Multiplicity of Solutions for the Kirchhoff equation with critical nonlinearity in high dimension

BiYun Tang, YongYi Lan*
School of Science, Jimei University, Xiamen 361021, People's Republic of China

Abstract: This paper is devoted to the following Kirchhoff type of problems:

$$\begin{cases} \Big(a + b\int_\Omega [|\nabla u|^2 + V(x)u^2]\,\mathrm{d}x\Big)[-\Delta u + V(x)u] = \lambda u^q + u^{2^*-1}, & u > 0 \quad x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domian in $\mathbb{R}^N$, $a > 0, b > 0$ are real constants, $\lambda \in \mathbb{R}, N \geq 5$, $q \in [1, 2^* - 1)$, $2^* = \frac{2N}{N-2}$. Under some suitable assumptions on $V(x)$, we will prove the multiplicity of solutions for the Kirchhoff-type equation by the variational method.

Keywords: Kirchhoff; Critical nonlinearity; Nehari manifold; Fibering maps.

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1 Introduction and Main Results

Consider the following Kirchhoff type with Dirichlet boundary value problem:

$$\begin{cases} \Big(a + b\int_\Omega [|\nabla u|^2 + V(x)u^2]\,\mathrm{d}x\Big)[-\Delta u + V(x)u] = \lambda u^q + u^{2^*-1}, & u > 0 \quad x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domian in $\mathbb{R}^N$, $a > 0, b > 0$ are real constants, $\lambda \in \mathbb{R}, N \geq 5$, $q \in [1, 2^* - 1)$, $2^* = \frac{2N}{N-2}$.

It is a stationary problem of the Kirchhoff type wave equation:

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)\Delta u = f(x, u). \tag{1.2}$$

Kirchhoff [1] introduced the original form of (1.2) in 1883 to study the free vibration of the elastic strings. In (1.2), u denotes the displacement, $f(x, u)$ the external force, and b the initial tension

*Corresponding author. E-mail address: lanyongyi@jmu.edu.cn (Y.-Y. Lan).

while a is related to the intrinsic properties of the string, such as Young modulus. Moreover, after the works [2] and [3], many mathematicians had been discussing the solvability and asymptotic behavior of (1.2) for decades. There were also previous works on the nonlocal parabolic type problem involving the Dirichlet energy in [4] and [5]. As is well-known, because of the lack of the compactness of the associated Sobolev embedding, a typical difficulty occured in proving the Palais-Smale (PS) condition. Applying the pioneering argument by Brezis-Nirenberg [6] with the concentration compactness lemma by P.L Lions [7], the authors got existence result.

Recently, many researchers were interested in the existence of (1.1) with critical Sobolev exponent when $V(x) = 0$. For example, F.F and C.F [8] considered the existence of the Kirchhoff type equation by the sequentially weakly lower semicontinuity and the Palais-Smale property of the energy functional involving the critical Sobolev exponent. Xu et al.[9] established the existence of the positive solutions set of a nonlocal problem of the Kirchhoff type by the local and global bifurcation techniques, a priori bounds for elliptic equation, and the properties of the principal eigenvalues in $N \leq 3$. Jin et al.[10] studied sign-changing solutions for nonlinear elliptic problem with Carrier type by using the fixed-point index method. Li et al.[11] studied ground-state solutions to Kirchhoff-type transmission problems with critical perturbation. Unlike other studies on elliptical equations with critical growth, a ground-state solution was obtained using a perturbation method instead of verifying that the mountain pass level is lower than the critical energy, which was used to verify the PS condition.

Moreover, many researchers had obtained existence of solutions with critical Sobolev exponent when $V(x) \neq 0$ (see[12-18] and the references therein). Liu et al.[13] considered a Kirchhoff type equation involving two potential by the Nehari manifold for the following Kirchhoff equation $-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\, \mathrm{d}x\right)\Delta u + V(x)u = K(x)u^{2^*-1}$. Lin et al.[14] studied existence and concentration of ground state solutions for the following singularly perturbed Kirchhoff-type problem $-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^N} |\nabla u|^2\, \mathrm{d}x\right)\Delta u + V(x)u = f(x)$. In [15], the author had shown the effect of suitable singular potential $V(x)$ on the existence of multiple solutions of $-\Delta u = \lambda V(x)u + |u|^{2^*-2}u$ in bounded main. In [17], the existence of (1.1) was obtained with the aid of the mountain pass theorem when $f(x, u) = u^q$ and $K(x) = 1$. In [18], the author proved the existence of the multiple positive solutions.

More recently, more and more authors began to pay attention to the higher dimensional problem when $V(x) \neq 0$(See [19-23]). Sun et al.[24] studied a class of superlinear Kirchhoff type equations with steep potential well and established the existence of two positive solutions by proposing new techniques and introducing new superlinear hypotheses on f. Li et al.[25] considered the existence and nonexistence of energy minimizer of the Kirchhoff-Schr$\ddot{o}$dinger energy function in dimension four. In [26], the authors studied the bounded state solution of Kirchhoff type equation with critical exponent in dimension four, with the concentration compactness argument for PS sequence. However, as is suggested Remark 4.4 in [26], the critical problem in high dimension had not been completely solved yet. As we known, if $N \geq 5$, the critical exponent was strictly less than four, which made the energy structure of the associated functional drastically different from the original semilinear problem. Then we readily expected the multiplicity of solutions. But serious difficulties occured in dealing with the PS sequence. First, the weak limit was not a solution of the original problem in Kirchhoff type problem, thus we could not get a solution even if we proved that it was nontrivial. Moreover, it was an essential fact on our problem that the limiting problem lacked the uniqueness of the solutions. Furthermore, one of them might have a negative energy. In addition, since the weak limit might also have the negative energy, it seemed too hard to control the PS sequence by the usual energy argument. This implied that the typical proof was no longer valid for our problem and we needed a new strategy. In [27], the author had obtained the existence of multiplicity of positive solutions of the Kirchhoff type critical problem in high dimension.

Motivated by the ideas in [26]-[27], we proved the existence of two solutions for (1.1) by adding the potential $V(x)$. For a mountain pass type solution, we utilized the limit function of the fibering maps of the concentrating PS sequence, basing on Nehari type sets. For global minimum solution, we needed a suitable modification to a concentrating minimizing sequence.

Throughout this paper, we make the following assumption:

(V) $V \in C(\overline{\Omega}, \mathbb{R}), \inf\limits_{x \in \Omega} V(x) = V_0 > 0$.

Namely, it is the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \Big(\int_\Omega (|\nabla u|^2 + V(x)u^2)\ \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Moreover, it is easy to check that $\|\cdot\|$ is equivalent to the usual Sobolev norm.

