Long-time behavior of global weak solutions for a Beris-Edwards type model of nematic liquid crystals

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Abstract

We consider a generalization of the standard Beris-Edwards system modeling incompressible liquid crystal flows of nematic type. This couples a Navier-Stokes system for the fluid velocity with an evolution equation for the Q-tensors variable describing the direction of liquid crystal molecules. The convergence at infinite time for global solutions is studied and we prove that whole trajectory goes to a single equilibrium by using a Lojasiewicz-Simon's result.

Keywords: Liquid crystals; Landau-De Gennes theory; Navier-Stokes system; Large-time behavior for dissipative systems.

1 Introduction

We deal with a system, which contains the Navier-Stokes equations with an additional forcing term and an evolution equation of parabolic type for the unknowns velocity, $\boldsymbol{u}$, pressure, p, and tensor parameter order, Q, satisfying: $(\boldsymbol{u}, p, Q) : (0,T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{3\times 3}$,

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\tau(Q) + \nabla\cdot\sigma(H,Q) \\ \nabla\cdot\boldsymbol{u} = 0 \\ \partial_t Q + (\boldsymbol{u}\cdot\nabla)Q - S(\nabla\boldsymbol{u}, Q) = -\gamma\, H(Q) \end{cases} \tag{1}$$

in the time-space cylinder $\Omega\times(0,T)$, subject to the initial and boundary conditions,

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad Q|_{t=0} = Q_0 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \qquad \partial_{\boldsymbol{n}} Q|_{\partial\Omega} = 0 \quad \text{in } (0,T). \tag{3}$$

The set $\Omega\subset\mathbb{R}^3$ is a smooth and bounded domain, the constant $\nu>0$ is the viscosity coefficient and $\gamma>0$ is a material-dependent elastic constant. The tensors $\tau=\tau(Q)\in\mathbb{R}^{3\times 3}$

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and $\sigma = \sigma(H,Q) \in \mathbb{R}^{3\times 3}$ are defined by

$$\begin{cases} \tau_{ij}(Q) & := & -\varepsilon\,(\partial_j Q : \partial_i Q) = -\varepsilon\,\partial_j Q_{kl}\,\partial_i Q_{kl}, \\ \sigma(H,Q) & := & H\,Q - Q\,H, \end{cases}$$

where $\varepsilon > 0$ and the tensor $H = H(Q)$ is related to the variational derivative in $L^2(\Omega)$ of a free energy functional, in fact

$$E(Q) := \frac{\varepsilon}{2}|\nabla Q|^2 + F(Q), \qquad \mathcal{E}(Q) := \int_\Omega E(Q)\,dx, \qquad H := \frac{\delta\mathcal{E}(Q)}{\delta Q}. \tag{4}$$

Here, $A : B = A_{ij}\,B_{ij}$ denote the scalar product of matrices (using the Einstein summation convention over repeated indices) and the potential function $F(Q)$ is defined by

$$F(Q) := \frac{a}{2}\,|Q|^2 - \frac{b}{3}\,(Q^2 : Q) + \frac{c}{4}\,|Q|^4, \tag{5}$$

with a, b, $c \in \mathbb{R}$ and $c > 0$. We denote by $|Q| = (Q : Q)^{1/2}$ the matrix euclidean norm. Then, from (4) and (5),

$$H = H(Q) = -\varepsilon\,\Delta Q + f(Q) \tag{6}$$

where

$$f(Q) = \frac{\partial F}{\partial Q}(Q) = a\,Q - \frac{b}{3}\,\left(Q^2 + QQ^t + Q^tQ\right) + c\,|Q|^2\,Q.$$

Note that H uses the one-constant approximation for the Oseen-Frank energy of liquid crystals together with a Landau-DeGennes expression for the bulk energy given by $f(Q)$.

Finally,

$$S(\nabla\boldsymbol{u}, Q) = \nabla\boldsymbol{u}\,Q^t - Q^t\,\nabla\boldsymbol{u}$$

is the so-called stretching term.

The vector $\boldsymbol{n}$ denotes the normal outwards vector on the boundary $\partial\Omega$,

The configurations of liquid crystals can be described by a director field as minimizers of an energy functional following the Ossen-Frank theory. In an Ericksen-Leslie model the dynamic of the problem is considered, the evolution of the director field is coupled with a Navier-Stokes-type equation for the underlying flow field. In the Landau-De Gennes theory, the director vector is replaced by a symmetric and traceless matrix Q, which measures the deviation of the second moment tensor from its isotropic value. Different expressions of the Q-tensor order parameter allows to represent a uniaxial, biaxial or isotropic behavior of the molecules of the nematic crystal. The corresponding dynamic model is called Beris-Edwards model. The system (1)-(3) is a modified version of this type of models studied by Paicu & Zarnescu in [13] and Abels et al. in [1] and was introduced in [10] and [11]. This model retains the essential difficulties of the models in [13] and [1]. In fact, the results obtained here can be extended to those models.

The large-time behavior of some models for Nematic liquid crystals with unknown vector director are studied in [16], [9] (without stretching terms), in [12], [8], [15] (with stretching terms) and in [14] (where different results are deduced depending on considering or not the stretching terms).

On the other hand, the large-time behavior is also analyzed for others related models, for example in [7] for a Cahn-Hilliard-Navier-Stokes system in $2D$ domains, in [6] for a chemotaxis model, and in [4] and [3], where a Cahn-Hilliard-Navier-Stokes vesicle model and a smectic-A liquid crystals model are studied respectively. In these articles, the variables phase field, orientation vector in the liquid crystal case or chemical density in the chemotaxis case are unidimensional. The model presented in [3] of liquid crystals follows the Ossen-Frank theory.

