Existence of Periodic Solutions for a Class of Fourth-order Difference Equation

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Abstract

We apply the continuation theorem of Mawhin to ensure that a fourth-order nonlinear difference equation of the form

$$\Delta^4 u(k-2) - a(k)u^{\alpha}(k) + b(k)u^{\beta}(k) = 0,$$

with periodic boundary conditions possesses at least one nontrivial positive solution, where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\alpha, \beta \in \mathbb{N}^+$ and $\alpha \neq \beta$. $a(k), b(k)$ are T-periodic functions and $a(k)b(k) > 0$. As applications, we will give some examples to illustrate the application of these theorems.

Keywords: Multiple solutions; Nonlinear difference equation; Continuation theorem

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1. Introduction

In recent years, the theory of nonlinear difference equations has been widely used in the study of discrete models in the fields of economics, neural networks, ecology and etc.. In particular, there are many authors have discussed the existence and multiplicity of periodic solutions for discrete boundary value problem by exploiting various methods, including the method of upper and lower solutions, Leray-Schauder degree, fixed point theory, critical theory and variational methods, see Mawhin [2], Cabada [4], Graef [8, 9] et al.[5, 12, 13, 16, 18, 19] and the references for more details.

Let $\mathbb{N}^+$, Z and $\mathbb{R}$ denote the sets of all positive integers, integers and real numbers respectively. This paper considers the following fourth-order nonlinear difference equation

$$\Delta^4 u(k-2) - a(k)u^{\alpha}(k) + b(k)u^{\beta}(k) = 0, \tag{1}$$

where $\alpha, \beta \in \mathbb{N}^+$ and $\alpha \neq \beta$, $a(k), b(k)$ are T-periodic functions, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator.

The equation (1) can be considered as a discrete analogue of a special case of the following fourth-order nonlinear differential equation

$$u'''' - a(t)u^{\alpha} + b(t)u^{\beta} = 0, \qquad t \in \mathbb{R}, \tag{2}$$

which has been studied in [3, 17] when $\alpha = 1, \beta = 3$. In [20], Yang and Han proved the existence of periodic solution to equation (2) when $\alpha = n, \beta = n+2$, where n is a positive integer.

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When $a(k) \equiv 0, \beta = 1$, Peterson and Ridenhour [15] considered the disconjugacy of the following equation

$$\Delta^4 u(k-2) + b(k)u(k) = 0, \quad k \in Z.$$

In 2005, Cai et al. [5] studied the fourth-order nonlinear difference equation

$$\Delta^4 u(k-2) + f(k, u(k)) = 0, \quad k \in Z. \tag{3}$$

By applying linking theorem, they obtained some criteria for the existence and multiplicity of periodic solutions of the equation (3).

Motivated by the above works, the main *aim* of this paper is to investigate the existence of at least one positive T-periodic solution of (1). In order to obtain the main results of (1), we assume that the coefficient functions $a(k)$ and $b(k)$ satisfy the following condition:

$(\mathbf{F_1})$ Suppose $a(k), b(k)$ are $T-$periodic functions and $a(k)b(k) > 0$ for all $k \in Z$. Furthermore, we assume that there exist positive constants a, A, b, B such that

$$a = \min_{k \in Z} |a(k)|, \ A = \max_{k \in Z} |a(k)|, \ b = \min_{k \in Z} |b(k)|, \ B = \max_{k \in Z} |b(k)|.$$

The main results in this paper are next Theorems 1.1-1.2.

Theorem 1.1. *Let* $(\mathbf{F_1})$ *hold, if* $\alpha < \beta$ *and the period* T *satisfies*

$$0 < T^4 \leq \frac{16}{\kappa(AR_1^{\alpha-1} + BR_1^{\beta-1})},$$

where κ *is a positive constant such that* $\|\mathbf{u}\| \leq \kappa \cdot \max_{1 \leq l,j \leq T} |u(l) - u(j)|$ *for* $\mathbf{u} \not\equiv const$, $R_1 = (\frac{A}{b})^{\frac{1}{\beta-\alpha}} + \rho$, *where* $\rho > 0$ *small enough such that* $(\frac{a}{B})^{\frac{1}{\beta-\alpha}} - \rho > 0$. *Then the equation (1) admits at least one positive* T*-periodic solution.*

