^{4/3} \log R}$.

2000 Mathematics Subject Classification. 35B60; 35R25; 35K10.

Key Words. Three-ball inequality, Quantitative uniqueness, Bi-Laplace operator, Carleman estimate.

1 Introduction

Let Ω be a connected open subset of $\mathbb{R}^n$ $(n \geq 2)$. Without loss of generality, we assume that $0 \in \Omega$ and denote $\mathcal{B}_r \overset{\triangle}{=} \{x \in \mathbb{R}^n |\ |x| < r\}$. For any complex number c, we denote by $\bar{c}$ and $\mathrm{Re}\, c$, its complex conjugate and real part, respectively.

In this paper we are interested in y satisfying the following equation:

$$\Delta^2 y = V_0 y + V_1 \cdot \nabla y + V_2 \Delta y + V_3 \cdot \nabla \Delta y, \tag{1.1}$$

*This work is partially supported by the NSF of China under grant 11971333, 11931011.

†School of Mathematics, Sichuan University, Chengdu 610064, China. E-mail address: xiaoyufu@scu.edu.cn

‡School of Mathematics, Sichuan University, Chengdu 610064, China. E-mail address: zhonghualiao@yeah.net

where $V_0,\ V_2 \in L^\infty(\Omega;\mathbb{C})$ and $V_1,\ V_3 \in L^\infty(\Omega;\mathbb{C}^n)$.

The main purpose of this paper is to investigate optimal three-ball inequality of equation (1.1) and its applications. In the literature, such kind of inequality may date back to the following classic Hadamard three-circle theorem (e.g. [12]):

Let $f(z)$ be a complex-valued analytic function defined on the closed annulus $\{z\in\mathbb{C}: r_1\le|z|\le r_2\}$. Denote $||f||_r^2=\int_{|z|=r}|f(z)|^2dz$. Then,

$$||f||_r \le ||f||_{r_1}^\alpha ||f||_{r_2}^{1-\alpha},$$

where $r_1<r<r_2$ and $\alpha=\log(r_2/r)/\log(r_2/r_1)$, i.e. $r=r_1^\alpha r_2^{1-\alpha}$.

This theorem has many generalizations, numerous results of this type inequality have been proved for solutions of second order partial differential equations (see [10, 27, 19] and the rich references cited therein). Now, the three-ball inequality is well known to imply that nonconstant solutions of elliptic equations cannot have a zero of infinite order at interior points and have applications to inverse problems (e.g. [1]), i.e. once we established a three-ball inequality for the partial differential operator we considered, then strong unique continuation property (SUCP for short) follows immediately.

The SUCP is well understood for second order elliptic operators. The classical paper by Carleman [3] established the SUCP for second order elliptic operators which need not to have analytic coefficients. The powerful technique he used, the so called "Carleman weighted inequality" has become one of the major tools in the study of SUCP, uniqueness and stability of Cauchy problems (see [13, 14, 16, 25, 26, 27, 28, 29] and the rich references therein). Besides Carleman estimate method, frequency function method is another one of the powerful tools to obtain sucp results for solutions of partial differential equations (see [10, 21] and the rich references therein).

In connection with the SUCP, a natural question is: How fast is a solution y allowed to vanish, before it mush vanish identically? We refer this type of question as "quantitative unique continuation", or simply the quantitative uniqueness. Meanwhile, we say the vanishing order of solution at x_0 is k, if k is the largest integer such that $D^\alpha y(x_0)=0$ for all $|\alpha|\le k$, we call the fastest rate the *maximal order of vanishing*. To see the *maximal order of vanishing* clearly, we first recall the following known result for the second order elliptic equation. Suppose that for some $K,M>>1$, $|W|_{L^\infty(\mathcal{B}_{10})}\le K$ and $|V|_{L^\infty(\mathcal{B}_{10})}\le M$, if $y:\mathcal{B}_{10}\to\mathbb{C}$ is a solution to

$$\Delta y + W\cdot\nabla y + Vy = 0$$

in $\mathcal{B}_{10}$ with $|u|_{L^\infty(\mathcal{B}_1)}\ge 1$ and $|u|_{L^\infty(\mathcal{B}_1)}\le C_0$, then a quantitative form of strong unique continuation asserts that

$$|u|_{L^\infty(\mathcal{B}_r)} \ge cr^{C(K^2+M^{2/3})} \quad \text{as } r\to 0,$$

which implies that the maximal order of vanishing for y at origin is less than $C(K^2+M^{2/3})$. In case of $W=0$, this kind of maximal order of vanishing estimate was proved by Bourgain and Kenig in [2]. They used this result to establish estimates at infinity that were relevant to their work on Anderson localization. Meshkov's examples in [23] imply that the power of $2/3$ is optimal for complex-valued functions. In case of $W\ne 0$, we refer to [5] for the second order Laplace operator and [20] for variable coefficient elliptic operator.

Higher order elliptic equation is an important model in the study of partial differential equations. It appeared in the study of continuum mechanics, in the related field of elasticity , and applications in engineering design as well (see [22] for example). In case of SUCP of higher order elliptic equations, we refer to [4, 18] and the related references therein for the strong unique continuation results of higher order elliptic equations by using Carleman estimates. In case of three-ball inequality and quantitative uniqueness of higher order elliptic equations, there are few references addressing these kind of problems (see [11, 24, 30, 31]).

In [11], the authors considered the unique continuation property of the following perturbed fourth order elliptic operator $\mathcal{L}_{A,q}u = 0$, where

$$\mathcal{L}_{A,q}(x, D) = \sum_{j=1}^{n} D_{x_j}^4 + \sum_{j=1}^{n} A_j D_{x_j} + q, \quad (A, q) \in W^{1,\infty}(\Omega, \mathbb{C}^n) \times L^\infty(\Omega, \mathbb{C}).$$

Meanwhile, the authors also proved the SUCP holds in 2-dimension in the sense of H^1-norm, while in three and higher dimensions, the SUCP does not holds.