We look for the weak solutions of (1.1) which are the same as the critical points of the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x - \frac{1}{2^*}\int_\Omega u_+^{2^*}\,\mathrm{d}x.$$

I is of $C^1(\Omega)$ and $u, v \in H_0^1(\Omega)$ with derivatives given by

$$\langle I'(u), v\rangle = (a + b\|u\|^2)\int_\Omega [\nabla u \cdot \nabla v + V(x)uv]\,\mathrm{d}x - \lambda\int_\Omega u^q v\,\mathrm{d}x - \int_\Omega u^{2^*-1} v\,\mathrm{d}x.$$

Theorem 1.1. Let (V) holds, then there exists a constant $b_0 > 0$ such that (1.1) has a solution u with $I(u) > 0$ for any $b \in (0, b_0), 0 < \lambda < a\lambda_1$ if $q = 1$, and $\lambda > 0$ if $1 < q < 2^* - 1$. where $\lambda_1 := \inf\limits_{0 \neq u \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x}$ be the principal eigenvalue of $-\Delta + V$ on Ω.

Theorem 1.2.Assume (V) holds, then there exists $b_1 > 0$ such that (1.1) admits a solution v with $I(v) < 0$ for all $b \in (0, b_1]$ and $\lambda > 0$.

This paper is organized as follows. In Section 2, we put some preliminaries for our main argument. In Section 3, we construct a PS sequence on a suitable one and give the proof of our main theorems.

2 Preliminary lemmas

For each $u \in H_0^1(\Omega)$, we define the fibering map:

$$f_u(t) := I(tu) = \frac{at^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \frac{\lambda t^{q+1}}{q+1}\int_\Omega u_+^{q+1}\,\mathrm{d}x - \frac{t^{2^*}}{2^*}\int_\Omega u_+^{2^*}\,\mathrm{d}x.$$

Consider the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega)\backslash\{0\} : \langle I'(u), u\rangle = 0\} = \{u \in H_0^1(\Omega)\backslash\{0\} : f_u'(1) = 0\}.$$

Therefore, we can split the Nehari manifold $\mathcal{N}$ into two parts, that is,

$$\mathcal{N}^- = \left\{u \in \mathcal{N} : \frac{d^2}{dt^2}I(tu)\mid_{t=1} < 0\right\} = \{u \in \mathcal{N} : f_u''(1) < 0\},$$

$$\mathcal{N}^0 = \left\{u \in \mathcal{N} : \frac{d^2}{dt^2} I(tu)\mid_{t=1} = 0\right\} = \{u \in \mathcal{N} : f_u''(1) = 0\}.$$

Lemma 2.1. Let $0 < \lambda < a\lambda_1$ if $q = 1$ and $\lambda > 0$ if $1 < q < 2^* - 1$, then for each $u \in H_0^1(\Omega)\backslash\{0\}$, either one of the next (i)-(iii) holds.

(i) $f_u(t)$ has an critical point in $(0, \infty)$. Moreover, $f_u'(t) > 0$ in $(0, \infty)$.

(ii) $f_u(t)$ possesses a unique critical point in $(0, \infty)$ such that $f_u'(t_0) = f_u''(t_0) = 0$ and $f_u'(t) > 0$ in $(0, t_0) \cup (t_0, \infty)$.

(iii) $f_u(t)$ admits two critical points $0 < t_0 < t_1$ such that $f_u'(t) > 0$ in $(0, t_0) \cup (t_1, \infty)$, $f_u'(t) < 0$ in (t_0, t_1) and $f_u''(t_0) < 0 < f_u''(t_1)$.

Proof. If $u_+ = 0$, then $f_u(t) = \frac{at^2}{2}\|u\|^2 + \frac{bt^4}{4}\|u\|^4$, which implies (i). On the contrary, we assume $u_+ \neq 0$. For $q = 1$, put $H(t) = t^{2^*-2}\int_\Omega u^{2^*}\,\mathrm{d}x - bt^2\|u\|^4$ and consider the equation $f_u'(t) = 0$ for $t > 0$, which is equivalent to

$$a\|u\|^2 - \lambda \int_\Omega u_+^2\,\mathrm{d}x = H(t). \tag{2.1}$$

Notice that the left hand side is strictly positive by $\lambda < a\lambda_1$ and Poincare inequality. We can easily compute that $H'(t) = (2^* - 2)t^{2^*-3}\int_\Omega u^{2^*}\,\mathrm{d}x - 2bt\|u\|^4$. It is easy to verify that there exists a unique constant $t^* > 0$ such that $H(t^*) = 0$ and

$$H'(t)\begin{cases} > 0 \text{ for } & t \in (0, t^*), \\ < 0 \text{ for } & t \in (t^*, \infty). \end{cases} \tag{2.2}$$

and $\lim\limits_{t\to+\infty} H(t) = -\infty$, so t^* attains the maximum of H on $t > 0$.

(i) If $a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x > H(t^*)$, which is equivalent to $H(t) > H(t^*)$, there exists no solution of (2.1).

(ii) Assume $a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x = H(t^*)$, then taking $t_0 = t^*$, noting (2.2) and the fact $H(t) < a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x$, for all $t \in (0, t^*) \cup (t^*, \infty)$.

(iii) Suppose $a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x < H(t^*)$, which is equivalent to $H(t) < H(t^*)$. Then there exist just two solutions $t_0 < t^* < t_1$ of (2.1). Since $H(t) < a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x$ for $t \in (0, t_0) \cup (t_1, \infty)$, $H(t) > a\|u\|^2 - \lambda\int_\Omega u_+^2\,\mathrm{d}x$ for $t \in (t_0, t_1)$, and $H'(t_1) < 0 < H'(t_0)$ by (2.2). This concludes the case $q = 1$.