In [11], some results of local in time regularity and uniqueness of the model (1)-(3) are proved.

In the present paper we study the large-time behavior for a system when the variable Q which models the orientation of the molecules of the crystal, is a tensor following the Landau-De Gennes theory.

Sections 2 and 3 describe the model and the weak solution concept (more details can be seen in [10]). In Section 3, two suitable energy inequalities are proved, a time-integral version for all time t and a time-differential version for almost every time. These inequalities as far as we know, have not been proved before in the liquid crystal case and they will be essential in the proof (they are cited in [14], [2] but do not proved). The used argument is valid only for weak solutions. In fact, the standard argument of obtaining regularity for big viscosity is not clear in this case. In section 4 the convergence at infinite time for global weak solutions is studied. Firstly, we prove that the ω-limit set defined only for weak solutions (strong solution is necessary in standard methods) consists of critical points of the free-energy. In the last section, the convergence of the whole trajectory to a single equilibrium as time goes to infinity is proved via a Lojasiewicz-Simon's lemma.

A first reduced version of this paper appears in [5]. We prove moreover now the existence of a special regularized energy satisfying the energy's law inequality for all interval of time without which it is not possible to prove the convergence of the trajectory to a unique point.

Notations

The notation can be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H^1 = H^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(0,T;X)$ the Banach space $L^p(0,T;X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$, and the type $\mathbb{L}^2 = \mathbb{L}^2(\Omega)^{N\times N}$ for the tensors.

We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla\cdot\boldsymbol{u} = 0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

From now on, $C > 0$ will denote different constants, depending only on data of the problem.

2 Weak solutions

We start arguing in a formal manner, assuming a regular enough solution $(\boldsymbol{u}, p, Q)$ of (1)-(3). For more detailed calculations in this section, see [10].

Variational formulation

By testing (1) by any $\widetilde{\boldsymbol{u}} : \Omega \to \mathbb{R}^3$ with $\widetilde{\boldsymbol{u}}|_{\partial\Omega} = \mathbf{0}$ and $\nabla \cdot \widetilde{\boldsymbol{u}} = 0$ in Ω, we arrive at the following variational formulation of (1):

$$\langle \partial_t \boldsymbol{u}, \widetilde{\boldsymbol{u}} \rangle + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \widetilde{\boldsymbol{u}}) + \nu(\nabla \boldsymbol{u}, \nabla \widetilde{\boldsymbol{u}}) - ((\widetilde{\boldsymbol{u}} \cdot \nabla)Q, H) + (\sigma(H, Q), \nabla \widetilde{\boldsymbol{u}}) = 0, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ is the duality product between V' and V and $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.

On the other hand, testing the Q-equation of (1) by any $\widetilde{H}$ and the system $-\varepsilon\,\Delta Q + f(Q) = H$ by any $\widetilde{Q}$, we get the following variational formulation:

$$\begin{cases} (\partial_t Q, \widetilde{H}) + ((\boldsymbol{u} \cdot \nabla)Q, \widetilde{H}) - (S(\nabla \boldsymbol{u}, Q), \widetilde{H}) + \gamma\,(H, \widetilde{H}) = 0, \\ \varepsilon\,(\nabla Q, \nabla \widetilde{Q}) + (f(Q), \widetilde{Q}) - (H, \widetilde{Q}) = 0, \end{cases} \tag{8}$$

for any $\widetilde{H}$, $\widetilde{Q} : \Omega \to \mathbb{R}^{3\times 3}$.

From (8), we obtain, in particular that:

$$(\partial_t Q, \widetilde{Q}) + ((\boldsymbol{u} \cdot \nabla)Q, \widetilde{Q}) - (S(\nabla \boldsymbol{u}, Q), \widetilde{H}) - \varepsilon\,\gamma\,(\Delta Q, \widetilde{Q}) + \gamma\,(f(Q), \widetilde{Q}) = 0. \tag{9}$$

Dissipative energy law and global in time a priori estimates

By taking $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$ in (7) and $(\widetilde{H}, \widetilde{Q}) = (H, \partial_t Q)$ in (8) the following "energy equality" holds:

$$\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega E(Q)\,d\boldsymbol{x}\right) + \nu\|\nabla \boldsymbol{u}\|^2_{\mathbb{L}^2} + \gamma\|H\|^2_{\mathbb{L}^2} = 0. \tag{10}$$

Observe that $\displaystyle\int_\Omega E(Q)\,d\boldsymbol{x}$ is not a positive term due to $F(Q)$. However, it is possible to find a large enough constant $\mu > 0$ depending on parameters a, b and c given in the definition of $F(Q)$ in (5), such that

$$F_\mu(Q) := F(Q) + \mu \geq \frac{c}{8}\,|Q|^4. \tag{11}$$

By replacing $E(Q)$ in (10) by $E_\mu(Q) := \dfrac{1}{2}|\nabla Q|^2 + F_\mu(Q) \geq 0$, and denoting the kinetic and the free energy of Q-tensor as

$$\mathcal{E}_k(\boldsymbol{u}(t)) := \frac{1}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2} \quad \text{and} \quad \mathcal{E}_\mu(Q) := \int_\Omega E_\mu(Q)\,d\boldsymbol{x}$$

and the total energy as $\mathcal{E}(\boldsymbol{u}, Q) := \mathcal{E}_k(\boldsymbol{u}) + \mathcal{E}_\mu(Q)$, then (10) implies