In [24], the authors proved a three sphere inequality for solutions to the equation

$$\mathcal{L}u \triangleq P_4(u) + Q(u),$$

where $n \geq 2$, Q is a third order operator with bounded coefficients and P_4 is a fourth order elliptic operator such that $P_4 = L_2 L_1$, where L_1 and L_2 are two second order uniformly elliptic operator with real and $C^{1,1}(\overline{\Omega})$ coefficients. In this situation, the authors obtained a three-ball inequality in the sense of H^3-norm (see [24, Theorem 5.3]). Based on this inequality, the authors further established three sphere inequalities for the plate equation

$$\mathcal{L}u \triangleq \partial_{ij}^2 (C_{ijkl} \partial_{kl}^2 u) = 0, \quad \text{ in } \mathcal{B}_R$$

in $\mathbb{R}^2$. The first version of the three sphere inequality given in [24, Theorem 6.2] in the sense of H^3 norm, the second version of the three sphere inequality given in [24, Theorem 6.5] in the sense of H^2 norm, and the third version of the three sphere inequality given in [24, Theorem 6.6] has the following form

$$\int_{\mathcal{B}_\rho} |u|^2 dx \leq C\Big(\int_{\mathcal{B}_r} |u|^2 dx\Big)^\theta \Big(\sum_{k=0}^{4} \rho_1^{2k} \int_{\mathcal{B}_{r_1}} |\nabla^k u|^2 dx\Big)^{1-\theta}$$

for some $0 < s < 1$, $\quad \rho_1 \in (0, sR)$ and $r < \rho < \rho_1$.

In [30], the author considered the quantitative uniqueness of higher order elliptic equation

$$(-\Delta)^m u(x) = V(x)u(x), \quad \text{ in } \mathcal{B}_{10}.$$

Under some assumptions on the potential V and the solution u, based on a variant of frequency function, the author proved that for $n \geq 4m$, the vanishing order of u is less than $C||V_0||_{L^\infty(\Omega)}$. In [31], the author considered the quantitative uniqueness of general higher order elliptic equation with singular coefficients

$$(-\Delta)^m u(x) + \sum_{|\alpha|=1}^{\alpha_0} V_\alpha(x) \cdot D^\alpha u + V_0(x)u(x) = 0, \quad \text{ in } \mathcal{B}_{10}.$$

It should be point out that in [31] if m is a positive even integer, the value $\alpha_0 \leq [3m/2]-1$, i.e., in case of $m = 2$, $\alpha_0 \leq 2$. As far as we know, three-ball inequality for bi-Laplace equations (1.1) in the sense of L^2-norm with third order terms and quantitative uniqueness havn't been discussed yet. In this paper, based on the Carleman estimate, we will establish a three-ball inequality for system (1.1). As its applications, we will investigate some quantitative uniqueness problems of solutions to bi-Laplace equations with lower order terms. We consider two kinds of quantitative uniqueness problems. First, we quantify the strong unique continuation property by estimating the vanishing order of solutions to (1.1). Second, we derive a minimal decaying rate around the point at infinity for solutions of the bi-Laplace equations in $\mathbb{R}^n$.

The rest of this paper is organized as follows. In Section 2, we state our main results. In Section 3 we give the proof of three-ball inequality for system (1.1), which is useful in the proof of quantitative uniqueness results (see Theorems 2.2 and 2.3) in Section 4.

2 Main results

To obtain the three-ball inequality of (1.1), here we adopt the Carleman weighted estimate. To this aim, we first give our choice of the weight functions.

We define

$$\varphi(r) = r \exp\Big(\int_0^r \frac{e^{-t}-1}{t} dt\Big), \qquad r > 0. \tag{2.1}$$

For $\lambda > 0$, put

$$\sigma(x) = |x|, \quad w(x) = \varphi(\sigma(x)), \quad \ell(x) = -\lambda \ln w(x). \tag{2.2}$$

Let $R_* > 0$ be such that $\mathcal{B}_{R_*} \subset \Omega$. Throughout of this paper, we will use $C = C(n, \Omega)$ to denote a generic positive constant which may vary from line to line. And for the sake of simplicity, we denote the L^∞-norm of y in Ω by $|y|_\infty$.

We have the following three-ball inequality.

Theorem 2.1 *Let $r_0 \in (0, R_*]$ and $\mathcal{B}_{R_*} \subset \Omega$. Assume that $y \in H^4(\mathcal{B}_{r_0})$ be a solution to (1.1) with $\max\limits_{0\leq j\leq 3} |V_j|_\infty \leq M$ for some constant $M \gg 1$. Then there exists a constant $C > 0$ such that for all $0 < r_2 < r_1 < 2r_1 < r_0$, it holds that*

$$|y|_{L^2(\mathcal{B}_{r_1})} \leq CM^4 r_1^{-10} |y|^{\varepsilon_0}_{L^2(\mathcal{B}_{r_2})} |y|^{1-\varepsilon_0}_{L^2(\mathcal{B}_{r_0})} + e^{\mathcal{C}(V_0,V_1,V_2,V_3)\ln(\varphi(2r_0/3)/\varphi(r_2/2))} |y|_{L^2(\mathcal{B}_{r_2})}, \tag{2.3}$$

where φ is given by (2.1),

$$\mathcal{C}(V_0, V_1, V_2, V_3) = C\Big(|V_0|_\infty^{\frac{1}{3}} + |V_1|_\infty^{\frac{1}{2}} + |V_2|_\infty^{\frac{2}{3}} + |V_3|_\infty^2\Big) \tag{2.4}$$

and

$$\varepsilon_0 = \frac{\ln \varphi(2r_0/3) - \ln \varphi(r_1)}{\ln \varphi(2r_0/3) - \ln(r_2/2)}. \tag{2.5}$$

Remark 2.1 *Three-ball inequality (2.3) is optimal in the sense explained in [6].*

Remark 2.2 *By Theorem 2.1, one can obtain the following strong unique continuation property of (1.1):*

Let $y \in H^4_{loc}(\Omega)$ be a solution of (1.1). If y satisfies

$$|y|_{L^2(\mathcal{B}_r)} = O(r^\nu) \quad as \quad r \to 0, \quad for\ every \quad \nu \in \mathbb{N}, \tag{2.6}$$

then $y \equiv 0$ in Ω.

Based on Theorem 2.1, we have the following two quantitative unique continuation result.