For $1 < q < 2^* - 1$, define the function $\widetilde{H}(t) = \lambda t^q \int_\Omega u_+^{q+1}\,\mathrm{d}x + t^{2^*-1}\int_\Omega u^{2^*}\,\mathrm{d}x - bt^2\|u\|^4$ and consider the equation $f_u'(t) = 0$ for $t > 0$, which is equivalent to

$$a\|u\|^2 = \widetilde{H}(t) \tag{2.3}$$

We can easily compute that $\widetilde{H}'(t) = \lambda q t^{q-1}\int_\Omega u_+^{q+1}\,\mathrm{d}x + (2^* - 1)t^{2^*-2}\int_\Omega u^{2^*}\,\mathrm{d}x - 2bt\|u\|^4$. Since $q-1 < 2^*-2 < 2$, we can verify that there exists a unique constant $t_1^* > 0$ such that $\widetilde{H}(t_1^*) = 0$ and

$$\widetilde{H}'(t)\begin{cases} > 0 \text{ for } & t \in (0, t_1^*),\\ < 0 \text{ for } & t \in (t_1^*, \infty).\end{cases} \tag{2.4}$$

and $\lim\limits_{t\to+\infty}\widetilde{H}(t) = -\infty$, so t^* attains the maximum of $\widetilde{H}$ on $t > 0$.
(i) If $a\|u\|^2 > \widetilde{H}(t_1^*)$, which is equivalent to $\widetilde{H}(t) > \widetilde{H}(t_1^*)$, there exists no solution of (2.3).
(ii) Assume $a\|u\|^2 = \widetilde{H}(t_1^*)$, then taking $t_0 = t_1^*$, noting (2.4) and the fact $H(t) < a\|u\|^2$, for all $t \in (0, t_1^*) \cup (t_1^*, \infty)$.
(iii) Suppose $a\|u\|^2 < \widetilde{H}(t_1^*)$, which is equivalent to $\widetilde{H}(t) < \widetilde{H}(t_1^*)$. Then there exist just two solutions $t_0 < t_1^* < t_1$ of (2.3). Since $\widetilde{H}(t) < a\|u\|^2$ for $t \in (0, t_0) \cup (t_1, \infty)$, $\widetilde{H}(t) > a\|u\|^2$ for $t \in (t_0, t_1)$, and $\widetilde{H}'(t_1) < 0 < \widetilde{H}'(t_0)$ by (2.4). □

Proposition 2.2.([26],Theorem 4.1). Let $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ be a bounded PS sequence for I, that is, $I(u_n) \le c,\ I'(u_n) \to 0$ in $H^{-1}(\Omega)$ and $\|u_n\|_{H_0^1(\Omega)}$ is bounded. Then $\{u_n\}_{n\in\mathbb{N}}$ has a subsequence which converges strongly in $H_0^1(\Omega)$, or otherwise, there exist a nonnegative function $u_0 \in H_0^1(\Omega)$ which is a weak limit of $\{u_n\}_{n\in\mathbb{N}}$, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \cdots, k\}$, a sequence of radii $\{R_n^i\}_{n\in\mathbb{N}} \subset (0, \infty)$, points $\{x_n^i\}_{n\in\mathbb{N}} \subset \overline{\Omega}$ and a nonnegative function $v_i \in D^{1,2}(\mathbb{R}^N)$ which is solutions of the "limiting problem", satisfying the following

$$-\left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k}\|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\right\}\triangle u_0 = \lambda u_0^q + u_0^{2^*-1} \text{ in } \Omega, \tag{2.5}$$

$$-\left\{a + b\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k}\|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\right\}\triangle v_i = v_i^{2^*-1} \text{ in } \mathbb{R}^N, \tag{2.6}$$

such that up to subsequences, $R_n^i dist(x_n^i, \partial\Omega) \to \infty$ as $n \to \infty$,

$$\left\| u_n - u_0 - \sum_{i=1}^{k} (R_n^i)^{\frac{N-2}{2}} v_i(R_n^i(\cdot - x_n^i)) \right\|_{D^{1,2}(\mathbb{R}^N)} = o(1),$$

$$\|u_n\|^2_{H_0^1(\Omega)} = \|u_0\|^2_{H_0^1(\Omega)} + \sum_{i=1}^{k} \|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + o(1),$$

and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^{k} \tilde{I}_\infty(v_i) + o(1),$$

where $o(1) \to 0$ as $n \to \infty$ and we define

$$\begin{aligned} \tilde{I}(u_0) \ &:= \ \frac{a}{2}\|u_0\|^2_{H_0^1(\Omega)} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\|u_0\|^2_{H_0^1(\Omega)} \\ &\quad -\frac{\lambda}{q+1}\int_\Omega u_0^{q+1}\,\mathrm{d}x - \frac{1}{2^*}\int_\Omega u_0^{2^*}\,\mathrm{d}x, \end{aligned}$$

$$\tilde{I}_\infty(v_i) := \frac{a}{2}\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} + \frac{b}{4}\left(\|u_0\|^2_{H_0^1(\Omega)} + \sum_{j=1}^{k} \|v_j\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\|v_i\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{1}{2^*}\int_{\mathbb{R}^N} v_i^{2^*}\,\mathrm{d}x.$$

Here, let us recall the well-known facts on the Talenti function in [28]. For any $\varepsilon > 0$, define

$$\widetilde{U}_\varepsilon(x) := \frac{\varepsilon^{\frac{N-2}{2}}}{\left(\varepsilon^2 + |x|^2\right)^{\frac{N-2}{2}}} \qquad \text{for} \ \ x \in \mathbb{R}^N.$$

and put S is the usual Sobolev constant defined by

$$S \ = \inf_{U\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla U|^2\,\mathrm{d}x}{(\int_{\mathbb{R}^N} |U|^{2^*}\,\mathrm{d}x)^{2/2^*}} = \inf_{U\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|U\|^2_{1,2}}{\|U\|^2_{2^*}},$$

then we have $\|\widetilde{U}_\varepsilon\|^2_{1,2} = S\|\widetilde{U}_\varepsilon\|^2_{2^*}$.

Furthermore, put

$$U_\varepsilon(x) := (N(N-2))^{\frac{N-2}{4}}\,\widetilde{U}_\varepsilon(x),$$

then $U_\varepsilon(x)$ is a solution of the semilinear critical problem:

$$\begin{cases} -\Delta U = U^{2^*-1}, U > 0 \ \text{in} \ \ \mathbb{R}^N, \\ U \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{2.7}$$

Now, as in [6], we define the Talenti function scaled and cut off approprately. Assume $0 \in \Omega$ and ξ is a smooth cut off function compactly supported in Ω such that $0 \le \xi \le 1$ and $\xi = 1$ on some neighborhood of 0. Define

$$\tilde{u}_\varepsilon(x) := \widetilde{U}_\varepsilon(x)\xi(x) \in H_0^1(\Omega),$$

Then we put $v_\varepsilon := \tilde{u}_\varepsilon / \|\tilde{u}_\varepsilon\|_{2^*}$ and estimate

$$\begin{cases} \int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d}x = S + O(\varepsilon^{N-2}), \\ \int_\Omega v_\varepsilon^{2^*} \, \mathrm{d}x = 1, \\ \int_\Omega v_\varepsilon^{q+1} \, \mathrm{d}x = \alpha\varepsilon^\beta + O(\varepsilon^{N-2}). \end{cases} \tag{2.8}$$

where $\alpha > 0$ and $0 < \beta = \beta(q, N) \le 2$.