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}(t), Q(t)) + \nu\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2} + \gamma\|H\|^2_{\mathbb{L}^2} = 0. \tag{12}$$

This energy equality shows the dissipative character of the model with respect to the total free-energy $\mathcal{E}(\boldsymbol{u}(t), Q(t))$. In fact, assuming finite total energy of initial data, i.e.

$$\int_\Omega E_\mu(Q_0)\, d\boldsymbol{x} + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} < +\infty,$$

then the following regularity hold:

$$\begin{gathered}\boldsymbol{u} \in L^\infty(0,+\infty; \boldsymbol{L}^2(\Omega)) \cap L^2(0,+\infty; \mathbf{H}^1(\Omega)),\\ \nabla Q \in L^\infty(0,+\infty; \mathbb{L}^2(\Omega)), \quad F_\mu(Q) \in L^\infty(0,+\infty; L^1(\Omega)),\\ H \in L^2(0,+\infty; \mathbb{L}^2(\Omega)).\end{gathered} \tag{13}$$

From (11) and (13), we deduce that $Q \in L^\infty(0,+\infty; \mathbb{L}^4(\Omega))$, $Q \in L^\infty(0,+\infty; \mathbb{H}^1(\Omega))$ and, in particular

$$Q \in L^\infty(0,+\infty; \mathbb{L}^6(\Omega)). \tag{14}$$

Since $f(Q)$ is a third order polynomial function, $|f(Q)| \leq C(a,b,c)\left(|Q| + |Q|^2 + |Q|^3\right)$ which, together with (14), gives $f(Q) \in L^\infty(0,+\infty; \mathbb{L}^2(\Omega))$.

From $H(Q) = -\varepsilon\,\Delta Q + f(Q)$ we obtain that $\Delta Q \in L^2(0,T; \mathbb{L}^2(\Omega))$ for all $T > 0$. Finally, by using the H^2-regularity of the Poisson problem:

$$\left\{\begin{array}{rcl} -\varepsilon\,\Delta Q + Q &=& f(Q) + Q \quad \text{in } \Omega,\\ \partial_{\boldsymbol{n}} Q|_\Gamma &=& 0 \end{array}\right.$$

we deduce that:

$$Q \in L^2(0,T; \mathbb{H}^2(\Omega)) \quad \forall T > 0.$$

Definition 1 (Weak solution) *It will be said that $(\boldsymbol{u}, Q)$ is a weak solution in $(0,+\infty)$ of problem (1)-(3) if*

$$\left\{\begin{array}{l} \boldsymbol{u} \in L^\infty(0,+\infty; \mathbf{H}) \cap L^2(0,+\infty; \mathbf{V}),\\ Q \in L^\infty(0,+\infty; \mathbb{H}^1(\Omega)) \cap L^2_{loc}(0;+\infty; \mathbb{H}^2(\Omega)) \quad \forall T > 0, \end{array}\right. \tag{15}$$

satisfies the variational formulation (7) and (8), the initial conditions (2), the boundary conditions (3) and the following energy inequality a.e. $t_1, t_0; t_1 \geq t_0 \geq 0$:

$$\mathcal{E}(\boldsymbol{u}(t_1), Q(t_1)) - \mathcal{E}(\boldsymbol{u}(t_0), Q(t_0)) + \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}(s)\|^2_{\mathbb{L}^2} + \gamma\|H(s)\|^2_{\mathbb{L}^2})\, ds \leq 0 \tag{16}$$

for the total energy.

Note that the regularity imposed in (15) is satisfied up to infinite time excepting the $\mathbb{H}^2(\Omega)$-regularity for Q.

By applying the regularity (15) to the systems (7) and (9), we have

$$\partial_t \boldsymbol{u} \in L^{4/3}_{loc}([0,+\infty); \mathbf{V}') \quad \text{and} \quad \partial_t Q \in L^{4/3}_{loc}([0,+\infty); \mathbb{L}^2(\Omega)).$$

Hence, the following time-continuity can be deduced:

$$\boldsymbol{u} \in C([0,+\infty); \mathbf{V}') \cap C_w([0,+\infty); \mathbf{H}),\ Q \in C([0,+\infty); \mathbb{L}^2(\Omega)) \cap C_w([0,+\infty); \mathbb{H}^1).$$

In particular, the initial conditions (2) have sense.

Theorem 2 (Existence of weak solutions) *If* $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1(\Omega)$, *there exists a weak solution* $(\boldsymbol{u}, Q)$ *of system (1)-(3) in* $(0, +\infty)$.