Theorem 1.2. *Let* $(\mathbf{F_1})$ *holds, if* $\alpha > \beta$ *and the period* T *satisfies*

$$0 < T^4 \leq \frac{16}{\iota(AQ_1^{\alpha-1} + BQ_1^{\beta-1})},$$

where κ *is a positive constant such that* $\|\mathbf{u}\| \leq \iota \cdot \max_{1 \leq l,j \leq T} |u(l) - u(j)|$ *for* $\mathbf{u} \not\equiv const$, $Q_1 = (\frac{B}{a})^{\frac{1}{\alpha-\beta}} + \tau$, *where* $\tau > 0$ *small enough such that* $(\frac{b}{A})^{\frac{1}{\alpha-\beta}} - \tau > 0$. *Then the equation (1) admits at least one positive* T*-periodic solution.*

Remark 1.3. *The conclusion of Theorems 1.1-1.2 require that* $\mathbf{u} \not\equiv const$. *In fact, if* $\mathbf{u} \equiv const$, *the conclusions of the existence of periodic solutions is still valid. See Sections 3-4 for detailed proof.*

Theorem 1.4. *Suppose* $a(k)b(k) \leq 0$ *and* $a(k), b(k)$ *are not identical to zero for all* $k \in Z$, *then the equation (1) has no positive solution.*

This paper is organized as follows: In Section 2, we give some lemmas needed to prove the main results. Section 3 contains the proof of the Theorem 1.1. Section 4 contains the proof of the Theorem 1.2. Section 5 contains the proof of the Theorem 1.4.

2. Preliminary results

In this section, we introduce some notations and well-known results which will be used in the subsequent section.

Definition 2.1. (See. [7]) Let X, Y be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping. The mapping L is said to be a Fredholm mapping of index zero if

(a) $\operatorname{Im} L$ is closed in Y;

(b) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L,$$
$$\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

It follows that the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$

Definition 2.2. (See. [7]) If Ω is a bounded open subset of X, N is called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.3. *(Mawhin's Continuation Theorem [7]). Let L be a Fredholm mapping of index zero, $\Omega \subset X$ is an open bounded set and let N is $L-$compact on $\overline{\Omega}$. Suppose*

(1) $L\mathbf{u} \neq \lambda N\mathbf{u}$ for all $\mathbf{u} \in \partial\Omega \cap \operatorname{Dom} L$, and all $\lambda \in (0, 1)$;

(2) $QN\mathbf{u} \neq 0$, for all $\mathbf{u} \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism.

Then the equation $L\mathbf{u} = N\mathbf{u}$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Let X be all real T-periodic sequences of the form $\mathbf{u} = \{u(k)\}_{k\in Z}$. Then X is a Banach space under the norm $\|\mathbf{u}\| = \max\limits_{k\in[2,T+1]_Z} |u(k)|$.

Define the operator $L : X \to X$ by setting

$$L\mathbf{u} = \Delta^4 u(k-2), \quad \mathbf{u} \in \operatorname{Dom} L,$$

where $\operatorname{Dom} L = \{\mathbf{u}|\mathbf{u} \in X, \Delta u(k+T) = \Delta u(k), \Delta^2 u(k+T) = \Delta^2 u(k) \text{ and } \Delta^3 u(k+T) = \Delta^3 u(k)\}$. By a simple calculation, we know that $\operatorname{Ker} L = \mathbb{R}$ and $\operatorname{Im} L = \{\mathbf{u} : \sum\limits_{i=2}^{T+1} u(i) = 0\}$. Since $\dim X = T$ and L is a linear mapping, by the knowledge of linear algebra, we know that $\dim \operatorname{Ker} L \bigoplus \dim \operatorname{Im} L = \dim X$. It is easy to see that $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$, and $\dim \operatorname{Im} L = T - 1$. It follows that $\operatorname{Im} L$ is closed in X. Therefore, the operator L is a Fredholm operator with index zero.

