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RESEARCH ARTICLE

The existence and uniqueness of fractional boundary value problems of the Riesz-Caputo differential equations with nonlocal conditions

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Abstract

By using the fixed point theorems, we give sufficient conditions for the existence and uniqueness of solutions for the nonlocal fractional boundary value problem of nonlinear Riesz-Caputo differential equation. The boundedness assumption on the nonlinear term is replaced by growth conditions or by a continuous function. Finally, some examples are presented to illustrate the applications of the obtained results.

KEYWORDS:
Fractional boundary value problem, Riesz-Caputo fractional derivative, existence and uniqueness, fixed point, nonlocal conditions

1 | INTRODUCTION

Fractional differential equations can be thought as an extension of the ordinary differential equation of real order. Fractional calculus is as old as differential calculus which goes back to Leibniz and Newton. In recent years, there has been an active movement in fractional differential equations which have been used for modelling real world phenomena in different fields. The reason is that they represent better these phenomena than ordinary differential equations. Geometric and physical interpretation of fractional differentiation and integration can be found in the paper [1]. Very recently, the existence of the solutions for fractional differential equations have attracted a good deal of attention and have been developed by many authors; see the books [2,3,4] and papers [5,6,7,8,9,10,11] and the references therein. A large number of studies on fractional differential equations has been presented for the existence and uniqueness of initial value problems. Multi-point and nonlocal boundary value problems for fractional differential equation are sparse and have received attention in the last decades [7,12,13,14].

It should be pointed out that the most of the papers and monographs on fractional calculus have focused on the fractional differential equations involving Riemann-Lioville and Caputo derivatives in the literature. Both two fractional operators are one sided operator and thus, they hold either past or future memory effects. In contrast, the main feature of the Riesz fractional operator is that it is both left and right sided operator which holds both the history and future non-local memory effects. This property of the Riesz fractional operator is important in the mathematical modelling for physical processes on a finite domain because the present states depend both on the past and future memory effects. As an example, Riesz fractional derivative has been used for the memory effects in both past and future concentrations in the anomalous diffusion problem [15,16].

A variety of papers are devoted to numerical solutions of the fractional calculus, specifically in the anomalous diffusion that involves the Riesz derivative [17,18,16,16,15]. Recently, there are papers on existence and positive solutions for the fractional boundary value problems of Riesz-Caputo derivative. [19,20].

To the best of our knowledge, there does not exist a paper on the fractional boundary value problems (FBVP) of the Riesz-Caputo differential equations with nonlocal boundary conditions. In this paper, we investigate the existence and uniqueness of solutions for the following nonlocal boundary value problems of the Riesz-Caputo fractional differential equations

$$\begin{aligned} {}^{RC}_{0}D^{\nu}_{T}u(\eta) &= F(\eta, u(\eta)) \quad \nu \in (1,2], \quad 0 \le \eta \le T, \\ u(0) &= g(u), \quad u(T) = u_T, \end{aligned} \tag{1}$$

where ${}^{RC}_{0}D^{\nu}_{T}$ is the Riesz-Caputo derivative defined below and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C[0,T] \to \mathbb{R}$ is a continuous function and $u_T \in \mathbb{R}$.

Byszewski[21] first time investigated the existence and uniqueness of a solution of nonlocal Cauchy problems. It should be noted that some psychical processes can be better described by the nonlocal boundary conditions than the usual initial/boundary conditions[22]. For instance, the initial condition $g(u)$ can be taken as

$$g(u) = \sum_{k=1}^{n} a_k u(t_k)$$

where $a_k, k = 1, 2, \ldots, n$ constant and $0 < t_1 < t_2 < \cdots < t_n \le T$.

The remainder of paper is organized as follows. Section 2 introduces some preliminaries, definitions and lemmas which are useful in proving main results. Section 3 provides some sufficient conditions for the existence and the uniqueness of solutions of the problem (1) with nonlocal boundary conditions. We establish these results by using the contraction principle in the Banach space and Schaefer's fixed point theorem and Leray–Schauder fixed point theorem, respectively. Finally, some numerical examples are given to illustrate the applications of the main results.

2 | PRELIMINARIES

In this section, we give some useful definitions and lemmas that will be used in this paper.