Proof: The first part of this theorem is proved in [10] by means of a Galerkin approximation. Therefore, we only going to prove (16). We start from the following energy equality satisfied by the Galerkin approximate solutions (see [10]) for all t, t_0 with $t \geq t_0 \geq 0$:

$$\mathcal{E}(\boldsymbol{u}_m(t), Q_m(t)) - \mathcal{E}(\boldsymbol{u}_m(t_0), Q_m(t_0)) + \int_{t_0}^{t} (\nu \|\nabla \boldsymbol{u}_m(s)\|_{L^2}^2 + \gamma \|H_m(s)\|_{\mathbb{L}^2}^2)\, ds \leq 0. \qquad (17)$$

Moreover, $\boldsymbol{u}_m(t)$ and $Q_m(t)$ have sufficient estimates to obtain

$$\mathcal{E}(\boldsymbol{u}_m(t), Q_m(t)) \to \mathcal{E}(\boldsymbol{u}(t), Q(t)) \quad \text{in } L^1(0,T), \text{ and in particular a.e. } t \geq 0. \qquad (18)$$

Since $\boldsymbol{u}_m \to \boldsymbol{u}$ weakly in $L^2(0,T; \boldsymbol{H}^1)$ and $H_m \to H$ weakly in $L^2(0,T;\mathbb{L}^2)$,

$$\liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu \|\nabla \boldsymbol{u}_m(s)\|_{\mathbb{L}^2}^2 + \gamma \|H_m(s)\|_{\mathbb{L}^2}^2)\, ds \geq \int_{t_0}^{t_1} (\nu \|\nabla \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \gamma \|H(s)\|_{\mathbb{L}^2}^2)\, ds \qquad (19)$$

for all $t_1, t_0 : t_1 \geq t_0 \geq 0$.

By taking $\liminf_{m\to+\infty}$ in (17), we obtain that for all $t_1 \geq t_0 \geq 0$,

$$\begin{aligned} \liminf_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t), Q_m(t)) + \liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu \|\nabla \boldsymbol{u}_m(s)\|_{L^2}^2 + \gamma \|H_m(s)\|_{L^2}^2)\, ds \\ \leq \limsup_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t_0), Q_m(t_0)). \end{aligned} \qquad (20)$$

By using (18) and (19) in (20), we obtain (16).

3 An improved energy inequality

In this section, we obtain an improved time-integral energy inequality for all time, in a rigorous manner, for the weak solutions got from the Galerkin approximations. From this integral version we also obtain a time-differential version for almost every time.

Lemma 3 *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0, +\infty)$ *of problem (1)-(3) then, there exists an appropriate function* $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(t) \in \mathbb{R}$ *defined for all* $t \geq 0$, *which satisfies the following integral inequality for all* $t_1, t_0 : t_1 \geq t_0 \geq 0$:

$$\widetilde{\mathcal{E}}(t_1) - \widetilde{\mathcal{E}}(t_0) + \int_{t_0}^{t_1} (\nu \|\nabla \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \gamma \|H(s)\|_{\mathbb{L}^2}^2)\, ds \leq 0, \qquad (21)$$

and the following differential version a.e. $t \geq 0$:

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \nu \|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \gamma \|H(t)\|_{\mathbb{L}^2}^2 \leq 0. \qquad (22)$$

Proof: Since the inequality (16) is satisfied for all $t_0, t_1 \in [0, +\infty)\backslash N$, where N is a set of null Lebesgue measure, then the map $t \in [0,+\infty)\backslash N \to \mathcal{E}(\boldsymbol{u}(t), Q(t)) \in \mathbb{R}$ is a real decreasing (and bounded) function. Then, we can define a special function $\widetilde{\mathcal{E}}(t)$ for all $t \in [0, +\infty)$ as:

$$\widetilde{\mathcal{E}}(0) := \mathcal{E}(u_0, Q_0), \qquad \widetilde{\mathcal{E}}(t) := \lim_{\substack{s\to t^- \\ s\in[0,+\infty)\backslash N}} \mathcal{E}(\boldsymbol{u}(s), Q(s)).$$

The function $\widetilde{\mathcal{E}}$, thus defined, is "continuous from the left" and decreasing for all $t \geq 0$. Indeed, for any $t_1, t_2 \in [0,+\infty)$, for instance $t_1 < t_2$, we can choose sequences $\{s_n^1\}, \{s_n^2\} \subset [0,+\infty)\backslash N$ such that $s_n^1 \to t_1^-$, $s_n^2 \to t_2^-$ and, $s_n^1 \leq s_n^2$ for all $n \geq n_0$. Since s_n^1 and s_n^2 are not in N, we know that $\mathcal{E}(\boldsymbol{u}(s_n^1), Q(s_n^1)) \geq \mathcal{E}(\boldsymbol{u}(s_n^2), Q(s_n^2))$. By taking limit as $s_n^1 \to t_1^-$ and $s_n^2 \to t_2^-$, we obtain that $\widetilde{\mathcal{E}}(t_1) \geq \widetilde{\mathcal{E}}(t_2)$.

Since $\widetilde{\mathcal{E}}(t)$ is decreasing for all $t \in [0,+\infty)$, it is differentiable almost everywhere $t \in (0,+\infty)$.

Since the inequality (16) is satisfied for all $t_0, t_1 \in [0,+\infty) \setminus N$ where the measure of N is zero, given any $t_0 < t_1$, we can take $\delta_n > 0$ and $\eta_n > 0$ such that $t_0 - \delta_n$, $t_1 - \eta_n \notin N$ and $\delta_n, \eta_n \to 0$, hence

$$\widetilde{\mathcal{E}}(t_1 - \eta_n) - \widetilde{\mathcal{E}}(t_0 - \delta_n) + \int_{t_0-\delta_n}^{t_1-\eta_n} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds \leq 0.$$

By taking $\delta_n \to 0$ and $\eta_n \to 0$, we obtain (21).