Theorem 2.2 *Let y be a solution to (1.1) with $|V_j|_\infty \le M_j$ and $1 \le |y|_\infty \le C_0$ for some constants $M_j >> 1$ for $0 \le j \le 3$ and $C_0 \ge 1$. Then there exist positive constants $C_1 = C_1(n, \Omega, C_0)$ and $C_2 = C_2(n, \Omega, C_0)$ such that*

$$m(r) = \sup_{|x|\le r} |y(x)| \ge C_1 r^{C_2\left(M_0^{\frac{1}{3}} + M_1^{\frac{1}{2}} + M_2^{\frac{2}{3}} + M_3^2\right)}, \quad \forall r \in (0, 1]. \tag{2.7}$$

As an application of Theorem 2.2, we have the following decay rate for nontrivial solution of (1.1).

Theorem 2.3 *Suppose that $y \in L^\infty(\mathbb{R}^n)$ is a solution of (1.1) with $\max_{0\le j\le 3} |V_j|_\infty \le 1$, $|y|_{L^\infty(\mathbb{R}^n)} \le C_0$ for a given constant $C_0 > 0$ and $y(0) = 1$. Then there exist a constant $C > 0$ such that*

$$M(R) \equiv \inf_{|x_0|=R} \sup_{x\in\mathcal{B}(x_0,1)} |y(x)| \ge Ce^{-CR^2 \log R}, \qquad \forall\ R > 0, \tag{2.8}$$

where

$$\mathcal{B}(x_0, 1) = \{x \in \mathbb{R}^n \,|\, |x - x_0| < 1\}. \tag{2.9}$$

In particular, if $V_1 = V_3 = (0, \cdots, 0)$, (2.8) can be improved to the following form:

$$M(R) \equiv \inf_{|x_0|=R} \sup_{x\in\mathcal{B}(x_0,1)} |y(x)| \ge Ce^{-CR^{4/3} \log R}, \qquad \forall\ R > 0. \tag{2.10}$$

3 Proof of three-ball inequality

In this section, we will give the proof of Theorem 2.1. To this aim, we first establish Carleman estimate for the bi-Laplace operator, which can be obtained by virtue of an alternative way for the second order Laplace operator. Further, we establish Some Caccioppoli inequalities which is a key to yield our three-ball inequality.

3.1 Carleman estimate for bi-Laplace operator

We have the following Carleman estimate for the bi-Laplace operator.

Theorem 3.1 *Let w be given in (2.2). one can find a constant $\lambda_0 > 0$ so that for all $y \in C_0^4\big(\Omega \setminus \{0\}; \mathbb{C}\big)$ and $\lambda \geq \lambda_0$, it holds that*

$$\begin{aligned}&\lambda \int_\Omega w^{1-2\lambda} |\nabla \Delta y|^2 dx + \lambda^3 \int_\Omega w^{-1-2\lambda} |\Delta y|^2 dx \\ &+\lambda^4 \int_\Omega \big(w^{-2-2\lambda}|\nabla y|^2 + \lambda^2 w^{-4-2\lambda}|y|^2\big) dx \leq C \int_\Omega w^{2-2\lambda}|\Delta^2 y|^2 dx.\end{aligned} \tag{3.1}$$

Remark 3.1 *The reason for the above choice of ℓ is two folds. First, it is strongly pseudo-convex in $\Omega \setminus \{0\}$ (in the sense of [13]). Second, $w(x) = O(|x|)$ (as $|x| \to 0$), which is a key point in deriving the three-ball inequality for solutions of elliptic equations.*

Before giving the proof of Theorem 3.1, we first recall the following known result.

Lemma 3.1 *([7, 8]) Let w be given in (2.2). one can find a constant $\lambda_* > 0$ so that for all $z \in C_0^2(\Omega \setminus \{0\}; \mathbb{C})$ and $\lambda \geq \lambda_*$, it holds that*

$$\lambda \int_\Omega \big(w^{1-2\lambda}|\nabla z|^2 + \lambda^2 w^{-1-2\lambda}|z|^2\big) dx \leq C \int_\Omega w^{2-2\lambda}|\Delta z|^2 dx. \tag{3.2}$$

Remark 3.2 *To obtain Carleman estimate for the elliptic operator with variable coefficients in the principal part, in [8, Theorem 2.2], the function φ has the following more general form:*

$$\varphi(r) = r \exp\Big(\int_0^r \frac{e^{-\mu t} - 1}{t}\Big)$$

with $\mu > 0$. In case of Laplace operator, $\mu = 1$ is enough to give the desired Carleman estimate.

Proof of Theorem 3.1. Put $u = \Delta y$. Then, it is easy to see that

$$\Delta u = \Delta^2 y. \tag{3.3}$$

In Lemma 3.1, by taking $u = z$, one can find a constant $\lambda_* > 0$ so that for all $u \in C_0^2(\Omega \setminus \{0\})$ and $\lambda \geq \lambda_0$, it holds that

$$\lambda \int_\Omega \big(w^{1-2\lambda}|\nabla u|^2 + \lambda^2 w^{-1-2\lambda}|u|^2\big) dx \leq C \int_\Omega w^{2-2\lambda}|\Delta u|^2 dx. \tag{3.4}$$

Next, noting that $u = \Delta y$, it is easy to see that

$$\int_\Omega w^{-1-2\lambda}|u|^2 dx = \int_\Omega w^{-1-2\lambda}|\Delta y|^2 dx = \int_\Omega w^{2-2(\lambda+3/2)}|\Delta y|^2 dx. \tag{3.5}$$

Then, by taking $z = y$ in Lemma 3.1, one can find a constant $\lambda^* > 0$ so that for all $y \in C_0^2(\Omega \setminus \{0\})$ and $\lambda \geq \lambda^*$, it holds that

$$\lambda \int_\Omega \big(w^{1-2(\lambda+3/2)}|\nabla y|^2 + \lambda^2 w^{-1-2(\lambda+3/2)}|y|^2\big) dx \leq C \int_\Omega w^{2-2(\lambda+3/2)}|\Delta y|^2 dx. \tag{3.6}$$

By (3.4)–(3.6), we conclude that there exists a constant $\lambda_0 = \max\{\lambda^*, \lambda_*\}$ and $R_* > 0$ so that for all $y \in C_0^2(\Omega \setminus \{0\})$ and $\lambda \geq \lambda_0$, it holds that

$$\begin{aligned} &\lambda \int_\Omega \big(w^{1-2\lambda}|\nabla u|^2 + \lambda^2 w^{-1-2\lambda}|u|^2\big)dx \\ &\quad + \lambda^4 \int_\Omega \big(w^{-2-2\lambda}|\nabla y|^2 + \lambda^2 w^{-4-2\lambda}|y|^2\big)dx \leq C \int_\Omega w^{2-2\lambda}|\Delta u|^2 dx. \end{aligned} \tag{3.7}$$

By (3.7) and noting that $u = \Delta y$, one can get the desired result immediately. □

3.2 Proof of Theorem 2.1

Based on Theorem 3.1, in this subsection, we will give the proof of Theorem 2.1.