Next, we concern with the "limiting problem" for our case,

$$\begin{cases} \left(a + b\int_{\mathbb{R}^N}[|\nabla\theta|^2 + V(x)\theta^2]\,\mathrm{d}x\right)[-\Delta\theta + V(x)\theta] = \theta^{2^*-1}, \theta > 0 \text{ in } \mathbb{R}^N, \\ \theta \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{2.9}$$

and we have the associated functional,

$$I^\infty(\theta) = \frac{a}{2}\|\theta\|_{1,2}^2 + \frac{b}{4}\|\theta\|_{1,2}^4 - \frac{1}{2^*}\int_{\mathbb{R}^N}|\theta|^{2^*}\,\mathrm{d}x(\theta \in D^{1,2}(\mathbb{R}^N)).$$

Consider the function on $t > 0$,

$$g(t) = I^\infty(tU_\varepsilon) = \frac{a}{2}S^{N/2}t^2 + \frac{b}{4}S^N t^4 - \frac{S^{N/2}}{2^*}t^{2^*}.$$

We note the equation for $t > 0$,

$$a + bS^{N/2}t^2 - t^{2^*-2} = 0, \tag{2.10}$$

which is equivalent to $g'(t) = 0$. If $a = 1$ and $b = 0$, $t = 1$ are the unique critical point of g on $t > 0$, $g(1) = I^\infty(U_\varepsilon) = S^{N/2}/N$. Define $b_2 := \inf\{b > 0 \mid \text{ there exists no solution of (2.8)}\}$. If $b > 0$ is large, (2.8) does not have any solution because of $2^* < 4$. So we consider $b \in (0, b_2)$.

Lemma 2.3. The following assertions are true.

(i) For all $b \in (0, b_2)$, $g(t)$ has just two critical points for $t > 0$, say $0 < \tau_b^- < \tau_b^+$, satisfying $g''(\tau_b^-) < 0 < g''(\tau_b^+)$.

(ii) There exists a value $b_1 \in (0, b_2)$, which depends only on N, such that $g(\tau_b^+) \le 0$ for any $b \in (0, b_1]$.

Proof. (i) Noting the definition of $b_2 > 0$, the proof of (i) is similar to the proof of Lemma 2.1(iii).

(ii) If $b \in (0, b_2)$ is small enough and $2^* < 4$, then there exists a τ_b^+ such that $g(\tau_b^+) \le 0$. By (i), we know that $g(\tau_b^+)$ is minimum value. So τ_b^+ attains the nonpositive minimum value of g on $t > 0$. □

Lemma 2.4. Assume $b \in (0, b_2)$, $1 \le q < 2^* - 1$, $\lambda > 0$ and define $f_\varepsilon(t) := I(tv_\varepsilon)$ for $t > 0$. then the following assertions are true.

(i) we have constants $0 < t_{b,\varepsilon}^- < t_{b,\varepsilon}^+$ such that $f_\varepsilon'(t_b^\pm) = 0$ and $f_\varepsilon''(t_{b,\varepsilon}^-) < 0 < f_\varepsilon''(t_{b,\varepsilon}^+)$ for small $\varepsilon > 0$.

(ii) If we take $\varepsilon > 0$ smaller if necessary, we get $I(t_b^- v_\varepsilon) < g(\tau_b^-)$ and $I(t_b^+ v_\varepsilon) < g(\tau_b^+)$.

Proof. (i) The first part is proved by Lemma 2.3.

(ii) Noting Lemma 2.3 and (2.8), we have a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $b \in (0, b_2)$, there exist just two solutions $0 < t_{b,\varepsilon}^- < t_{b,\varepsilon}^+$ of $f_\varepsilon'(t) = 0$ such that

$$a\|v_\varepsilon\|^2 + b(t_{b,\varepsilon_n}^\pm)^2\|v_\varepsilon\|^4 - \lambda(t_{b,\varepsilon_n}^\pm)^{q-1}\int_\Omega v_\varepsilon^{q+1}\,\mathrm{d}x - (t_{b,\varepsilon_n}^\pm)^{2^*-2} = 0, \tag{2.11}$$

and

$$f_\varepsilon''(t_{b,\varepsilon}^-) < 0 < f_\varepsilon''(t_{b,\varepsilon}^+). \tag{2.12}$$

First, we prove that for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$, $t_{b,\varepsilon_n}^\pm$ is bounded. By contradiction, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $t_{b,\varepsilon_n}^\pm \to \infty$ as $n \to \infty$. From (2.8) and (2.11), we have $a(S+O(\varepsilon_n^{N-2}))+b(t_{b,\varepsilon_n}^\pm)^2(S^2+O(\varepsilon_n^{N-2}))-\lambda(t_{b,\varepsilon_n}^\pm)^{q-1}(\alpha\varepsilon_n^\beta+O(\varepsilon_n^{N-2}))-(t_{b,\varepsilon_n}^\pm)^{2^*-2} = 0$. Then $aS+bS^2(t_{b,\varepsilon_n}^\pm)^2-(t_{b,\varepsilon_n}^\pm)^{2^*-2} = 0$, which implies that $\frac{aS}{(t_{b,\varepsilon_n}^\pm)^2}+bS^2 = \frac{(t_{b,\varepsilon_n}^\pm)^{2^*-2}}{(t_{b,\varepsilon_n}^\pm)^2}$. Since $2^* < 4$, we have the contradiction as $n \to \infty$. Therefore we may assume that there exist constants $t_b^\pm > 0$ such that $t_{b,\varepsilon_n}^\pm \to t_b^\pm$ as $n \to \infty$.

By (2.8), (2.11), (2.12) and Lemma 2.3 (i), we conclude that $0 < t_b^- < t_b^+$ are the solutions to

$$aS + bS^2t^2 - t^{2^*-2} = 0 (t > 0). \tag{2.13}$$

Here notice that

$$t_b^{\pm} = \tau_b^{\pm} S^{(N-2)/4}, \tag{2.14}$$

with $\tau_b^{\pm} > 0$ in Lemma 2.3. Put

$$\delta_n^{\pm} \triangleq t_{b,\varepsilon_n}^{\pm} - t_b^{\pm}, \tag{2.15}$$

then $\delta_n^{\pm} \to 0$ as $n \to \infty$. Now, using (2.8) and (2.13)-(2.15), we estimate,

$$I(t_{b,\varepsilon_n}^{\pm} v_{\varepsilon_n}) = g(\tau_b^{\pm}) - \lambda C \varepsilon_n^{\beta} + o(\delta_n^{\pm}) + o(\varepsilon_n^{\beta})$$

for some constant $C > 0$, where g is defined as in Lemma 2.3.