In particular, by choosing $t_0 = t$ and $t_1 = t + h$ in (21), we obtain

$$\frac{\widetilde{\mathcal{E}}(t+h) - \widetilde{\mathcal{E}}(t)}{h} + \frac{1}{h}\int_t^{t+h} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds \leq 0, \quad \forall\, t, h \geq 0. \tag{23}$$

Observe that

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2)\, ds = \nu\|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \gamma\|\nabla H(t)\|_{L^2}^2,$$

a.e. $t \geq 0$ because the map, $s \in [0,+\infty) \to \nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|\nabla H(s)\|_{L^2}^2 \in \mathbb{R}$, belongs to $L^1(0,+\infty)$. Accordingly, by taking $h \to 0$ in (23), we obtain (22) a.e. $t \geq 0$. □

4 Convergence at infinite time.

Let $(\boldsymbol{u}, Q)$ be a weak solution of (1)-(3) in $(0,+\infty)$ associated to an initial data $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1(\Omega)$ (see Definition 1) satisfying Lemma 3. From the energy inequality (16), there exists a real number $E_\infty \geq 0$ such that the total energy evaluated in the trajectory $(\boldsymbol{u}(t), Q(t))$ satisfies

$$\mathcal{E}(\boldsymbol{u}(t), Q(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{ as } t \uparrow +\infty. \tag{24}$$

Let us define the ω-limit set of this global weak solution $(\boldsymbol{u}, Q)$ as follows:

$$\omega(\boldsymbol{u}, Q) = \{(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1 : \exists \{t_n\} \uparrow +\infty \text{ s.t.}$$

$$(\boldsymbol{u}(t_n), Q(t_n)) \to (\boldsymbol{u}_\infty, Q_\infty) \text{ weakly in } \boldsymbol{L}^2 \times \mathbb{H}^1\}.$$

Observe that this ω-limit set is defined with a weak convergence.

Let $\mathcal{S}$ be the set of critical points of the energy $\mathcal{E}(Q)$ defined in (4), that is

$$\mathcal{S} = \{Q \in \mathbb{H}^2 : -\varepsilon\Delta Q + f(Q) = 0 \text{ in } \Omega,\ \partial_n Q|_\Gamma = 0\}.$$

Theorem 4 *Assume that $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1$. Fixed $(\boldsymbol{u}, Q)$ a weak solution of (1)-(3) in $(0,+\infty)$ satisfying Lemma 3, then $\omega(\boldsymbol{u}, Q)$ is nonempty and $\omega(\boldsymbol{u}, Q) \subset \{0\} \times \mathcal{S}$. Moreover, for any $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, it holds*

$$\mathcal{E}_\mu(Q_\infty) = E_\infty.$$

In particular, $\boldsymbol{u}(t) \to 0$ *weakly in* $\boldsymbol{L}^2$ *and* $\mathcal{E}_\mu(Q(t)) \to \mathcal{E}_\mu(Q_\infty)$ *in* $\mathbb{R}$ *as* $t \uparrow +\infty$.

Proof: Observe that since

$$(\boldsymbol{u}, Q) \in L^\infty(0,+\infty; \boldsymbol{H} \times \mathbb{H}^1),$$

for any sequence $\{t_n\} \uparrow +\infty$ there exists a subsequence (equally denoted) and suitable limit functions $(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1$, such that

$$\boldsymbol{u}(t_n) \to \boldsymbol{u}_\infty \text{ weakly in } \boldsymbol{H}, \quad Q(t_n) \to Q_\infty \text{ weakly in } \mathbb{H}^1. \tag{25}$$

We consider the initial and boundary-value problem associated to (1)-(3) restricted on the time interval $[t_n, t_n+1]$ with initial values $\boldsymbol{u}(t_n)$ and $Q(t_n)$. If we define

$$\boldsymbol{u}_n(s) := \boldsymbol{u}(s+t_n),\ Q_n(s) := Q(s+t_n),\ H_n(s) := H(s+t_n)$$

for a.e. $s \in [0,1]$, then, $(\boldsymbol{u}_n, Q_n)$ is a weak solution to the problem (1)-(3) in the time interval $[0,1]$. From the energy inequality (16), we have that

$$\begin{aligned} \int_0^1 (\nu \|\nabla \boldsymbol{u}_n(s)\|^2_{\mathbf{L}^2} + \gamma \|H_n(s)\|^2_{\mathbb{L}^2})\, ds &= \int_{t_n}^{t_n+1} (\nu \|\nabla \boldsymbol{u}(t)\|^2_{\mathbf{L}^2} + \gamma \|H(t)\|^2_{\mathbb{L}^2})\, dt \\ &\le \mathcal{E}_\mu(Q(t_n)) - \mathcal{E}_\mu(Q(t_n+1)) \longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned}$$

hence,

$$\nabla \boldsymbol{u}_n \to 0 \text{ strongly in } L^2(0,1; \boldsymbol{L}^2)$$

and

$$H_n \to 0 \text{ strongly in } L^2(0,1; \mathbb{L}^2).$$

In particular, by using Poincaré inequality, one has

$$\boldsymbol{u}_n \to 0 \ \text{ strongly in } L^2(0,1;\, \boldsymbol{V})$$

and

$$H_n \to 0 \ \text{ strongly in } L^2(0,1; \mathbb{L}^2).$$

Moreover, since $\boldsymbol{u}_n$ and $\partial_t \boldsymbol{u}_n$ are bounded in $L^\infty(0,1;\boldsymbol{H})$ and $L^{4/3}(0,1;\, \boldsymbol{V}')$ respectively, then $\boldsymbol{u}_n \to 0$ in $C([0,1];\, \boldsymbol{V}')$. In particular, $\boldsymbol{u}(t_n) = \boldsymbol{u}_n(0) \to 0$ in $\boldsymbol{V}'$, hence $\boldsymbol{u}_\infty = 0$ (owing to (25)). Consequently, the whole trajectory $\boldsymbol{u}(t) \to 0$ as $t \to +\infty$.