Definition 1. [2] Let $\nu > 0$. The left and right Riemann-Liouville fractional integral of a function $f \in C[a,b]$ of order ν defined as, respectively

$$I^{\nu}_{a} f(x) = \frac{1}{\Gamma(\nu)} \int_{a}^{x} (x-s)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

$$_{b}I^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{b} (s-x)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

Definition 2. (Riesz Fractional İntegral) Let $\nu > 0$. The Riesz fractional integral of a function $f \in C[a,b]$ of order ν defined as

$$_{b}I^{\nu}_{a} f(x) = \frac{1}{2\Gamma(\nu)} \int_{a}^{b} |x-s|^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

Note that the Riesz fractional integral operator can be written as

$$_{b}I^{\nu}_{a} f(x) = \frac{1}{2}\Big(I^{\nu}_{a} f(x) +_{b} I^{\nu} f(x)\Big) \tag{2}$$

Definition 3. [2] Let $\nu \in (n, n+1], n \in \mathbb{N}$. The left and right Caputo fractional derivative of a function $f \in C^{n+1}[a,b]$ of order ν defined as, respectively

$${}^{C}_{a}D^{\nu}_{x} f(x) = \frac{1}{\Gamma(n+1-\nu)} \int_{a}^{x} (x-s)^{n-\nu} f^{(n+1)}\, ds = (I^{n+1-\nu}_{a} D^{n+1})u(x).$$

$${}^{C}_{x}D^{\nu}_{b} f(x) = \frac{(-1)^{n+1}}{\Gamma(n+1-\nu)} \int_{x}^{b} (s-x)^{n-\nu} f^{(n+1)}\, ds = (-1)^{n+1}({}_{b}I^{n+1-\nu} D^{n+1})u(x).$$

where D is ordinary differential operator.

Definition 4. Let $\nu \in (n, n+1], n \in \mathbb{N}$. The Riesz-Caputo fractional derivative ${}^{RC}_{a}D^{\nu}$ of order ν of a function $f \in C^{n+1}[a,b]$ defined by

$$\begin{aligned} {}^{RC}_{a}D^{\nu}_{b} f(x) &= \frac{1}{\Gamma(n+1-\nu)} \int_a^b |x-s|^{n-\nu} f^{(n+1)}(s)\,ds \\ &= \frac{1}{2}\Big({}^{C}_{a}D^{\nu}_{x} f(x) + (-1)^{n+1}\,{}^{C}_{x}D^{\nu}_{b} f(x)\Big) \\ &= \frac{1}{2}\Big((I^{n+1-\nu}_{a} D^{n+1})u(x) + (-1)^{n+1}({}_{b}I^{n+1-\nu} D^{n+1})u(x)\Big). \end{aligned}$$

In the case when $\nu \in (1,2]$ and $f(x) \in C^2(a,b)$ we then have

$$ {}^{RC}_{a}D^{\nu}_{b} f(x) = \frac{1}{2}\Big({}^{C}_{a}D^{\nu}_{x} f(x) + {}^{C}_{x}D^{\nu}_{b} f(x)\Big). \tag{3}$$

Lemma 1. [2] Let $f \in C^n[a,b]$ and $\nu \in (n, n+1]$. Then we have the following relations

$$ I^{\nu}_{a}\,{}^{C}_{a}D^{\nu}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

$$ {}_{b}I^{\nu}\,{}^{C}_{x}D^{\nu}_{b} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k.$$

Lemma 2. [19] Assume that $u \in C[0,T]$ satisfies

$$ |u(t)| \le c_1 + c_2 \int_0^T |t-s|^{\nu-1} |u(s)|^{\beta}\,ds + c_3 \int_0^T (T-s)^{\nu-2} |u(s)|^{\beta}\,ds,$$

where $\nu \in (1,2], \beta \in (0,\sigma)$ for some $0 < \sigma < \nu - 1$ and $c_i, (i = 1,2,3)$ are positive constants. Then there is a positive constant C such that

$$ |u(t)| \le C.$$

3 | EXISTENCE RESULTS

Let $E = C[0,T]$ denote the Banach space with the norm defined as $\|u\| = \sup\{|u(t)| : t \in J = [0,T]\}$.

We say that $u \in C^2(J)$ with ${}^{RC}_{0}D^{\nu}_{T}$ exists on J is a solution of the problem (1) if u solves the equation ${}^{RC}_{0}D^{\nu}_{T}u(t) = F(t, u(t))$ for each $t \in J$ and the conditions $u(0) = g(u)$ and $u(T) = u_T$ are fulfilled.