Let us define $N : X \to X$ by

$$(Nu)(k) = a(k)u^{\alpha}(k) - b(k)u^{\beta}(k). \tag{4}$$

We define $P : X \to \operatorname{Ker} L$ and $Q : X \to X$ as follows

$$(Pu)(k) = (Qu)(k) = \frac{1}{T}\sum_{i=2}^{T+1} u(i). \tag{5}$$

The operators P and Q are projections. Hence, $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ has an inverse which is denoted by K_P. In view of (4) and (5), for any $\mathbf{u} \in X$, we can see that

$$(QNu)(k) = \frac{1}{T}\sum_{i=2}^{T+1}[a(i)u^{\alpha}(i) - b(i)u^{\beta}(i)], \tag{6}$$

and

$$((I-Q)Nu)(k) = a(k)u^{\alpha}(k) - b(k)u^{\beta}(k) - \frac{1}{T}\sum_{i=2}^{T+1}[a(i)u^{\alpha}(i) - b(i)u^{\beta}(i)]. \tag{7}$$

Since K_P is linear. By virtue of (6)-(7), it is not difficult to see that QN and $K_P(I-Q)N$ are continuous on X. Hence, we know that if Ω is an open and bounded subset of X, then $QN(\overline{\Omega})$ is bounded. It is follows that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Therefore, the mapping N is L-compact on $\overline{\Omega}$ with any open and bounded subset $\Omega \subset X$.

Lemma 2.4. *(see [18]) Let $\{u(k)\}_{k\in Z}$ be a real T-periodic sequence, then*

$$\max_{2\le i,j\le T+1} |u(i) - u(j)| \le \frac{T^3}{16}\sum_{k=2}^{T+1} |\Delta^4 u(k-2)|.$$

3. Proof of Theorem 1.1

Proof. In the preceding assumption, we assume that $\alpha < \beta$. From the condition $(\mathbf{F_1})$ we know that $a(k)b(k) > 0$, which include both positive and negative cases. So we need to classify the cases where both $a(k)$ and $b(k)$ are positive and both negative.

Case 1: If coefficient functions $a(k)$ and $b(k)$ are positive T-periodic functions. In view of $(\mathbf{F_1})$, we have that $0 < a \le a(k) \le A$, $0 < b \le b(k) \le B$.

Let

$$\Omega_1 = \{\mathbf{u} \in X : H_1 < u(k) < R_1\}, \tag{8}$$

which is an open set in X, where

$$R_1 = R + \rho,\ \ R = (\frac{A}{b})^{\frac{1}{\beta-\alpha}}, \tag{9}$$

$$H_1 = H - \rho,\ H = (\frac{a}{B})^{\frac{1}{\beta-\alpha}}, \tag{10}$$

where $\rho > 0$ small enough such that $(\frac{a}{B})^{\frac{1}{\beta-\alpha}} - \rho > 0$. Obviously, H_1 and R_1 are well defined. By $\alpha < \beta$, $0 < a \le a(k) \le A$ and $0 < b \le b(k) \le B$, we obtain

$$0 < H_1 < H \le (\frac{a(k)}{b(k)})^{\frac{1}{\beta-\alpha}} \le R < R_1 \tag{11}$$

uniformly for $k \in Z$.

We prove that the condition (1) of Lemma 2.3 holds. Let $0 < \lambda < 1$ and $\mathbf{u}$ be such that

$$\Delta^4 u(k-2) - \lambda a(k) u^{\alpha}(k) + \lambda b(k) u^{\beta}(k) = 0.$$

Summing from 2 to $T+1$, we can see that

$$\sum_{i=2}^{T+1} [\Delta^4 u(i-2) - \lambda a(i) u^{\alpha}(i) + \lambda b(i) u^{\beta}(i)] = 0.$$