Next, we prove $\delta_n^{\pm} = O(\varepsilon_n^{\beta})$. In order to prove $I(t_{b,\varepsilon_n}^{\pm} v_{\varepsilon_n}) < g(\tau_b^{\pm})$, we need to prove $-\lambda C \varepsilon_n^{\beta} + o(\delta_n^{\pm}) + o(\varepsilon_n^{\beta}) < 0$, which implies that we need to prove $\delta_n^{\pm} = O(\varepsilon_n^{\beta})$. By (2.8), (2.11), (2.13) and (2.15), we have

$$\begin{aligned} 0 &= a\|v_{\varepsilon_n}\|^2 + b(t_{b,\varepsilon_n}^{\pm})^2 \|v_{\varepsilon_n}\|^4 - \lambda (t_{b,\varepsilon_n}^{\pm})^{q-1} \int_{\Omega} v_{\varepsilon_n}^{q+1} \, \mathrm{d}x - (t_{b,\varepsilon_n}^{\pm})^{2^*-2} \\ &= -\lambda C' \varepsilon_n^{\beta} + C_b^{\pm} \delta_n^{\pm} + o(\delta_n^{\pm}) + o(\varepsilon_n^{\beta}), \end{aligned} \tag{2.16}$$

where $C' > 0$ is some constant and $C_b^{\pm} = 2bS^2 t_b^{\pm} - (2^* - 2)(t_b^{\pm})^{2^*-3}$. By (2.14) and Lemma 2.3 (i), we have $g''(t_b^-) = C_b^- < 0 < C_b^+ = g''(t_b^+)$. By (2.16), we have

$$\begin{aligned} 0 &= -\lambda C' + C_b^{\pm} \frac{\delta_n^{\pm}}{\varepsilon_n^{\beta}} + \frac{o(\delta_n^{\pm})}{\varepsilon_n^{\beta}} + \frac{o(\varepsilon_n^{\beta})}{\varepsilon_n^{\beta}} \\ &= -\lambda C' + C_b^{\pm} \frac{\delta_n^{\pm}}{\varepsilon_n^{\beta}} + \frac{o(\varepsilon_n^{\beta})}{\delta_n^{\pm}} \frac{\delta_n^{\pm}}{\varepsilon_n^{\beta}} \\ &= -\lambda C' + \left[C_b^{\pm} + \frac{o(\varepsilon_n^{\beta})}{\delta_n^{\pm}} \right] \frac{\delta_n^{\pm}}{\varepsilon_n^{\beta}} \end{aligned}$$

Since C' and $C_b^{\pm}$ are constants, we have $\frac{\delta_n^{\pm}}{\varepsilon_n^{\beta}} \to \frac{\lambda C'}{C_b^{\pm}} \neq 0$ as $n \to \infty$. By the definition of infinitesimal of the same order, we have $I(t_b^{\pm} v_{\varepsilon}) < g(\tau_b^{\pm})$. □

3 Proof of Theorem 1.1 and 1.2

To construct a PS sequence, we define $\sigma = \sigma(b, \lambda)$ by

$$\sigma := \inf \left\{ \liminf_{n \to \infty} I(u_n) : \{u_n\}_{n \in \mathbb{N}} \in \mathcal{M} \right\},$$

$$\mathcal{M} := \left\{ \{u_n\}_{n\in\mathbb{N}} \in \mathcal{N} : \lim_{n\to\infty} f''_{u_n}(1) = 0 \right\}.$$

Put

$$c^- = c^-(b,\lambda) := \inf_{u\in\mathcal{N}^-} I(u), \quad c^0 = c^0(b,\lambda) := \inf_{u\in\mathcal{N}^0} I(u).$$

Here we define $\sigma = \infty$ if $\mathcal{M} = \emptyset$ and $c^0 = \infty$ if $\mathcal{N} = \emptyset$. Then $\sigma \le c^0$. With the assumptions in Lemma 2.1, standard arguments show that

$$\inf_{u\in\mathcal{N}} \|u\| > 0 \tag{3.1}$$

and $c^-, c^0 > 0$ (as long as each restriction is nonempty). Furthermore, it is obvious from Lemma 2.4 that $\mathcal{N}^- \neq \emptyset$ for all $b \in (0, b_2)$ and all $\lambda \in (0, a\lambda_1)$ if $q = 1$ and all $\lambda > 0$ if $q \in (1,3)$.

Lemma 3.1. Suppose $1 < q < 2^* - 1$, then

$$\sigma \ge \frac{a^2(q-1)^2}{4(q+1)(3-q)b}.$$

Proof. By the definition of σ, take $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{M}$ and put $c_n = f''_{u_n}(1)$, then we have

$$c_n = f''_{u_n}(1) - qf'_{u_n}(1) = -a(q-1)\|u_n\|^2 + (3-q)b\|u_n\|^4 - (2^* - q - 1)\int_\Omega (u_n)_+^{2^*}\ \mathrm{d}x, \tag{3.2}$$

and

$$\begin{aligned} I(u_n) &= I(u_n) - \frac{1}{q+1} f'_{u_n}(1) \\ &= \frac{a(q-1)}{2(q+1)}\|u_n\|^2 - \frac{(3-q)b}{4(q+1)}\|u_n\|^4 + \frac{2^* - q - 1}{2^*(q+1)}\int_\Omega (u_n)_+^{2^*}\ \mathrm{d}x. \end{aligned}$$

Dividing (3.2) by $2^*(q+1)$, we get

$$I(u_n) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\frac{a(q-1)}{q+1}\|u_n\|^2 + \left(\frac{1}{2^*} - \frac{1}{4}\right)\frac{(3-q)b}{q+1}\|u_n\|^4 - \frac{c_n}{2^*(q+1)}. \tag{3.3}$$

On the other hand, it follows from (3.2) that

$$-a(q-1)\|u_n\|^2 + (3-q)b\|u_n\|^4 = (2^* - q - 1)\int_\Omega (u_n)_+^{2^*}\ \mathrm{d}x + c_n \ge c_n.$$

This suggests

$$\|u_n\|^2 \ge \frac{a(q-1)}{(3-q)b} + \frac{c_n}{(3-q)b\|u_n\|^2}.$$

Applying this inequality to (3.1), (3.3) and taking $n \to \infty$, we conclude

$$\liminf_{n\to\infty} I(u_n) \geq \frac{a^2(q-1)^2}{4(q+1)(3-q)b}.$$

Since $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{M}$ is arbitrary, we finish the proof. □

Proposition 3.2. Let $1 < q < 2^* - 1$, there exists a constant $b_3 \in (0, b_2]$ depending only on N and q such that

$$c^- < \frac{a^2(q-1)^2}{4(q+1)(3-q)b} \leq \sigma$$

holds for any $b \in (0, b_3)$ and $\lambda > 0$.