In order to prove the existence results for the problem (1), the following lemmas are useful.

Lemma 3. [19] Assume that $h \in C[0,T]$ and $\nu \in (1,2]$. Then the following boundary value problem of Riesz-Caputo fractional differential equation

$$\begin{cases} {}^{RC}_{0}D^{\nu}_{T}u(t) = h(t), & 0 \le t \le T, \\ u(0) = g(u), \quad u(T) = u_T, \end{cases}$$

has a unique solution $u(x)$ given by

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\nu)} \int_0^T |t-s|^{\nu-1} h(s)\,ds + \frac{T-2t}{T\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} h(s)\,ds \\ &- \frac{T-t}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} h(s)\,ds + (1-\frac{t}{T})g(u) + \frac{t}{T}u_T. \end{aligned}\tag{4}$$

Proof. From the definitions and Lemma 1 we have for $\nu \in (1,2]$

$$ {}_{b}I^{\nu}_{a}\,{}^{RC}_{0}D^{\nu}_{T}u(t) = u(t) - \frac{1}{2}(u(0) + u(T)) - \frac{t}{2}(u'(0) + u'(T)) + \frac{T}{2}u'(T).$$

This implies that

$$u(t)=\frac{1}{2}(u(0)+u(T))+\frac{t}{2}(u'(0)+u'(T))-\frac{T}{2}u'(T)+\frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}h(s)\,ds. \tag{5}$$

We compute the first derivative of u

$$u'(t)=\frac{1}{2}(u'(0)+u'(T))+\frac{1}{\Gamma(\nu-1)}\int_0^t (t-s)^{\nu-2}h(s)\,ds-\frac{1}{\Gamma(\nu-1)}\int_t^T (s-t)^{\nu-2}h(s)\,ds. \tag{6}$$

The equation (5) can be rewritten as follows

$$u(t)=u(0)+\frac{T}{2}(u'(0)-u'(T))+\frac{t}{2}(u'(0)+u'(T))+\frac{1}{\Gamma(\nu)}\int_0^T [(T-s)^{\nu-1}+|t-s|^{\nu-1}]h(s)\,ds. \tag{7}$$

From the equation (6), we have

$$u'(T)=u'(0)+\frac{2}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}h(s)\,ds. \tag{8}$$

We plug the equation above (8) into the equation (7) to obtain

$$u(t)=u(0)+tu'(0)-\frac{T-t}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}h(s)\,ds+\frac{1}{\Gamma(\nu)}\int_0^T [(T-s)^{\nu-1}+|t-s|^{\nu-1}]h(s)\,ds. \tag{9}$$

Applying the boundary conditions to the equation (9) yields the desired result (4).

□

Lemma 4. [23] Let X be a Banach space and B be a closed and convex subset of X. If C is a open subset of B and $T : C \to C$ is a continuous and compact operator, then one of the following hols:

1. The operator has a fixed point in C,
2. There is a point $c \in \partial C$ with $0 < \mu < 1$ such that $c = \mu T(c)$.

Lemma 5. [24] Let K be a cone of a Banach space X. Assume that Ω_1 and Ω_2 are two open subset of X with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be completely continuous operator. İf one of the following conditions holds

1. $\|Tx\| \le \|x\|$ for all $x \in K \cap \partial\Omega_1$ and $\|Tx\| \ge \|x\|$ for all $x \in K \cap \partial\Omega_2$,
2. $\|Tx\| \ge \|x\|$ for all $x \in K \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$ for all $x \in K \cap \partial\Omega_2$,

then T has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

By making use of Lemma 3, we consider the operator $K : C[0,1] \to C[0,1]$ defined as

$$\begin{aligned} K(u)(t) = {} & \frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}F(s,u(s))\,ds+\frac{T-2t}{T\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}F(s,u(s))\,ds \\ & -\frac{T-t}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}F(s,u(s))\,ds+(1-\frac{t}{T})g(u)+\frac{t}{T}u_T. \end{aligned} \tag{10}$$

We now state and prove the first existence result by using Banach contraction principle.