Firstly, we *claim* that for each $\lambda \in (0,1)$ and $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$, $L\mathbf{u} \neq \lambda N \mathbf{u}$. In fact, in view of (8), if $\mathbf{u} \not\equiv$ const and $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$, then $H_1 \le \|\mathbf{u}\| \le R_1$. Further,

$$\begin{aligned}
0 =& \sum_{i=2}^{T+1} \Big| \Delta^4 u(i-2) - \lambda a(i) u^{\alpha}(i) + \lambda b(i) u^{\beta}(i) \Big| \\
\ge& \sum_{i=2}^{T+1} \Big| \Delta^4 u(i-2) \Big| - \sum_{i=2}^{T+1} \Big| \lambda a(i) u^{\alpha}(i) + \lambda b(i) u^{\beta}(i) \Big| \\
>& \sum_{i=2}^{T+1} \Big| \Delta^4 u(i-2) \Big| - \sum_{i=2}^{T+1} \Big| a(i) u^{\alpha}(i) + b(i) u^{\beta}(i) \Big| \\
\ge& \frac{16}{T^3} \max_{2 \le l,j \le T+1} |u(l) - u(j)| - T(AR_1^{\alpha-1} + BR_1^{\beta-1}) \|\mathbf{u}\| \\
\ge& \frac{16}{T^3 \kappa} \|\mathbf{u}\| - T(AR_1^{\alpha-1} + BR_1^{\beta-1}) \|\mathbf{u}\| \\
=& \Big[\frac{16}{T^3 \kappa} - T(AR_1^{\alpha-1} + BR_1^{\beta-1}) \Big] \|\mathbf{u}\| \\
\ge& 0,
\end{aligned}$$

where κ is a constant such that $\|\mathbf{u}\| \le \kappa \cdot \max\limits_{2 \le l,j \le T+1} |u(l) - u(j)|$ for $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$. This is a contradiction.

If $\mathbf{u} \equiv$ const, since $\mathbf{u} \in \partial\Omega_1$, it follows from (9), (10) and (11) that

$$a(k) - b(k) H_1^{\beta-\alpha} \ge a - BH_1^{\beta-\alpha} > 0.$$

$$a(k) - b(k)R_1^{\beta-\alpha} \le A - bR_1^{\beta-\alpha} < 0.$$

Therefore,

$$a(k) - b(k)H_1^{\beta-\alpha} > 0, \quad a(k) - b(k)R_1^{\beta-\alpha} < 0, \tag{12}$$

which is the desired conclusion.

If $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Ker} L$, then $\mathbf{u} = \{H_1\}_{k\in Z}$ or $\mathbf{u} = \{R_1\}_{k\in Z}$. By virtue of (12) we conclude that

$$(QNu)(k) = \frac{1}{T}\sum_{i=2}^{T+1} u^\alpha(i)[a(i) - b(i)u^{\beta-\alpha}(i)] \ne 0.$$

Hence, $QN\mathbf{u} \ne 0$ for each $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Ker} L$.

Next let us consider $\frac{H_1+R_1}{2}$, the arithmetic mean of H_1 and R_1. We define $G : X \times \mathbb{R} \to X$ as follows

$$G(\mathbf{u},\mu) = -(1-\mu)\left(\mathbf{u} - \frac{H_1+R_1}{2}\right) + \mu\frac{1}{T}\sum_{i=2}^{T+1} u^\alpha(i)[a(i) - b(i)u^{\beta-\alpha}(i)],\ \ \mu \in [0,1].$$

Clearly, we find that

$$G(\mathbf{u},\mu) \ne 0, \quad \forall\ \mathbf{u} \in \partial\Omega_1 \cap \operatorname{Ker} L.$$

By using the homotopy invariance theorem, it is easy to see that

$$\begin{aligned}\deg(QN,\Omega_1\cap\operatorname{Ker} L,0) &= \deg\left(G(\mathbf{u},1),\Omega_1\cap\operatorname{Ker} L,0\right)\\ &= \deg\left(G(\mathbf{u},0),\Omega_1\cap\operatorname{Ker} L,0\right)\\ &= -1 \ne 0.\end{aligned}$$

Therefore, conditions (1)-(3) of Lemma 2.3 hold for Ω_1.

Furthermore, according to the above reasoning, we deduce that (1) has at least one positive solution in $\overline{\Omega}_1$.