Proof of Theorem 2.1. The proof is divided into several steps.

Step 1. For all $0 < r_2 < r_1 < 2r_1 < r_0$. Let $\xi \in C_0^4([0, r_0]; [0, 1])$ be a cut-off function satisfying

$$\xi(r) = \begin{cases} 0, & \text{if } r \in [0, \frac{r_2}{2}] \cup [\frac{3r_0}{4}, r_0], \\ 1, & \text{if } r \in [\frac{3r_2}{4}, \frac{2r_0}{3}], \end{cases} \tag{3.8}$$

and that

$$\begin{cases} \Big|\dfrac{d^j\xi}{dr^j}\Big| \leq C_\xi / r_2^j, \quad j = 1,\ 2,\ 3,\ 4, \quad & \text{in } \Big[\dfrac{r_2}{2}, \dfrac{3r_2}{4}\Big], \\ \Big|\dfrac{d^j\xi}{dr^j}\Big| \leq C_\xi / r_0^j, \quad j = 1,\ 2,\ 3,\ 4, \quad & \text{in } \Big[\dfrac{2r_0}{3}, \dfrac{3r_0}{4}\Big], \end{cases} \tag{3.9}$$

where C_ξ is a constant.

Let $\{y_n\}_{n=1}^\infty$ be a sequence in $C_0^\infty(\mathcal{B}_{r_0})$, which converges to y in $H^4(\mathcal{B}_{r_0})$. By applying the inequality (3.1) to $y_n\zeta$ where $\zeta(x) = \xi(|x|)$ and passing to the limit, for all $\lambda \geq \lambda_0$, we obtain that

$$\begin{aligned} &\lambda \int_{\mathcal{B}_{r_0}} w^{1-2\lambda}|\nabla\Delta(\zeta y)|^2 dx + \lambda^3 \int_{\mathcal{B}_{r_0}} w^{-1-2\lambda}|\Delta(\zeta y)|^2 dx \\ &\quad + \lambda^4 \int_{\mathcal{B}_{r_0}} \big(w^{-2-2\lambda}|\nabla(\zeta y)|^2 + \lambda^2 w^{-4-2\lambda}|\zeta y|^2\big)dx \leq C \int_{\mathcal{B}_{r_0}} w^{2-2\lambda}|\Delta^2(\zeta y)|^2 dx. \end{aligned} \tag{3.10}$$

Step 2. To estimate the terms in both side of (3.10), let us divide $\mathcal{B}_{r_0}$ into several parts according to the weight function. Put

$$\begin{aligned} \mathcal{K}_1 &= \Big\{x \in \mathbb{R}^n : \frac{3r_2}{4} \leq |x| \leq \frac{2r_0}{3}\Big\}, \\ \mathcal{K}_2 &= \Big\{x \in \mathbb{R}^n : \frac{2r_0}{3} \leq |x| \leq \frac{3r_0}{4}\Big\}, \\ \mathcal{K}_3 &= \Big\{(t, x) \in \mathbb{R}^{1+n} : \frac{r_2}{2} \leq |x| \leq \frac{3r_2}{4}\Big\}, \\ \mathcal{K}_4 &= \mathcal{B}_{r_0} \setminus \bigcup_{j=1}^3 \mathcal{K}_j. \end{aligned}$$

Clearly, it holds that $\mathcal{B}_{r_0} = \bigcup_{j=1}^{4} \mathcal{K}_j$ and $\mathcal{K}_j \cap \mathcal{K}_k = \emptyset$ for $j \neq k,\ j,k = 1,\ 2,\ 3,\ 4$.

From (3.10) and (1.1), noting that $\zeta = 1$ in $\mathcal{K}_1$ and $r_0 \in (0, R_*]$, we obtain that for every $\lambda \geq \lambda_*$,

$$
\begin{aligned}
&\lambda \int_{\mathcal{K}_1} w^{1-2\lambda} |\nabla\Delta y|^2 dx + \lambda^3 \int_{\mathcal{K}_1} w^{-1-2\lambda} |\Delta y|^2 dx \\
&\quad + \lambda^4 \int_{\mathcal{K}_1} \left(w^{-2-2\lambda} |\nabla y|^2 + \lambda^2 w^{-4-2\lambda} |y|^2 \right) dx \\
&\leq C \int_{\mathcal{B}_{r_0}} w^{2-2\lambda} |\Delta^2 \zeta y + 4\nabla\Delta\zeta \cdot \nabla y + 4\nabla\zeta \cdot \nabla\Delta y \\
&\qquad + 2\Delta\zeta\Delta y + 4 \sum_{j,k=1}^{n} \zeta_{x_j x_k} y_{x_j x_k} + \zeta \Delta^2 y|^2 dx \\
&\leq C \int_{\bigcup_{j=2}^{3} \mathcal{K}_j} w^{2-2\lambda} \mathcal{J} dx + C \int_{\bigcup_{j=1}^{3} \mathcal{K}_j} w^{2-2\lambda} \left[|V_3|_{L^\infty}^2 |\nabla\Delta y|^2 \right. \\
&\qquad \left. + |V_2|_{L^\infty}^2 |\Delta y|^2 + |V_1|_{L^\infty}^2 |\nabla y|^2 + |V_0|_{L^\infty}^2 |y|^2 \right] dx,
\end{aligned}
\tag{3.11}
$$

where

$$
\mathcal{J} = |\Delta^2 \zeta y|^2 + |\nabla\Delta\zeta \cdot \nabla y|^2 + |\Delta\zeta\Delta y|^2 + |\nabla\zeta \cdot \nabla\Delta y|^2 + \Big| \sum_{j,k=1}^{n} \zeta_{x_j x_k} y_{x_j x_k} \Big|^2. \tag{3.12}
$$

Noting that $w(x) = \varphi(|x|)$ and φ is an increasing function, it is easy to see that

$$
w^{2-2\lambda}(x) = w^6(x) w^{-4-2\lambda}(x) \leq \varphi^6\Big(\frac{3r_0}{4}\Big) w^{-4-2\lambda}(x) \leq \varphi^6(R_*) w^{-4-2\lambda}(x),\ x \in \mathcal{K}_2 \cup \mathcal{K}_3.
$$