Theorem 1. Assume that the following assumptions hold

A1 The function F is Lipschitz continuous in the second variable, that is, there is a positive constant C_1 such that

$$|F(t,z)-F(t,y)| \le C_1|z-y|, \quad \textit{for each} \quad t \in J, \quad \textit{and} \quad \forall z,y \in \mathbb{R}.$$

A2 The function g is Lipschitz continuous, that is, there is a positive constant C_2 such that

$$|g(z)-g(y)| \le C_2|z-y|, \quad \textit{for each} \quad t \in J, \quad \textit{and} \quad \forall z,y \in C(J).$$

Assume also that

$$\frac{(3+\nu)T^{\nu}}{\Gamma(\nu+1)}C_1 + C_2 < 1. \tag{11}$$

Then the problem (1) has a unique solution.

Proof. Obviously, the solutions of the problems (1) are the fixed point of the operator K. We will show that the operator K is a contraction. To this end, let $u, v \in C(J)$. Then for $t \in J$ we get

$$\begin{aligned}
|K(u)(t) - K(v)(t)| \le & \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |F(s,u(s)) - F(s,v(s))|\, ds \\
& + \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |F(s,u(s)) - F(s,v(s))|\, ds \\
& + \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s)) - F(s,v(s))|\, ds \\
& + \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s)) - F(s,v(s))|\, ds + |g(u) - g(v)| \\
\le & \frac{C_1\|u-v\|}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1}\, ds + \frac{C_1\|u-v\|}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1}\, ds \\
& + \frac{C_1\|u-v\|}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1}\, ds + \frac{TC_1\|u-v\|}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2}\, ds + C_2\|u-v\| \\
\le & \Big(\frac{C_1 t^{\nu}}{\Gamma(\nu+1)} + \frac{C_1(T-t)^{\nu}}{\Gamma(\nu+1)} + \frac{C_1 T^{\nu}}{\Gamma(\nu+1)} + \frac{C_1 T^{\nu}}{\Gamma(\nu)} + C_2 \Big) \|u-v\| \\
\le & \Big(\frac{(3+\nu)T^{\nu}}{\Gamma(\nu+1)} C_1 + C_2 \Big) \|u-v\|.
\end{aligned}$$

Therefore we arrive at

$$\|K(u) - K(v)\| \le \frac{(3+\nu)T^{\nu}}{\Gamma(\nu+1)} C_1 + C_2 \Big) \|u-v\|.$$

This shows that K is a contraction operator. Banach fixed point theorem implies that K has at least one fixed point u which is a unique solution of the problem (1). □

Next we present the second existence theorem in the next theorem.

Theorem 2. Assume that the following conditions are satisfied

A3 $F \in C([0,T] \times \mathbb{R})$, that is, F is a continuous function.

A4 There is a positive constant L_1 and $\beta \in (0,\sigma)$ for some $0 < \sigma < \nu - 1$ such that

$$|F(t,z)| \le L_1(1 + |z|^{\beta}) \quad \textit{for each} \quad t \in J \quad \textit{and} \quad \forall z \in \mathbb{R}.$$

A5 There exists a positive constant L_2 such that

$$|g(z)| \le L_2 \quad \forall z \in C[0,T].$$

Then the problem (1) has at least one solution on J.

Proof. We will show that K has a fixed point by using the Schaefer fixed point theorem. We first show K is continuous operator. To show this, consider a sequence $\{u_n\}$ with the limit $u_n \to u \in C[0,1]$. Then for $t \in J$, we get

$$
\begin{aligned}
|K(u_n)(t) - K(u)(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |F(s,u_n(s)) - F(s,u(s))| \, ds \\
& + \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |F(s,u_n(s)) - F(s,u(s))| \, ds \\
& + \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u_n(s)) - F(s,u(s))| \, ds \\
& + \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u_n(s)) - F(s,u(s))| \, ds + |g(u_n) - g(u)| \\
\leq & \Big(\frac{t^\nu}{\Gamma(\nu+1)} + \frac{(T-t)^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu)} \Big) \| F(s,u_n(s)) - F(s,u(s)) \| + \| g(u_n) - g(u) \| \\
\leq & \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} \| F(s,u_n(s)) - F(s,u(s)) \| + \| g(u_n) - g(u) \|.
\end{aligned}
$$

The continuity of the functions F and g yields

$$
\| K(u_n) - K(u) \| \to 0, \quad as \quad n \to \infty,
$$

which shows that K is continuous.