Proof. First we clearly see that $g(\tau_b^-)$ is nondecreasing with respect to $b \in (0, b_2)$. Furthermore, there is a constant $b_3 \in (0, b_2]$ such that

$$g(\tau_{b_3}^-) \leq \frac{a^2(q-1)^2}{4(q+1)(3-q)b_3},$$

then using Lemma 2.4 and Lemma 3.1, for all $b \in (0, b_3)$ and $\lambda > 0$, we have

$$c^- < g(\tau_b^-) \leq g(\tau_{b_3}^-) \leq \frac{a^2(q-1)^2}{4(q+1)(3-q)b_3} < \frac{a^2(q-1)^2}{4(q+1)(3-q)b} \leq \sigma. \qquad \square$$

Lemma 3.3. Let $q = 1$ and $0 \leq \lambda \leq a\lambda_1$, then there exist constants $C(N) > 0$ depending on N and $C(N, \Omega) > 0$ depending on N, Ω such that

$$\sigma \geq \min\left\{C(N)b^{-1}, C(N,\Omega)b^{-2^*/(4-2^*)}\right\}.$$

Proof. Similarly to the proof of Lemma 3.1, we take $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{M}$ and put $c_n = f''_{u_n}(1)$, then

$$c_n = f''_{u_n}(1) - f'_{u_n}(1) = 2b\|u_n\|^4 - (2^* - 2)\int_\Omega (u_n)_+^{2^*}\, \mathrm{d}x, \tag{3.4}$$

and

$$I(u_n) = I(u_n) - \frac{1}{2}f'_{u_n}(1) = \frac{4-2^*}{4\cdot 2^*}b\|u_n\|^4. \tag{3.5}$$

Here we divide the proof into two cases:

$$\textbf{Case1:}\ \lambda\int_\Omega (u_n)_+^2\, \mathrm{d}x \leq \frac{a}{2}\|u_n\|^2, \qquad \textbf{Case2:}\ \lambda\int_\Omega (u_n)_+^2\, \mathrm{d}x \geq \frac{a}{2}\|u_n\|^2.$$

Case 1. Using $f'_{u_n}(1) = 0$ and (3.4), we obtain

$$\frac{a}{2}\|u_n\|^2 \le a\|u_n\|^2 - \lambda \int_\Omega (u_n)_+^2 \,\mathrm{d}x = \int_\Omega (u_n)_+^{2^*} \,\mathrm{d}x - b\|u_n\|^4 = \frac{4-2^*}{4\cdot 2^*} b\|u_n\|^4 - \frac{c_n}{2^*-2}.$$

It follows that

$$\|u_n\|^2 \ge \frac{(2^*-2)a}{2(4-2^*)b} + \frac{c_n}{(4-2^*)b\|u_n\|^2}.$$

Therefore, applying this inequality to (3.5), we have

$$I(u_n) \ge \frac{4-2^*}{4\cdot 2^*} b\left(\frac{(2^*-2)a}{2(4-2^*)b} + \frac{c_n}{(4-2^*)b\|u_n\|^2}\right)^2.$$

Case 2. Noting $\lambda \le a\lambda_1$, the Holder inequality and (3.4), we estimate

$$\begin{aligned}
\frac{a}{2}\|u_n\|^2 \quad & \le \lambda \int_\Omega (u_n)_+^2 \,\mathrm{d}x \le a\lambda_1 \left(\int_\Omega \,\mathrm{d}x\right)^{1-2/2^*} \left(\int_\Omega (u_n)_+^{2^*} \,\mathrm{d}x\right)^{2/2^*} \\
&= a\lambda_1 |\Omega|^{1-2/2^*} \left(\frac{2b\|u_n\|^4}{2^*-2} - \frac{c_n}{2^*-2}\right)^{2/2^*} \\
&= a\lambda_1 |\Omega|^{1-2/2^*} \left(\frac{2}{2^*-2} - \frac{c_n}{(2^*-2)b\|u_n\|^4}\right)^{2/2^*} b^{2/2^*} \|u_n\|^{8/2^*}.
\end{aligned}$$

Thus we observe

$$\frac{a}{2}\|u_n\|^4 \ge \left(2a\lambda_1 |\Omega|^{1-2/2^*} \left(\frac{2}{2^*-2} - \frac{c_n}{(2^*-2)b\|u_n\|^4}\right)^{2/2^*} b^{2/2^*}\right)^{-2\cdot 2^*/(4-2^*)}.$$

Then (3.5) with this estimate suggest

$$I(u_n) \ge \frac{4-2^*}{4\cdot 2^*} b\left(2a\lambda_1 |\Omega|^{1-2/2^*} \left(\frac{2}{2^*-2} - \frac{c_n}{(2^*-2)b\|u_n\|^4}\right)^{2/2^*} b^{2/2^*}\right)^{-2\cdot 2^*/(4-2^*)}.$$

Combining Case1, Case2, the definition of $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{M}$ and (3.1), it follows that

$$\liminf_{n\to\infty} I(u_n) \ge \min\left\{\frac{(2^*-2)a^2}{16\cdot 2^*(4-2^*)b}, \quad \frac{4-2^*}{4\cdot 2^*}\left(\frac{(2^*-2)^{2/2^*}}{2a\lambda_1|\Omega|^{1-2/2^*}2^{2/2^*}}\right)^{2\cdot 2^*/(4-2^*)} b^{-2^*/(4-2^*)}\right\},$$

where $C(N) = \frac{(2^*-2)a^2}{16\cdot 2^*(4-2^*)}$, $C(N,\Omega) = \frac{4-2^*}{4\cdot 2^*}\left(\frac{(2^*-2)^{2/2^*}}{2a\lambda_1|\Omega|^{1-2/2^*}2^{2/2^*}}\right)^{2\cdot 2^*/(4-2^*)}$. □

Proposition 3.4. In the case $q = 1$, there exist a value $b_4 \in (0, b_2]$ depending on N, Ω, $C(N) > 0$ depending on N and $C(N,\Omega) > 0$ depending on N, Ω such that

$$c^- < \min\left\{C(N)b^{-1}, C(N,\Omega)b^{-2^*/(4-2^*)}\right\} \le \sigma$$

is ture for all $b \in (0, b_4)$ and $\lambda \in (0, a\lambda_1)$.