Similarly, for $x \in \mathcal{K}_2 \cup \mathcal{K}_3$,

$$
\begin{cases}
w^{2-2\lambda}(x) \leq \varphi(R_*) w^{1-2\lambda}, \\
w^{2-2\lambda}(x) \leq \varphi^3(R_*) w^{-1-2\lambda}, \\
w^{2-2\lambda}(x) \leq \varphi^4(R_*) w^{-2-2\lambda}.
\end{cases}
\tag{3.13}
$$

Next, taking λ_j (j=1, 2, 3, 4) as following

$$
\begin{cases}
\dfrac{\lambda_1^6}{2} = C|V_0|_\infty^2 \varphi^6(R^*), \\
\dfrac{\lambda_2}{2} = C|V_3|_\infty^2 \varphi(R^*), \\
\dfrac{\lambda_3^3}{2} = C|V_2|_\infty^2 \varphi^3(R^*), \\
\dfrac{\lambda_4^4}{2} = C|V_1|_\infty^2 \varphi^4(R^*).
\end{cases}
\tag{3.14}
$$

Then, choosing λ large enough such that $\lambda \geq \max_{0\leq j\leq 4} \lambda_j$. It follows from (3.11) that

$$\lambda^6 \int_{\mathcal{K}_1} w^{-4-2\lambda}|y|^2 dx \leq CM^2 \int_{\mathcal{K}_2\cup\mathcal{K}_3} w^{2-2\lambda}\Big[\mathcal{J} + |\nabla\Delta y|^2 + |\Delta y|^2 + |\nabla y|^2 + |y|^2\Big] dx. \tag{3.15}$$

By (3.9), we have that

$$\begin{aligned}&\int_{\frac{3r_2}{4}\leq|x|\leq r_1} w^{-4-2\lambda}|y|^2 dx\\ &\leq CM^2\Big[r_0^{-8}\int_{\frac{2r_0}{3}\leq|x|\leq\frac{3r_0}{4}} w^{2-2\lambda}\mathcal{H}dx + r_2^{-8}\int_{\frac{r_2}{2}\leq|x|\leq\frac{3r_2}{4}} w^{2-2\lambda}\mathcal{H}dx\Big],\end{aligned} \tag{3.16}$$

where

$$\mathcal{H} = |y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla\Delta y|^2 + \Big|\sum_{j,k=1}^{n} y_{x_j x_k}\Big|^2. \tag{3.17}$$

Again, recalling (2.2) for the definition of w, by (3.16), we have that

$$\begin{aligned}&\varphi^{-4-2\lambda}(r_1)\int_{\frac{3r_2}{4}\leq|x|\leq r_1} |y|^2 dx\\ &\leq CM^2\Big[r_0^{-8}\varphi^{2-2\lambda}(2r_0/3)\int_{\frac{2r_0}{3}\leq|x|\leq\frac{3r_0}{4}} \mathcal{H}dx + r_2^{-8}\varphi^{2-2\lambda}(r_2/2)\int_{\frac{r_2}{2}\leq|x|\leq\frac{3r_2}{4}} \mathcal{H}dx\Big].\end{aligned} \tag{3.18}$$

Step 3. In this step, we will prove some Caccioppoli-type inequalities to estimate "$\int_{\frac{2r_0}{3}\leq|x|\leq\frac{3r_0}{4}} \mathcal{H}dx$" and "$\int_{\frac{r_2}{2}\leq|x|\leq\frac{3r_2}{4}} \mathcal{H}dx$".

Let $\eta \in C_0^4(\Omega;[0,1])$ be a cut-off function satisfying

$$\begin{cases} \eta(x) = 0, & \text{if } |x| \in [0, 5r_0/8] \cup [5r_0/6, r_0],\\ \eta(x) = 1, & \text{if } |x| \in [2r_0/3, 3r_0/4],\\ |\nabla\eta| \leq C_\eta/r_0. \end{cases} \tag{3.19}$$

Multiplying (1.1) by $\eta^8\Delta\bar{y}$, we have

$$\begin{aligned}\eta^8\mathrm{Re}\Big(\Delta\bar{y}(\Delta^2 y)\Big) &= \eta^8\mathrm{Re}\Big(\Delta\bar{y}(V_0 y + V_1\cdot\nabla y + V_2\Delta y + V_3\cdot\nabla\Delta y)\Big)\\ &= \nabla\cdot\mathrm{Re}\Big(\eta^8\Delta\bar{y}\nabla\Delta y\Big) - \eta^8|\nabla\Delta y|^2 - 8\eta^7\mathrm{Re}\,(\nabla\eta\cdot\nabla\Delta\bar{y}\Delta y).\end{aligned} \tag{3.20}$$

Now, integrating (3.20) on Ω and notice that $\eta \leq 1$, we have

$$\int_\Omega \eta^8|\nabla\Delta y|^2 dx \leq C_1 M^2 r_0^{-2}\int_\Omega \eta^6(|\Delta y|^2 + \eta^2|\nabla y|^2 + \eta^2|y|^2)dx. \tag{3.21}$$

Next, multiplying (1.1) by $\eta^6\bar{y}$, we have that

$$\begin{aligned}\eta^6\mathrm{Re}\,(\bar{y}\Delta^2 y) &= \eta^6\mathrm{Re}\Big(\bar{y}(V_0 y + V_1\cdot\nabla y + V_2\Delta y + V_3\cdot\nabla\Delta y)\Big)\\ &= \nabla\cdot\mathrm{Re}\,(\eta^6\bar{y}\nabla\Delta y - \eta^6\nabla\bar{y}\Delta y) + \eta^6|\Delta y|^2 + 6\eta^5\mathrm{Re}\,(\nabla\eta\cdot\nabla\bar{y}\Delta y)\\ &\quad -6\eta^5\mathrm{Re}\,(\bar{y}\nabla\eta\cdot\nabla\Delta y).\end{aligned} \tag{3.22}$$