Furthermore, Q_n is bounded in $L^2(0,1;\mathbb{H}^2) \bigcap L^\infty(0,1;\mathbb{H}^1)$ and $\partial_t Q_n$ is bounded in $L^{4/3}(0,1;\mathbb{L}^2)$. Therefore, there exists a subsequence of Q_n (equally denoted) and a limit function $\overline{Q}$ such that $Q_n \to \overline{Q}$ strongly in $C^0([0,1];\mathbb{L}^2) \cap L^2(0,1;\mathbb{H}^1)$ and weakly in $L^2(0,1;\mathbb{H}^2)$.

In particular, $Q(t_n) = Q_n(0) \to \overline{Q}(0)$ in $C^0(\mathbb{L}^2)$, hence $\overline{Q}(0) = Q_\infty$ (owing to (25)) in $\mathbb{H}^1$. On the other hand, $\partial_t Q_n$ converges weakly to $\partial_t \overline{Q}$ in $L^{4/3}(0,1;\mathbb{L}^2)$, hence taking limits in the variational formulation:

$$\begin{aligned} &(\partial_t Q_n, \widetilde{Q}) + ((\boldsymbol{u}_n \cdot \nabla) Q_n, \widetilde{Q}) - (S(\nabla \boldsymbol{u}_n, Q_n), \widetilde{Q}) \\ &\qquad - \varepsilon\, \gamma\, (\Delta Q_n, \widetilde{Q}) + \gamma\, (f(Q_n), \widetilde{Q}) = 0. \end{aligned}$$

for all $\widetilde{Q} \in \mathbb{L}^2$, we have that $\partial_t Q_n \to 0$ in $L^{4/3}(0,1;\mathbb{L}^2)$ weakly. Therefore, $\partial_t \overline{Q} = 0$ and $\overline{Q}(t)$ is a constant function of $\mathbb{L}^1$ for all $t \in [0,1]$, hence since $\overline{Q}(0) = Q_\infty$, we have

$$\overline{Q}(t) = Q_\infty \in \mathbb{H}^1 \quad \text{for all } t \in [0,1]. \tag{26}$$

Finally, since $f(Q_n)$ converges weakly in $L^\infty(0,1;\mathbb{L}^2)$, by taking limit as $n \to +\infty$ in the variational formulation $(H_n, \widetilde{Q}) = \varepsilon\,(\nabla Q_n, \nabla \widetilde{Q}) + (f(Q_n), \widetilde{Q})$ for all $\widetilde{Q} \in \mathbb{H}^1$, we deduce

$$\varepsilon\,(\nabla \overline{Q}, \nabla \widetilde{Q}) + (f(\overline{Q}), \widetilde{Q}) = 0, \quad \forall\, \widetilde{Q} \in \mathbb{H}^1, \text{ a.e. } t \in (0,1).$$

Then, from (26), $Q_\infty \in \mathbb{H}^1$ and $\varepsilon\,(\nabla Q_\infty, \nabla \widetilde{Q}) + (f(Q_\infty), \widetilde{Q}) = 0$, $\forall\, \widetilde{Q} \in \mathbb{H}^1$, a.e. $t \in (0,1)$. Finally, by applying $\mathbb{H}^2$-regularity of the Poisson problem:

$$\begin{cases} -\varepsilon\,\Delta Q + Q &= f(Q) + Q \quad \text{in } \Omega, \\ \partial_{\boldsymbol{n}} Q|_\Gamma &= 0 \end{cases}$$

we deduce that $Q_\infty \in \mathbb{H}^2$, hence $Q_\infty \in \mathcal{S}$ and the proof is finished. $\square$

In the next theorem we apply the following Lojasiewicz-Simon's result that can be found in [14].

Lemma 5 (Lojasiewicz-Simon inequality) *Let $Q_* \in \mathcal{S}$ and $K > 0$ fixed. Then, there exists positive constants β_1, β_2 and C and $\theta \in (0, 1/2]$, such that for all $Q \in \mathbb{H}^2$ with $\|Q\|_{\mathbb{H}^1} \le K$, $\|Q - Q_*\|_{\mathbb{L}^2} \le \beta_1$ and $|\mathcal{E}(Q) - \mathcal{E}(Q_*)| \le \beta_2$, it holds*

$$|\mathcal{E}(Q) - \mathcal{E}(Q_*)|^{1-\theta} \le C\,\|H\|_{\mathbb{H}^{-1}}$$

where $H = H(Q)$ is defined in (6).

Theorem 6 *Assume that $\widetilde{\mathcal{E}}(t)$ belongs to the equivalence class of the energy function $\mathcal{E}(\boldsymbol{u}(t), Q(t))$, that is, $\widetilde{\mathcal{E}}(t) = \mathcal{E}(\boldsymbol{u}(t), Q(t))$ almost everywhere $t \ge 0$. Then, under the hypotheses of Theorem 4, there exists a unique limit $Q_\infty \in \mathcal{S}$ such that $Q(t) \to Q_\infty$ in $\mathbb{H}^1$-weakly as $t \uparrow +\infty$, i.e. $\omega(\boldsymbol{u}, Q) = \{(0, Q_\infty)\}$.*

Proof: Let $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, i.e. there exists $t_n \uparrow +\infty$ such that $\boldsymbol{u}(t_n) \to 0$ weakly in $\boldsymbol{L}^2$ and $Q(t_n) \to Q_\infty$ weakly in $\mathbb{H}^1$ (and strongly in $\mathbb{L}^2$).