Proof. Using Lemma 3.3, we give the proof similarly to that of Proposition 3.2. □

As we known, by Proposition 3.2 and 3.4, we construct the PS sequences in $\mathcal{N}^-$ at the level c^-. Next, using the above proposition , we prove the proof of our main from two parts. First, we consider the critical point with positive energy in $\mathcal{N}^-$. Second, we deal with the critical point with negative energy. Recall the constants $b_1, b_2 > 0, \tau_b^{\pm} > 0$ and the function g in Lemma 2.3.

Proof of Theorem 1.1. Let $b_3, b_4 \in (0, b_2]$ be constants determined in Proposition 3.2 and 3.4 respectively. Define $b_0 := b_3$ if $q > 1$ and $b_0 := b_4$ if $q = 1$. Then the proof lies in two steps.

First step. We need to construct a bounded PS sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{N}^-$ at the level c^-, that is, $I(u_n) = c^- + o(1)$, $I'(u_n) = o(1)$ in $H^{-1}(\Omega)$ as $n \to \infty$, and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$.

As $f_u'(1) = 0$ for all $u \in \mathcal{N}$, the Sobolev embedding implies

$$a\|u\|^2 + b\|u\|^4 = \lambda \int_\Omega u_+^{q+1}\,\mathrm{d}x + \int_\Omega u_+^{2^*}\,\mathrm{d}x \le C(\lambda\|u\|^{q+1} + \|u\|^{2^*}),$$

for some constant $C > 0$ which does not depend on u. Since $2 < q + 1 < 2^* < 4$, we conclude the boundedness of $\mathcal{N}^-$.

Now let us construct the desired PS sequence for I. As $2^* < 4$, I is coercive, that is, $I(u_n) \to \infty$ as $n \to \infty$. In particular, it is bounded from below on $\mathcal{N}^- \cup \mathcal{N}^0$. Therefore, by the Ekeland variational principle, we have a sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{N}^- \cup \mathcal{N}^0$ such that

$$I(u_n) \le \inf_{u\in\mathcal{N}^-\cup\mathcal{N}^0} I(u) + \frac{1}{n} \quad \text{and} \quad I(\omega) \ge I(u_n) - \frac{1}{n}\|u_n - \omega\|(\omega \in \mathcal{N}^- \cup \mathcal{N}^0). \tag{3.6}$$

Proposition 3.2 and 3.4 suggest that $\inf_{u\in\mathcal{N}^-\cup\mathcal{N}^0} I(u) = c^-$ and $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{N}^-$ for large n. Since the boundedness of $\mathcal{N}^-$, we have $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Next, we claim $\|I'(u_n)\|_{H^{-1}(\Omega)} = o(1)$ as $n \to \infty$. Noting the upper bounds for c^- in the propositions, the rest of the proof is done similarly to the argument in [29]. This shows the existence of the desired PS sequence.

Second step. We need to prove that there exists a critical point $u \in \mathcal{N}^-$ of I with $I(u) = c^-$, which implies that we need to show the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$.

By Lemma 2.4, we have $c^- \in (0, g(\tau_b^-))$. Now, we suppose that $\{u_n\}_{n\in\mathbb{N}}$ does not contain any subsequence which strongly converges in $H_0^1(\Omega)$ on the contrary. We consider the limit function of the fibering maps,

$$\lim_{n\to\infty} f_{u_n}(t) = \lim_{n\to\infty} I(tu_n) = \phi(t) + \psi(t).$$

Then from Proposition 2.2, we have

$$u_n = u_0 + \sum_{j=1}^{k} v_j + o(1) \quad \text{in} \;\; D^{1,2}(\mathbb{R}^N),$$

$$\phi(t) := \frac{a\|u_0\|^2}{2}t^2 + \frac{bA\|u_0\|^2}{4}t^4 - \frac{\lambda \int_\Omega u_0^{q+1}\,\mathrm{d}x}{q+1}t^{q+1} - \frac{\int_\Omega u_0^{2^*}\,\mathrm{d}x}{2^*}t^{2^*},$$
$$\psi(t) := \sum_{i=1}^{k}\left(\frac{a\|v_i\|_{1,2}^2}{2}t^2 + \frac{bA\|v_i\|_{1,2}^2}{4}t^4 - \frac{\int_{\mathbb{R}^N} v_i^{2^*}\,\mathrm{d}x}{2^*}t^{2^*}\right),$$

where $A := \lim_{n\to\infty} \|u_n\|^2$. Similarly we get $\lim_{n\to\infty} f'_{u_n}(t) = \phi'(t) + \psi'(t)$ and $\lim_{n\to\infty} f''_{u_n}(t) = \phi''(t) + \psi''(t)$. Since $u_n \in \mathcal{N}^-$ which is equivalent to $f''_{u_n}(1) < 0$ and $f'_{u_n}(1) = 0$, we have $f'_{u_n}(t) > 0$ for all $t \in (0, 1)$ by Lemma 2.1(iii). Thus $\lim_{n\to\infty} f'_{u_n}(t) = \phi'(t) + \psi'(t) \geq 0$ for all $t \in (0, 1)$, which means that $\phi(t) + \psi(t)$ is nondecreasing for all $t \in (0, 1)$.

Equation (2.5) and (2.6) show that $\phi'(1) = \psi'(1) = 0$. Furthermore, as $u_n \in \mathcal{N}^-$, $f''_{u_n}(1) < 0$, which implies $\phi''(1) + \psi''(1) \leq 0$. On the other hand, we calculate

$$\phi''(1) = B\|u_0\|^2 + (2^* - q - 1)\lambda \int_\Omega u_0^{q+1}\,\mathrm{d}x,$$

and

$$\psi''(1) = \sum_{i=1}^{k}(B\|v_i\|_{1,2}^2),$$

where $B := (2 - 2^*)a + (4 - 2^*)bA$. If $B \geq 0$, we obtain

$$\phi''(1) + \psi''(1) = BA + (2^* - q - 1)\lambda \int_\Omega u_0^{q+1}\,\mathrm{d}x > 0,$$

which is a contradiction. Therefore, $B < 0$. This implies $\psi''(1) < 0$. Then, by Lemma 2.1, we conclude that $\psi(t)$ is increasing on $(0, 1)$. In particular, $\psi(t) > 0$ for all $t \in (0, 1]$.