Secondly we will show that K transforms bounded sets to bounded sets in $C[0,T]$. Let $M_\ell = \{u \in C[0,T] : \|u\| \leq \ell\}$ be a bounded subset of $C[0,T]$. Our goal is to show that $\|K(z)\| \leq m$ for some constant m. For each $t \in J$ and $u \in M_\ell$ we have

$$
\begin{aligned}
|K(u)(t)| \leq & \frac{1}{\Gamma(\nu)} \int_0^T |t-s|^{\nu-1} |F(s,u(s))| \, ds + \frac{T-2t}{T\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))| \, ds \\
& + \frac{T-t}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))| \, ds + |(1 - \frac{t}{T}) g(u)| + |\frac{t}{T} u_T| \\
\leq & \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |F(s,u(s))| \, ds + \frac{1}{\Gamma(\nu)} \int_t^T (s-t)^{\nu-1} |F(s,u(s))| \, ds \\
& + \frac{1}{\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} |F(s,u(s))| \, ds + \frac{T}{\Gamma(\nu-1)} \int_0^T (T-s)^{\nu-2} |F(s,u(s))| \, ds + 2|g(u)| + |u_T| \\
\leq & \Big(\frac{t^\nu}{\Gamma(\nu+1)} + \frac{(T-t)^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu+1)} + \frac{T^\nu}{\Gamma(\nu)} \Big) L_1(1+\ell^\beta) + 2L_2 + |u_T| \\
\leq & \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + 2L_2 + |u_T|.
\end{aligned}
$$

Therefore we get

$$
\|K(z)\| \leq \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)} L_1(1+\ell^\beta) + 2L_2 + |u_T| := m,
$$

which proves the desired result.

Finally we will show that K transforms bounded sets to equicontinuous sets in $C[0,T]$. Again let $M_\ell = \{u \in C[0,T] : \|u\| \le \ell\}$ be a bounded subset of $C[0,T]$. We give a bound on the derivative of $K(u)'(t)$ for each $t \in J$ and $u \in M_\ell$ as follows

$$\begin{aligned}
|K(u)'(t)| \le& \frac{1}{\Gamma(\nu-1)}\int_0^t (t-s)^{\nu-2}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu-1)}\int_t^T (s-t)^{\nu-2}|F(s,u(s))|\,ds \\
&+ \frac{2}{T\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + |g(u)| + \frac{1}{T}|u_T| \\
\le& \Big(\frac{t^{\nu-1}}{\Gamma(\nu)} + \frac{(T-t)^{\nu-1}}{\Gamma(\nu)} + \frac{2T^{\nu-1}}{\Gamma(\nu+1)} + \frac{T^{\nu-1}}{\Gamma(\nu)}\Big)L_1(1+\ell^\beta) + 2L_2 + \frac{1}{T}|u_T| \\
\le& \frac{(2+3\nu)T^{\nu-1}}{\Gamma(\nu+1)}L_1(1+\ell^\beta) + L_2 + \frac{1}{T}|u_T|.
\end{aligned}$$

Set $L := \dfrac{(2+3\nu)T^{\nu-1}}{\Gamma(\nu+1)}L_1(1+\ell^\beta) + L_2 + \frac{1}{T}|u_T|$. Let $t_1, t_2 \in J$ with $t_1 < t_2$, then we have

$$|K(u)(t_1) - K(u)(t_2)| = \int_{t_1}^{t_2} |K(u)'(s)|\,ds \le L(t_2 - t_1).$$

Thus, $K(M_\ell)$ is equicontinuous in $C[0,T]$. Sp far we have shown that the operator K is completely continuous.

Lastly, we will show that the set

$$\mathcal{E}(K) = \{u \in C[0,T] : u = \mu K(u), \quad \mu \in (0,1)\}$$

is bounded. Let $u = \mu K(u)$ for $\mu \in (0,1)$. Then we have for $t \in J$

$$\begin{aligned}
|u(t)| \le& \frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds \\
&+ \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T| \\
\le& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}|F(s,u(s))|\,ds \\
&+ \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T| \\
\le& \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}L_1 + 2L_2 + |u_T| + \frac{L_1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|u(s)^\beta|\,ds + \frac{L_1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}|u(s)^\beta|\,ds \\
&+ \frac{L_1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|u(s)^\beta|\,ds + \frac{L_1 T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|u(s)^\beta|\,ds.
\end{aligned}$$