It can be assumed that $\widetilde{\mathcal{E}}(t) > \mathcal{E}_\mu(Q_\infty) (= E_\infty)$ for all $t > 0$, because otherwise, if it exists some $\widetilde{t} > 0$ such that $\widetilde{\mathcal{E}}(\widetilde{t}) = E_\infty$, then the energy inequality (21) implies

$$\widetilde{\mathcal{E}}(t) = E_\infty, \quad \|\nabla \boldsymbol{u}(t)\|^2_{\mathbb{L}^2} = 0 \quad \text{and} \quad \|H(t)\|^2_{\mathbb{L}^2} = 0, \quad \forall\, t \ge \widetilde{t}.$$

Therefore, $\boldsymbol{u}(t) = 0$ and $H(t) = 0$ for all $t \ge \widetilde{t}$, and by using the Q-equation of (1), $\partial_t Q(t) = 0$, hence $Q(t) = Q_\infty$ for all $t \ge \widetilde{t}$. Then, the convergence of the whole Q-trajectory towards Q_∞ is trivial and $\widetilde{\mathcal{E}}(t) > E_\infty$ is assumed for all $t \ge 0$.

The proof will be divided into three steps.

Step 1: *There exists a n_0 such that $\|Q(t) - Q_\infty\|_{\mathbb{L}^2} \le \beta_1$ and $|\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\mu(Q_*)| \le \beta_2$ for all $t \ge t_{n_0}$ (β_1, β_2 given in Lemma 5).*

Since $Q(t_n) \to Q_\infty$ strongly in $\mathbb{L}^2$ and $\mathcal{E}(\boldsymbol{u}(t_n), Q(t_n)) \searrow E_\infty = \mathcal{E}_\mu(Q_\infty)$ in $\mathbb{R}$ (see (24)), then for any $\delta \in (0, \beta_1)$, there exists an integer $N(\delta)$ such that, for all $n \geq N(\delta)$,

$$\|Q(t_n) - Q_\infty\|_{\mathbb{L}^2} \leq \delta \quad \text{ and } \quad \frac{1}{\theta}(\mathcal{E}_\mu(Q(t_n)) - E_\infty)^\theta \leq \delta. \tag{27}$$

For each $n \geq N(\delta)$, we define

$$\bar{t}_n := \sup\{t : t > t_n,\ \|Q(s) - Q_\infty\|_{\mathbb{L}^2} < \beta_1 \quad \forall s \in [t_n, t)\}.$$

It suffices to prove that $\bar{t}_{n_0} = +\infty$ for some n_0. Assume by contradiction that $t_n < \bar{t}_n < +\infty$ for all n, hence $\|Q(\bar{t}_n) - Q_\infty\|_{\mathbb{L}^2} = \beta_1$ and $\|Q(t) - Q_\infty\|_{\mathbb{L}^2} < \beta_1$ for all $t \in [t_n, \bar{t}_n)$. By applying Step 1 for all $t \in [t_n, \bar{t}_n]$, from (29) and (27) we obtain,

$$\int_{t_n}^{\bar{t}_n} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq C\delta, \quad \forall\, n \geq N(\delta).$$

Therefore,

$$\|Q(\bar{t}_n) - Q_\infty\|_{\mathbb{H}^{-1}} \leq \|Q(t_n) - Q_\infty\|_{\mathbb{H}^{-1}} + \int_{t_n}^{\bar{t}_n} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq (1 + C)\delta,$$

which implies that $\lim_{n\to+\infty} \|Q(\bar{t}_n) - Q_\infty\|_{\mathbb{H}^{-1}} = 0$.

On the other hand, $Q(\bar{t}_n)$ is bounded in $\mathbb{H}^1$. Indeed, from (24), $\widetilde{\mathcal{E}}(\boldsymbol{u}(\bar{t}_n), Q(\bar{t}_n))$ is bounded in $\mathbb{R}$, therefore in particular

$$\int_\Omega \mathcal{E}_\mu(Q(\bar{t}_n))\, dx = \int \Big(\frac{\varepsilon}{2}|\nabla Q(\bar{t}_n)|^2 + F_\mu(Q(\bar{t}_n))\Big)\, dx$$

is bounded. But, since $F_\mu(Q)$ is bounded in $L^\infty(\mathbb{L}^1)$, then $\nabla Q(\bar{t}_n)$ is bounded in $\mathbb{L}^2(\Omega)$ and $Q(\bar{t}_n)$ is bounded in $\mathbb{H}^1$.

Therefore, $Q(\bar{t}_n)$ is relatively compact in $\mathbb{L}^2$. There exists a subsequence of $Q(\bar{t}_n)$, also denoted $Q(\bar{t}_n)$, that converges to Q_∞ in $\mathbb{L}^2$-strong. Hence $\|Q(\bar{t}_n) - Q_\infty\|_{\mathbb{L}^2} < \beta_1$ for a sufficiently large n, which contradicts the definition of $\bar{t}_n$.