Integrating (3.22) on Ω and using integration by parts, for any $\varepsilon_1 > 0$, we have

$$\int_\Omega \eta^6|\Delta y|^2 dx \quad \leq C_{\varepsilon_1} M^2 r_0^{-4}\Big(\int_\Omega \eta^2|y|^2 + \eta^4|\nabla y|^2\Big)dx + \varepsilon_1 r_0^2 \int_\Omega \eta^8|\nabla\Delta y|^2 dx. \tag{3.23}$$

Hence, for sufficiently small ε_1, by (3.21), we have

$$\int_\Omega \eta^6|\Delta y|^2 dx \leq CM^2 r_0^{-4}\int_\Omega \eta^2\Big(y^2 + \eta^2|\nabla y|^2\Big)dx. \tag{3.24}$$

On the other hand, noting that

$$\begin{aligned}&\eta^6 \mathrm{Re}\,(\nabla\bar{y}\cdot\nabla\Delta y)\\ &= \sum_{j,k=1}^n \mathrm{Re}\,\Big((\eta^6\bar{y}_{x_j} y_{x_j x_k})_{x_k}\Big) - \eta^6\sum_{j,k=1}^n |y_{x_j x_k}|^2 - 6\eta^5\sum_{j,k=1}^n \mathrm{Re}\,\Big(\eta_{x_j}\bar{y}_{x_k} y_{x_j x_k}\Big).\end{aligned} \tag{3.25}$$

Therefore, for any $\varepsilon_2 > 0$, we have

$$\int_\Omega \eta^6\sum_{j,k=1}^n |y_{x_j x_k}|^2 dx \leq C_{\varepsilon_2}\int_\Omega \eta^4|\nabla y|^2 dx + \varepsilon_2\int_\Omega \eta^8|\nabla\Delta y|^2 dx. \tag{3.26}$$

Combining (3.26), (3.21) and (3.24), taking ε_2 small enough, we get

$$\int_\Omega \eta^6\Big|\sum_{j,k=1}^n y_{x_j x_k}\Big|^2 dx \leq CM^2 r_0^{-6}\int_\Omega \eta^2\Big(|y|^2 + \eta^2|\nabla y|^2\Big)dx. \tag{3.27}$$

Further, by (3.24), it is easy to see that

$$\begin{aligned}\int_\Omega \eta^4|\nabla y|^2 dx \quad &= -\int_\Omega \eta^4 \mathrm{Re}\,(\bar{y}\Delta y)dx - 4\int_\Omega \eta^3 \mathrm{Re}\,(\bar{y}\nabla\eta\cdot\nabla y)dx\\ &= -\int_\Omega \eta^4 \mathrm{Re}\,(\bar{y}\Delta y)dx + 2\int_\Omega \nabla\cdot(\eta^3\nabla\eta)|y|^2 dx\\ &\leq \varepsilon_3 r_0^{-4}\int_\Omega \eta^6|\Delta y|^2 dx + C_{\varepsilon_3} r_0^{-6}\int_\Omega \eta^2|y|^2 dx.\end{aligned} \tag{3.28}$$

Therefore, taking ε_3 small enough, we end up with

$$\int_\Omega \eta^4|\nabla y|^2 dx \leq C r_0^{-6}\int_\Omega \eta^2|y|^2 dx. \tag{3.29}$$

Combining (3.21), (3.24), (3.27) and (3.29), we end up with

$$\int_{\frac{2r_0}{3}\leq|x|\leq\frac{3r_0}{4}} \mathcal{H} dx \leq CM^2 r_0^{-8}\int_{\frac{5r_0}{8}\leq|x|\leq\frac{5r_0}{6}} |y|^2 dx. \tag{3.30}$$

Similarly, proceeding the same analysis as (3.30), it is easy to prove that

$$\int_{\frac{r_2}{2}\leq|x|\leq\frac{3r_2}{4}} \mathcal{H} dx \leq CM^2 r_0^{-8}\int_{\frac{r_2}{3}\leq|x|\leq r_2} |y|^2 dx. \tag{3.31}$$

Step 4. By (2.1) and (2.2), it follows that for a constant $C_1 > 0$, $\dfrac{r}{C_1} \le w(x) \le C_1 r$ with $r = |x|$. Then, for $\lambda \ge \max\limits_{0\le j\le 4} \lambda_j$, by (3.18) and (3.30)–(3.31), recall the definition of φ, we have that

$$\begin{aligned}
&\int_{\mathcal{B}_{r_1}} |y|^2 dx \\
&\le CM^4 \Big(\frac{r_0}{\varphi(2r_0/3)}\Big)^{-16} \Big(\frac{\varphi(2r_0/3)}{\varphi(r_1)}\Big)^{-14-2\lambda} \varphi^{-10}(r_1) \int_{\mathcal{B}_{r_0}} |y|^2 dx \\
&\quad + CM^4 \Big(\frac{r_2}{\varphi(r_2/2)}\Big)^{-16} \Big(\frac{\varphi(r_2/2)}{\varphi(r_1)}\Big)^{-14-2\lambda} \varphi^{-10}(r_1) \int_{\mathcal{B}_{r_2}} |y|^2 dx.
\end{aligned} \tag{3.32}$$

So for $\lambda \ge \max\limits_{0\le j\le 4} \lambda_i$, we have

$$\int_{\mathcal{B}_{r_1}} |y|^2 dx \le CM^4 r_1^{-10} \Big[\Big(\frac{\varphi(2r_0/3)}{\varphi(r_1)}\Big)^{-2\lambda} \int_{\mathcal{B}_{r_0}} |y|^2 dx + \Big(\frac{\varphi(r_2/2)}{\varphi(r_1)}\Big)^{-2\lambda} \int_{\mathcal{B}_{r_2}} |y|^2 dx \Big]. \tag{3.33}$$

Set

$$\lambda_5 = \frac{1}{2} \frac{\ln \int_{\mathcal{B}_{r_0}} |y|^2 dx - \ln \int_{\mathcal{B}_{r_2}} |y|^2 dx}{\ln \varphi(2r_0/3) - \ln \varphi(r_2/2)} \tag{3.34}$$

If $\lambda_5 \ge \max\limits_{0\le j\le 4} \lambda_j$, we have

$$\int_{\mathcal{B}_{r_1}} |y|^2 dx \le CM^4 r_1^{-10} \Big(\int_{\mathcal{B}_{r_2}} |y|^2 dx \Big)^{\varepsilon_0} \Big(\int_{\mathcal{B}_{r_0}} |y|^2 dx \Big)^{1-\varepsilon_0}, \tag{3.35}$$

where ε_0 is given by (2.5).