Lemma 2 implies that there is a positive constant C such that

$$\|u\| \le C.$$

This concludes that $\mathcal{E}(K)$ is bounded. By using Schaefer's fixed point theorem, we infer that K has a fixed point which is a solution of the problem (1). □

Theorem 3. Assume the condition $A5$ in the previous theorem and the following condition hold

A6 There are $\phi \in C[0,T]$ and $\Phi : [0,\infty) \to \mathbb{R}^+$ continuous and increasing functions such that $|F(t,z)| \leq \phi(t)\Phi(|z|)$ for each $t \in J$ and $\forall z \in \mathbb{R}$.

Assume also that there is a positive constant C_m such that

$$\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\Big)\frac{\phi_s\Phi(C_m)}{C_m} + \frac{2L_2+|u_T|}{C_m} < 1, \quad where \quad \phi_s = \sup_{t\in J}\phi(t). \tag{12}$$

Then The problem (1) has at least one solution on $[0,T]$.

Proof. Define $M_c = \{u \in C(0,T] : \|u\| \leq C_m\}$. Clearly, M_c is closed, convex and bounded subset of $C[0,T]$. For each $t \in J$ and $u \in M_c$ using the assumptions $A5$ and $A6$ we have

$$\begin{aligned}
|Ku(t)| \leq& \frac{1}{\Gamma(\nu)}\int_0^T |t-s|^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds\\
&+\frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T|\\
\leq& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}|F(s,u(s))|\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}|F(s,u(s))|\,ds + \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}|F(s,u(s))|\,ds + 2|g(u)| + |u_T|\\
\leq& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds + \frac{1}{\Gamma(\nu)}\int_t^T (s-t)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds\\
&+\frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}\phi(s)\Phi(|u(s)|)\,ds + \frac{T}{\Gamma(\nu-1)}\int_0^T (T-s)^{\nu-2}\phi(s)\Phi(|u(s)|)\,ds + 2|g(u)| + |u_T|\\
\leq& \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\phi_s\Phi(\|u\|) + 2L_2 + |u_T|\\
\leq& \frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\phi_s\Phi(C_m) + 2L_2 + |u_T|\\
\leq& C_m. \qquad \textit{(from the condition (12))}
\end{aligned}$$

We have shown that the operator $K : M_c \to M_c$ is continuous and completely continuous. İf there is $u \in \partial M_c$ with $\mu \in (0,1)$ satisfying $u = \mu K u$, then the we would get a contradiction from the discussion above. As a consequence of Leray–Schauder fixed point theorem (see Lemma (4)), K has a fixed point $u \in \overline{M_c}$. This implies that there exists one solution of the equation (1). Thus we complete the proof. □

4 | NUMERICAL EXAMPLES

This section is devoted to numerical examples to illustrate the application of the results presented in this paper.

Example 1. Consider the following differential equation with the Riesz-Caputo fractional derivative of order $\nu \in (1,2]$

$$\begin{aligned}
{}^{RC}_{0}D^{\nu}_{1}u(t) &= \frac{1}{12}u(t) + (1-t)(\frac{1}{6}+t), \quad t \in [0,1], \nu \in (1,2],\\
u(0) &= \frac{1}{2}u(\frac{1}{2}), \quad u(1) = 0,
\end{aligned} \tag{13}$$

Let $F(t,w)=\frac{1}{12}u(t)+(1-t)(\frac{1}{6}+t),\quad (t,w)\in[0,1]\times\mathbb{R}$, and $g(u)=\frac{1}{2}u(\frac{1}{2})$. Then for any $u,w\in\mathbb{R}$ and $t\in[0,1]$ we have

$$|F(t,u)-F(t,w)|\leq\frac{1}{12}|u-w|.$$

Moreover, we have

$$|g(u)-g(w)|\leq\frac{1}{2}|u-w|.$$

Thus, the conditions $A1$ and $A2$ are satisfied with $C_1=\frac{1}{12}$ and $C_2=\frac{1}{10}$. Taking $T=1$, we observe that

$$\frac{1}{12}\frac{(3+\nu)}{\Gamma(\nu+1)}+\frac{1}{2}<1$$

if and only if $\dfrac{(3+\nu)}{6}<\Gamma(\nu+1)$ which holds true since $\Gamma(\nu+1)>\frac{5}{6}$ when $\nu\in(1,2]$. So, the condition (11) is satisfied. Theorem 1 implies that the problem (13) has a unique solution in $[0,1]$.