Step 2: *Under the conditions of step 1, the following inequalities hold:*

$$\frac{d}{dt}\Big((\widetilde{\mathcal{E}}(t) - E_\infty)^\theta\Big) + C\,\theta\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \tag{28}$$

a.e. $t \in (t_1, \infty)$.

$$\int_{t_1}^{t_2} \|\partial_t Q\|_{\mathbb{H}^{-1}} \leq \frac{C}{\theta}(\widetilde{\mathcal{E}}(t_1) - E_\infty)^\theta, \tag{29}$$

for all $t_2 \in (t_1, \infty)$*, where* $\theta \in (0, 1/2]$ *is the constant appearing in Lemma 5.*

In this step, the hypothesis $\mathcal{E}(u(t), Q(t)) = \widetilde{\mathcal{E}}(t)$ for almost every t is a key point. In particular, this hypothesis implies that the integral and differential versions of the energy law (21) and (22) are satisfied by $\mathcal{E}(u(t), Q(t))$ a.e. in time. In fact, energy law (22), changing $\widetilde{\mathcal{E}}(t)$ by $\mathcal{E}(u(t), Q(t))$, is the crucial hypothesis imposed in Remark 2.4 of [14].

From the inequalities:

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - E_\infty) + C\left(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \|H(t)\|_{\mathbb{L}^2}^2\right) \leq 0, \text{ a.e. } t \in (t_1, \infty),$$

$$\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \|H(t)\|_{\mathbb{L}^2}^2 \geq \frac{1}{2}\left(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}\right)^2$$

and

$$\frac{1}{2}(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \geq C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}),$$

we obtain

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - E_\infty) + C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \quad \text{a.e. } t \geq 0$$

and, using the time derivative of the $(\widetilde{\mathcal{E}}(t) - E_\infty)^\theta$, we get

$$\begin{aligned} &\frac{d}{dt}\left((\widetilde{\mathcal{E}}(t) - E_\infty)^\theta\right) \\ &\quad +\theta(\widetilde{\mathcal{E}}(t) - E_\infty)^{\theta-1} C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0. \end{aligned} \tag{30}$$

almost everywhere $t \geq 0$.

On the other hand, since $|\mathcal{E}_k(\boldsymbol{u}(t))| = \dfrac{1}{2}\|\boldsymbol{u}(t)\|_{\mathbf{L}^2}^2$ and $\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \leq K$, we have that

$$|\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|_{\mathbf{L}^2}^{2(1-\theta)} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|_{\mathbf{L}^2}^{1-2\theta}\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \leq C\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} \quad a.e.\ t \geq 0.$$

This estimate together the Lojasiewicz-Simon inequality $|\mathcal{E}_\mu(Q(t)) - E_\infty|^{1-\theta} \leq C\|H\|_{\mathbb{H}^{-1}}$, give

$$\begin{aligned} (\mathcal{E}(u(t), Q(t)) - E_\infty)^{1-\theta} &\leq |\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} + |\mathcal{E}_\mu(Q(t)) - E_\infty|^{1-\theta} \\ &\leq C(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}) \quad \text{a.e. } t \geq t_1. \end{aligned}$$

Therefore,

$$(\mathcal{E}(u(t), Q(t)) - E_\infty)^{\theta-1}(\|\boldsymbol{u}(t)\|_{\mathbf{L}^2} + \|H(t)\|_{\mathbb{H}^{-1}}) \geq C \tag{31}$$

almost every where $t \geq t_1$. By applying (31) in (30),

$$\frac{d}{dt}\left((\mathcal{E}(u(t), Q(t)) - E_\infty)^\theta\right) + C\,\theta\,(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \leq 0, \quad \text{a.e. } t \geq t_1$$

and (28) is proved.

Secondly, for any $t_2 \in (t_1, +\infty)$, since $(\mathcal{E}(\boldsymbol{u}(t_2), Q(t_2)) - E_\infty)^\theta > 0$, integrating (28) into $[t_1, t_2]$ we have

$$\theta\, C \int_{t_1}^{t_2} (\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) dt \leq (\mathcal{E}(\boldsymbol{u}(t_1), Q(t_1)) - E_\infty)^\theta. \tag{32}$$

From (9), by using the weak regularity $Q \in L^\infty((0, +\infty) \times \Omega)$, we achieve

$$\|\partial_t Q(t)\|_{\mathbb{H}^{-1}} \leq C(\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2} + \|H(t)\|_{\mathbb{L}^2}) \qquad a.e.\ t \geq 0.$$

By integrating this inequality into $[t_1, t_2]$ and using (32), we attain (29).

Step 3: *There exists a unique Q_∞ such that $Q(t) \to Q_\infty$ weakly in $\mathbb{H}^1$ as $t \uparrow +\infty$.*

By using (29) for any $t_1, t_0 : t_1 > t_0 \geq t_{n_0}$,

$$\|Q(t_1) - Q(t_0)\|_{\mathbb{H}^{-1}} \leq \int_{t_0}^{t_1} \|\partial_t Q\|_{\mathbb{H}^{-1}} \to 0, \text{ as } t_0, t_1 \to +\infty.$$

Therefore, $(Q(t))_{t\geq t_{n_0}}$ is a Cauchy sequence in $\mathbb{H}^{-1}$ as $t\uparrow +\infty$, hence, there exists a unique $Q_\infty \in \mathbb{H}^{-1}$ such that $Q(t)\to Q_\infty$ in $\mathbb{H}^{-1}$ as $t\uparrow +\infty$. Finally, the convergence in $\mathbb{H}^1$-weak by sequences of $Q(t)$ proved in Theorem 4, yields to $Q(t)\to Q_\infty$ in $\mathbb{H}^1$-weak, and the proof is finished.

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