If $\lambda_5 < \max\limits_{0\le j\le 4} \lambda_j$, noting that λ_0 independent of V_k ($k = 0,\ 1,\ 2,\ 3$) and λ_j (j=1, 2, 3, 4) satisfying (3.14), by (2.4), we have

$$\begin{aligned}
\int_{\mathcal{B}_{r_1}} |y|^2 dx \le \int_{\mathcal{B}_{r_0}} |y|^2 dx \quad &\le e^{2\max_{0\le j\le 4} \lambda_j \ln(\varphi(2r_0/3)/\varphi(r_2/2))} \int_{\mathcal{B}_{r_2}} |y|^2 dx \\
&\le e^{\mathcal{C}(V_0,V_1,V_2,V_3) \ln(\varphi(2r_0/3)/\varphi(r_2/2))} \int_{\mathcal{B}_{r_2}} |y|^2 dx.
\end{aligned} \tag{3.36}$$

Combining (3.35) and (3.36), we get the desired result (2.3) immediately. $\square$

4 Proofs of Theorems 2.2 and 2.3

In the section, we will give the proof of our quantitative unique continuation results, we borrow some ideas from [2].

Proof of Theorem 2.2. Without loss generality, we assume that $\mathcal{B}_{12} \subset \Omega$. Choosing $r_2 = r \le 1$, $r_1 = 4$ and $r_0 = 12$ in Theorem 2.1, we have that

$$|y|_{L^2(\mathcal{B}_4)} \le CM^4 |y|^{\varepsilon_0}_{L^2(\mathcal{B}_r)} |y|^{1-\varepsilon_0}_{L^2(\mathcal{B}_{12})} + e^{\mathcal{C}(V_0,V_1,V_2,V_3) \ln(\varphi(8)/\varphi(r/2))} |y|_{L^2(\mathcal{B}_r)}, \tag{4.1}$$

where

$$\varepsilon_0 = \frac{\ln\varphi(8) - \ln\varphi(4)}{\ln\varphi(8) - \ln(\frac{r}{2})}. \tag{4.2}$$

Combining (4.1) with the elliptic regularity estimate $|y|_{L^\infty(\mathcal{B}_1)} \le C_n M^{n/2}|y|_{L^2(\mathcal{B}_4)}$, we have that

$$|y|_{L^\infty(\mathcal{B}_1)} \le \mathcal{J}_1 + \mathcal{J}_2, \tag{4.3}$$

where

$$\mathcal{J}_1 = CC_n M^{2n+4}|y|^{\varepsilon_0}_{L^2(\mathcal{B}_r)}|y|^{1-\varepsilon_0}_{L^2(\mathcal{B}_{12})}, \tag{4.4}$$

and

$$\mathcal{J}_2 = C_n M^{2n}\Big(\frac{C}{r}\Big)^{\mathcal{C}(V_0,V_1,V_2,V_3)}|y|_{L^2(\mathcal{B}_r)}. \tag{4.5}$$

On the one hand, if $\mathcal{J}_1 \le \mathcal{J}_2$, by (4.1) and (4.3), a short calculation shows that

$$\begin{aligned} |y|_{L^\infty(\mathcal{B}_1)} &\le 2C_n M^{2n}\Big(\frac{C}{r}\Big)^{\mathcal{C}(V_0,V_1,V_2,V_3)}|y|_{L^2(\mathcal{B}_r)} \\ &\le 2C_n M^{2n}\Big(\frac{C}{r}\Big)^{C\left(|V_0|_\infty^{\frac{1}{3}}+|V_1|_\infty^{\frac{1}{2}}+|V_2|_\infty^{\frac{2}{3}}+|V_3|_\infty^{2}\right)}\sup_{x\in\mathcal{B}_r}|y|, \end{aligned} \tag{4.6}$$

which gives the desired lower bound.

On the other hand, if $\mathcal{J}_1 \ge \mathcal{J}_2$, recalling that $|y|_{L^\infty(\mathcal{B}_1)} \ge 1$, it holds that

$$1 \le |y|_{L^\infty(\mathcal{B}_1)} \le 2CC_n M^{2n+4}|y|^{\varepsilon_0}_{L^2(\mathcal{B}_r)}|y|^{1-\varepsilon_0}_{L^2(\mathcal{B}_{12})}. \tag{4.7}$$

Raising both sides to $\dfrac{1}{\varepsilon_0}$ and using the bound $|y|_{L^\infty(\mathcal{B}_{12})} \le C_0$, we obtain

$$\begin{aligned} 1 \ &\le \Big(2CC_n M^{2n+4}\Big)^{\frac{1}{\varepsilon_0}}|y|_{L^\infty(\mathcal{B}_r)}C_0^{\frac{1-\varepsilon_0}{\varepsilon_0}} \\ &\le \Big(2CC_0C_n M^{2n+4}\Big)^{\frac{1}{\varepsilon_0}}|y|_{L^\infty(\mathcal{B}_r)} \end{aligned} \tag{4.8}$$

Recall the definition of ε_0 and φ, we find that $\frac{1}{\varepsilon_0} \le C\ln\frac{1}{r}$. Hence, the right hand side of (4.8) is bounded by $r^{-C\ln(CC_0C_nM^{2n+4})}|y|_{L^\infty(\mathcal{B}_r)}$. The result follows. □

Proof of Theorem 2.3. Fix an x_0 such that $|x_0| = R$ and that

$$M(R) = \inf_{|x_0|=R}\sup_{\mathcal{B}(x_0,1)}|y(x)| = \sup_{\mathcal{B}(x_0,1)}|y(x)|. \tag{4.9}$$

Set

$$y_R(x) = y(Rx + x_0) = y(R(x + x_0/R)). \tag{4.10}$$

By (1.1), we have

$$\begin{aligned} \Delta^2 y_R &= R^4(V_0y_R + R^{-1}V_1\cdot\nabla y_R + R^{-2}V_2\Delta y_R + R^{-3}V_3\cdot\nabla\Delta y_R) \\ &= R^4V_0y_R + R^3V_1\cdot\nabla y_R + R^2V_2\Delta y_R + RV_3\cdot\nabla\Delta y_R. \end{aligned} \tag{4.11}$$

So that

$$|R^4V_0|_\infty + |R^3V_1|_\infty + |R^2V_2|_\infty + |RV_3|_\infty \le R^4. \tag{4.12}$$