In general, the exact solutions of nonlinear fractional differential equation (even ordinary nonlinear differential equations) are not available. Thus, we use the method in[15] to plot the numerical solutions of problems. We report the numerical solution of problem (13) with $\nu=\frac{3}{2}$ in Figure 1 .

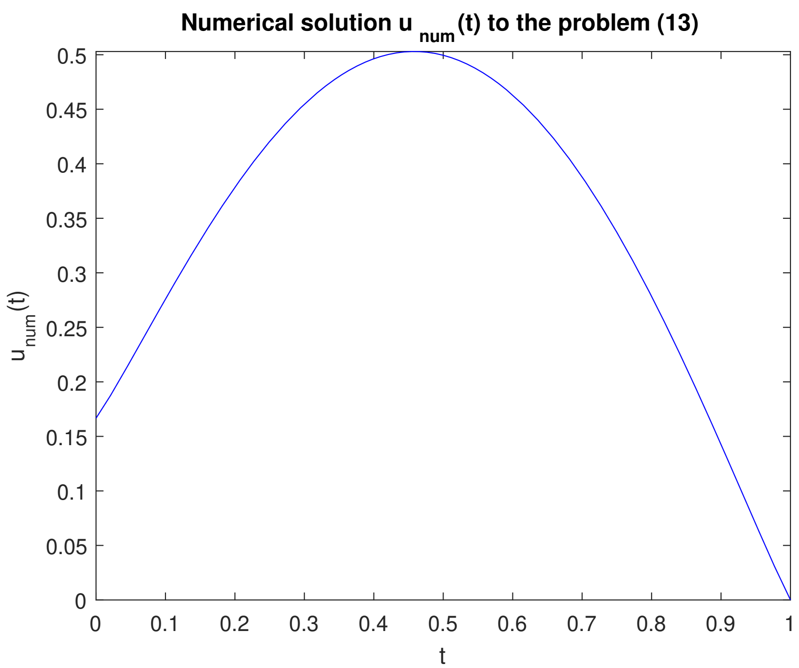

FIGURE 1 The numerical trajectory of the solution for Example 13 with $\nu=\frac{3}{2}$.

Example 2. Consider the following boundary value problem of the fractional Riezs-Caputo derivative,

$$\begin{aligned}{}^{RC}_{0}D^{\nu}_{1}u(t)&=\frac{|u(t)|}{(8+t^2)(1+|u(t)|)},\quad t\in[0,1],\nu\in(1,2]\\ u(0)&=\sum_{k=1}^{n}a_k u(t_k),\quad u(1)=0,\end{aligned}\tag{14}$$

where $0<t_1<t_2<\cdots<t_n<1$, and $a_k>0, k=0,1\ldots,n$ are constants satisfying $\sum_{k=1}^{n}a_k<\frac{1}{4}$.

Let $F(t,w) = \dfrac{w}{(8+t^2)(1+w)}$, $\quad (t,w) \in [0,1]\times[0,\infty)$, and $g(u) = \sum_{k=1}^{n} a_k u(t_k)$. Then for any $u, w \in [0,\infty)$ and $t \in [0,1]$ we have

$$\begin{aligned}|F(t,u) - F(t,w)| &= \frac{1}{8+t^2}\Big|\frac{u}{1+u} - \frac{w}{1+w}\Big| = \frac{1}{8+t^2}\frac{|u-w|}{(1+u)(1+w)}\\ &\leq \frac{1}{8}|u-w|.\end{aligned}$$

Moreover, we have

$$|g(u) - g(w)| \leq \sum_{k=1}^{n} a_k |u-w|.$$

Thus, the conditions $A1$ and $A2$ are satisfied with $C_1 = \frac{1}{10}$ and $C_2 \leq \frac{1}{4}$. We also have with $T = 1$

$$\frac{(3+\nu)}{\Gamma(\nu+1)}\frac{1}{8} + \frac{2}{8} \leq \frac{5+\nu}{8\Gamma(\nu+1)} \leq \frac{1}{\Gamma(\nu+1)} < 1$$

if and only if $\Gamma(\nu+1) > 1$ which holds true when $\nu \in (1,2]$. So, the condition (11) is satisfied. Theorem 1 implies that the problem (14) has a unique solution in $[0,1]$