Lastly, as $0 < \|u_0\|^2 < A$, we have

$$f'_{u_0}(1) = a\|u_0\|^2 + b\|u_0\|^4 - \lambda \int_\Omega u_0^{q+1}\,\mathrm{d}x - \int_\Omega u_0^{2^*}\,\mathrm{d}x < \phi'(1) = 0.$$

Then $f_{u_0}(t)$ has the unique local maximum $t_0 \in (0,1)$ and $t_0 u_0 \in \mathcal{N}^-$ by Lemma 2.1 (ii). As a consequence, we show

$$I(t_0u_0) = f_{u_0}(t_0) < \phi(t_0) < \phi(t_0) + \psi(t_0) \le \phi(1) + \psi(1) = c^-.$$

This contradicts the definition of c^-. So there exists $u \in \mathcal{N}^-$ such that $u_n \to u$ with $I(u) = c^-$. □

Proof of Theorem 1.2. First, we observe that for any $b \in (0, b_2)$, $c := \inf\limits_{u \in H_0^1(\Omega)} I(u) < g(\tau_b^+)$ by Lemma 2.4. Take $b_1 \in (0, b_2)$ from Lemma 2.3 (ii) so that $g(\tau_b^+) \le 0$ for all $b \in (0, b_1]$. Next, we need to prove that if $c < 0$, then there exists a critical point v with $I(v) = c$. Actually, since $2^* < 4$, I is coercive. Thus $c > -\infty$. For any minimizing sequence $\{v_n\}_{n\in\mathbb{N}}$ at the level c, there exists a PS sequence $\{u_n\}_{n\in\mathbb{N}}$ of I at the same level such that $u_n = v_n + o(1)$ in $H_0^1(\Omega)$ by the Ekeland variational principle. Obviously, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ from the coerciveness of I. Then our aim is to show the strong convergence of $\{u_n\}_{n\in\mathbb{N}}$. On the contrary, we suppose that $\{u_n\}_{n\in\mathbb{N}}$ does not contain any subsequence which strongly converges in $H_0^1(\Omega)$. Define $\tilde{u}_n := u_n - u_0 \in H_0^1(\Omega)$. Then we have $\lim\limits_{n\to\infty} \|\tilde{u}_n\|^2 = \sum_{i=1}^k \|v_i\|^2 > 0$ by Proposition 2.2. Take a constant $e > 0$ so that $\|u_0\|^2 - e \lim\limits_{n\to\infty} \|\tilde{u}_n\|^2 = 0$. Let $s \in \mathbb{R}$ be a parameter satisfying $s \in (-1, 1/e)$. We put

$$w_{n,s} := (1+s)^{1/2} u_0 + (1-es)^{1/2} \tilde{u}_n \in H_0^1(\Omega).$$

Then, for $s \in (-1, 1/e)$, we have

$$\begin{aligned}
\lim_{n\to\infty} \|w_{n,s}\|^2 &= \|(1+s)^{1/2}u_0\|^2 + \lim_{n\to\infty} \|(1-es)^{1/2}\tilde{u}_n\|^2 \\
&= (1+s)\|u_0\|^2 + (1-es)\lim_{n\to\infty} \|\tilde{u}_n\|^2 \\
&= \|u_0\|^2 + \lim_{n\to\infty} \|\tilde{u}_n\|^2 + s(\|u_0\|^2 - e\lim_{n\to\infty} \|\tilde{u}_n\|^2) \\
&= \lim_{n\to\infty} \|u_n\|^2 + s(\|u_0\|^2 - e\lim_{n\to\infty} \|\tilde{u}_n\|^2) = A,
\end{aligned}$$

where $A = \lim\limits_{n\to\infty} \|u_n\|^2$ as before. Now we consider

$$h(s) := \lim_{n\to\infty} I(w_{n,s}).$$

Note that $h(0) = \lim\limits_{n\to\infty} I(u) = c$. In addition, by Proposition 2.2 and the Vitali convergence theorem, we have

$$\begin{aligned} h(s) &= \frac{a\|u_0\|^2}{2}(1+s)+\frac{bA\|u_0\|^2}{4}(1+s)-\frac{\lambda\int_\Omega u_0^{q+1}\,\mathrm{d}x}{q+1}(1+s)^{(q+1)/2}-\frac{\int_\Omega u_0^{2^*}\,\mathrm{d}x}{2^*}(1+s)^{2^*/2} \\ &+ \sum_{i=1}^{k}\left(\frac{a\|v_i\|_{1,2}^2}{2}(1-es)+\frac{bA\|v_i\|_{1,2}^2}{4}(1-es)-\frac{\int_{\mathbb{R}^N} v_i^{2^*}\,\mathrm{d}x}{2^*}(1-es)^{2^*/2}\right). \end{aligned}$$

Direct calculations imply

$$\begin{aligned} h'(0) &= \frac{a\|u_0\|^2}{2}+\frac{bA\|u_0\|^2}{4}-\frac{\lambda\int_\Omega u_0^{q+1}\,\mathrm{d}x}{q+1}-\frac{\int_\Omega u_0^{2^*}\,\mathrm{d}x}{2^*} \\ &- e\sum_{i=1}^{k}\left(\frac{a\|v_i\|_{1,2}^2}{2}+\frac{bA\|v_i\|_{1,2}^2}{4}-\frac{\int_{\mathbb{R}^N} v_i^{2^*}\,\mathrm{d}x}{2^*}\right), \end{aligned}$$

and

$$h''(0)=-\frac{\lambda(q-1)\int_\Omega u_0^{q+1}\,\mathrm{d}x}{4}-\frac{(2^*-2)\int_\Omega u_0^{2^*}\,\mathrm{d}x}{2^*}-e^2\sum_{i=1}^{k}\frac{(2^*-2)\int_{\mathbb{R}^N} v_i^{2^*}\,\mathrm{d}x}{4},$$

which implies $h''(0)<0$. Moreover, from (2.5) and (2.6), we get

$$\begin{aligned} h'(0) &= \frac{a\|u_0\|^2}{2}+\frac{bA\|u_0\|^2}{4}-\frac{(a+bA)\|u_0\|^2}{2}-e\sum_{i=1}^{k}\left(\frac{a\|v_i\|_{1,2}^2}{2}+\frac{bA\|v_i\|_{1,2}^2}{4}\frac{(a+bA)\|v_i\|^2}{2}\right) \\ &= -\frac{bA\|u_0\|^2}{4}-e\sum_{i=1}^{k}\left(-\frac{bA\|v_i\|_{1,2}^2}{4}\right)=\frac{-bA}{4}\left(\|u_0\|^2-e\sum_{i=1}^{k}\|v_i\|_{1,2}^2\right)=0. \end{aligned}$$

Consequently, we see

$$h(0)=c, h'(0)=0, h''(0)<0.$$

Thus, for sufficiently small $s_0>0$, we have $h(s_0)<h(0)=c$. Therefore, for $s=s_0$, it follows that

$$I(w_{n,s_0})<c.$$

If n is sufficiently large, then this contradicts the definition of c. We finish the proof. □

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