Let $\tilde{x}_0 = -x_0/R$, then $|\tilde{x}_0| = 1$ and $y_R(\tilde{x}_0) = y(0) = 1$, thus $|y_R|_{L^\infty(\mathcal{B}_1)} \ge 1$ and $\sup\limits_{\mathcal{B}(x_0,1)} |y(x)| = \sup\limits_{\mathcal{B}_{1/R}} |y_R|$. So that using Theorem 2.2 with $M = R^4$, we have that

$$M(R) = \sup_{\mathcal{B}_{1/R}} |y_R| \ge C_1\Big(\frac{1}{R}\Big)^{C_2(R^2 + R^{\frac{4}{3}} + R^{\frac{3}{2}} + R^{\frac{4}{3}})} = Ce^{-CR^2 \log R}. \tag{4.13}$$

if $V_1 = V_3 = (0, \cdots, 0)$, we have that

$$M(R) \ge C_1\Big(\frac{1}{R}\Big)^{C_2 R^{\frac{4}{3}}} = Ce^{-CR^{\frac{4}{3}} \log R}. \tag{4.14}$$

This completes the proof of Theorem 2.3. $\square$

References

[1] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, *Optimal stability for inverse elliptic boundary value problems with unknown boundaries,* Ann. Scuola. Norm. Sci., **29** (2000), 755–806.

[2] J. Bourgain, C. E. Kenig, *On localization in the continuous AndersonBernoulli model in higher dimension,* Invent. Math., **161** (2005), 389–426.

[3] T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes,* Ark. Mat. Astr. Fys., **17** (1939), 1–9.

[4] F. Colombini, C. Grammatico, *Some remarks on strong unique continuation for the Laplace operator and its powers,* Commun. Part. Diff. Eq., **24** (1999), 1079–1094.

[5] B. Davey, *Some Quantitative Unique Continuation Results for Eigenfunctions of the Magnetic Schrödinger Operator,* Commun. Part. Diff. Eq., **39** (2014), 876–945.

[6] L. Escauriaza, F. J. Fernández, S. Vessella, *Doubling properties of caloric functions, Appl. Anal.,* 85(2006), 205–223.

[7] L. Escauriaza, S. Vessella, *Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients,* Inverse problems: theory and applications, (2002), 79–87.

[8] X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications,* Springer, 2019.

[9] A.V. Fursikov, O.Y. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, **34** (1996).

[10] N. Garofalo, F. H. Lin, *Unique continuation for elliptic operators: a geometric-variational approach, Commun. Pur. Appl. Math.*, **40** (1987), 347-366.

[11] A. Ghosh and T. Ghosh, *Unique continuation for a non bi-Laplacian fourth order elliptic operator,* 2019, https://arxiv.org/abs/1908.05882.

[12] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de Taylor, J. Math. Pure. Appl.,* **8** (1892), 101–186.

[13] L. Hörmander, *Uniqueness theorems for second order elliptic differential equations. Commun. Part. Diff. Eq.,* **8** (1983), 21–64.

[14] V. Isakov, *Inverse Problems for Partial Differential Equations,* Springer, 2006.

[15] C. L. Lin, *Strong unique continuation for m-th powers of a laplacian operator with singular coefficients. P. Am. Math. Soc.,* **135** (2007), 569–578.

[16] C. E. Kenig, *Carleman estimates, uniform Sobolev inequalities for second order differential operators, and unique continuation theorems,* Proceedings of the International Congress of Mathematicians, **2** (1986), 948–960.

[17] V. A. Kondratiev, E. M. Landis, *Qualitative properties of the solutions of a second-order nonlinear equation,* lau Commission on instruments, **2** (1992), 29.

[18] C. L. Lin, N. Sei and J. N. Wang, *Quantitative uniqueness for the power of the Laplacian with singular coefficients, Ann. Scuola. Norm. Sci.,* **10** (2011), 513–529.

[19] C. L. Lin, G. Uhlmann, J. N. Wang, *Optimal three-ball inequalities and quantitave uniqueness for the stokes system,* Discrete. Cont. Dyn. A., **28** (2010), 1273-1290.

[20] C. L. Lin, J. N. Wang. *Quantitative uniqueness estimates for the general second order elliptic equations, J. Funct. Anal.,* **266** (2014), 5108–5125.

[21] F.H. Lin, *Nodal sets of solutions of elliptic equations of elliptic and parabolic equations,Commun. Pur. Appl. Math.,* **44** (1991), 287–308.

[22] M. Luis, B.C. Campos, *Generalized calculus with applications to matter and forces. Mathematics and Physics for Science and Technology, CRC* Press, 2014.

[23] V. Z. Meshkov, *On the possible rate of decay at infinity of solutions of second order partial differential equations, Math. Ussr. Sb.,* **72** (1992), 343–361.

[24] A. Morassi, E. Rosset, S. Vessella, *Sharp three sphere inequality for perturbations of a product of two second order elliptic operators and stability for the Cauchy problem for the anisotropic plate equation, J. Funct. Anal., (2011),* doi:10.1016/j.jfa.2011.05.011.

[25] T. Nguyen, *Quantitative Unique Continuation for Second Order Elliptic Operators with Singular Coefficients,* Vietnam J. Math., *(2020),* https://doi.org/10.1007/s10013-020-00386-3.

[26] X. X. Tao, *Quantitative unique continuation for Neumann problems of elliptic equations with weight, Math. Meth. Appl. Sci.,* **33** (2010), 863–873.

[27] S. Vessella, *Carleman estimates, optimal three cylinder inequality, and unique continuation properties for solutions to parabolic equations. Commun. Part. Diff. Eq.,* **28** (2003), 637–676.

[28] S. Y. Zhang, *Three-sphere inequalities for second order singular partial differential equations, Acta. Math. Sci.,* **30** (2010), 993–1003.

[29] C. Zhang, *Quantitative unique continuation for the heat equation with Coulomb potentials, Math. Control. Relat. f.,* bf 8 (2018) : 1097-1116.

[30] J. Zhu,*Quantitative uniqueness of elliptic equations, Am. J. Math.* **138** (2006), 733–762.

[31] J. Zhu, *Quantitative unique continuation of solutions to higher order elliptic equations with singular coefficients, Calc. Var. Partial. Dif.,* **57** (2017), 1–35.