Example 3. Consider the following fractional differential equation of the fractional Riesz-Caputo derivative

$$\begin{aligned}{}^{RC}_{\;\;0}D^{\frac{8}{5}}_{1} u(t) &= \frac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)}, \quad t \in [0,1],\\ u(0) &= \sin(2\pi u(\frac{1}{2})), \quad u(1) = 0.\end{aligned} \tag{15}$$

Let $F(t,w) = \dfrac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)}$, $\quad (t,w) \in [0,1]\times[0,\infty)$, and $g(u) = \sin(2\pi u(\frac{1}{2}))$ with $\nu = \frac{8}{5}$. Let $\beta = \frac{1}{5}$ and $\sigma = \frac{2}{5}$, then $\beta \in (0,\sigma)$ and $0 < \sigma < \nu - 1$. Then for any $u \in [0,\infty)$ and $t \in [0,1]$ we have

$$\begin{aligned}|F(t,u)| &= \frac{|u(t)|^{\frac{1}{5}}}{(1+t^2)(1+|u(t)|)}\\ &\leq \frac{1}{2}(1+|u(t)|^{\frac{1}{5}}).\end{aligned}$$

Additionally, we have

$$|g(u)| \leq 1 = L_2.$$

Thus, the conditions $A3 - A4$ and $A5$ are satisfied. Then we infer from Theorem 2 that the problem (15) has at least one solution on J.

Let $\phi(t) = \frac{1}{1+t^2}$ and $\Phi(|u|) = |u(t)|^{\frac{1}{5}}$. Then we have $F(t,u(t)) \leq \phi(t)\Phi(|u|)$ with $\phi_s = \frac{1}{2}$. Let $C_m = 81$ and $u_T = 0$. We find that

$$\Big(\frac{(3+\nu)T^{\nu}}{\Gamma(\nu+1)}\Big)\frac{\phi_s\Phi(C_m)}{C_m} + \frac{2L_2 + |u_T|}{C_m} = \frac{23}{270\Gamma(\frac{13}{5})} + \frac{2}{81} < 1.$$

So the condition (12) is satisfied. Theorem 3 tells us there exists at least one solution to the problem (15).

Example 4. Consider the following three-point fractional boundary value problem

$$\begin{aligned}{}^{RC}_{\;\;0}D^{\frac{3}{2}}_{1} u(t) &= \frac{1}{4}t^2u^2(t)e^{-u^2(t)}, \quad t \in [0,1],\\ u(0) &= \frac{1}{32}e^{-u(\eta)}, \quad \eta \in (0,1], \quad u(1) = \frac{1}{16}.\end{aligned} \tag{16}$$

We will exhibit that the conditions $A5 - A6$ and (12) are satisfied.

Let $F(t,w) = \frac{1}{4}t^2u^2(t)e^{-u^2(t)}$, $\quad (t,w) \in [0,1]\times\mathbb{R}$, and $g(u) = e^{-u(\eta)}$, $\quad \eta \in (0,1]$ with $\nu = \frac{3}{2}$. For each $u \in \mathbb{R}$ and $t \in [0,1]$ we have

$$|F(t,u)| = |\frac{1}{4}t^2u^2(t)e^{-u^2(t)}| \leq \frac{1}{4}t^2u^2(t) = \phi(t)\Phi(u), \quad (t,u) \in [0,1]\times\mathbb{R},$$

where $\phi(t) = \frac{1}{4}t^2$ and $\Phi(u) = u^2$ with $\phi_s = \sup_{t\in[0,1]} |\phi(t)| = \frac{1}{4}$. The function $g(u)$ is bounded, that is,

$$|g(u)| \leq \frac{1}{32} = L_2.$$

Lastly we check the condition (12). Let $C_m = 1$ and $u_T = \frac{1}{16}$, then

$$\Big(\frac{(3+\nu)T^\nu}{\Gamma(\nu+1)}\Big)\frac{\phi_s\Phi(C_m)}{C_m} + \frac{2L_2 + |u_T|}{C_m} = \frac{9}{8\Gamma(\frac{5}{2})} + \frac{1}{8} < 1.$$

Again Theorem 2 implies that the problem (16) has at least one solution on $[0, T]$.

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