Case 2: If the coefficient functions $a(k), b(k)$ are negative T-periodic functions. In view of $(\mathbf{F_1})$, we have that $-A \le a(k) \le -a < 0$, $-B \le b(k) \le -b < 0$.

Let $\widetilde{a}(k) = -a(k), \widetilde{b}(k) = -b(k)$, then, we see that

$$0 < a \le \widetilde{a}(k) \le A, 0 < b \le \widetilde{b}(k) \le B. \tag{13}$$

It is obvious that (1) is equivalent to the equation

$$\Delta^4 u(k-2) + \widetilde{a}(k)u^\alpha(k) - \widetilde{b}(k)u^\beta(k) = 0. \tag{14}$$

Let $0 < \lambda < 1$ and $\mathbf{u}$ be such that

$$\Delta^4 u(k-2) + \lambda\widetilde{a}(k)u^\alpha(k) - \lambda\widetilde{b}(k)u^\beta(k) = 0.$$

Summing from 2 to $T+1$, we can see that

$$\sum_{i=2}^{T+1}[\Delta^4 u(i-2) + \lambda\widetilde{a}(i)u^\alpha(i) - \lambda\widetilde{b}(i)u^\beta(i)] = 0.$$

Firstly, we *claim* that for each $\lambda \in (0,1)$ and $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$, $L\mathbf{u} \neq \lambda N\mathbf{u}$. In fact, in view of (8), if $\mathbf{u} \not\equiv \text{const}$ and $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$, then $H_1 \le \|\mathbf{u}\| \le R_1$. Further,

$$\begin{aligned}
0 =& \sum_{i=2}^{T+1}\left|\Delta^4 u(i-2) + \lambda\widetilde{a}(i)u^\alpha(i) - \lambda\widetilde{b}(i)u^\beta(i)\right| \\
\ge& \sum_{i=2}^{T+1}\left|\Delta^4 u(i-2)\right| - \sum_{i=2}^{T+1}\left|\lambda\widetilde{a}(i)u^\alpha(i) + \lambda\widetilde{b}(i)u^\beta(i)\right| \\
>& \sum_{i=2}^{T+1}\left|\Delta^4 u(i-2)\right| - \sum_{i=2}^{T+1}\left|\widetilde{a}(i)u^\alpha(i) + \widetilde{b}(i)u^\beta(i)\right| \\
\ge& \frac{16}{T^3}\max_{2\le l,j\le T+1}|u(l)-u(j)| - T(AR_1^{\alpha-1} + BR_1^{\beta-1})\|\mathbf{u}\| \\
\ge& \frac{16}{T^3\kappa}\|\mathbf{u}\| - T(AR_1^{\alpha-1} + BR_1^{\beta-1})\|\mathbf{u}\| \\
=& \left[\frac{16}{T^3\kappa} - T(AR_1^{\alpha-1} + BR_1^{\beta-1})\right]\|\mathbf{u}\| \\
\ge& 0,
\end{aligned}$$

where κ is a constant such that $\|\mathbf{u}\| \le \kappa \cdot \max\limits_{2\le l,j\le T+1}|u(l)-u(j)|$ for $\mathbf{u} \in \partial\Omega_1 \cap \operatorname{Dom} L$. This is a contradiction.

If $\mathbf{u} \equiv \text{const}$, since $\mathbf{u} \in \partial\Omega_1$, it is easy to know that

$$-\widetilde{a}(k) + \widetilde{b}(k)H_1^{\beta-\alpha} < 0$$

and

$$-\widetilde{a}(k) + \widetilde{b}(k)R_1^{\beta-\alpha} > 0.$$

The remaining proof is similar to the proof of case 1, and so we omit it. Furthermore, according to the above reasoning, we deduce that (14) has at least one positive solution in $\overline{\Omega}_1$.

$\square$

4. Proof of Theorem 1.2

Similarly, in the case of $\alpha > \beta$, we need to discuss the case where the coefficient functions $a(k)$ and $b(k)$ are both positive and negative, respectively.

Case 1: If coefficient functions $a(k)$ and $b(k)$ are positive T-periodic functions, we have that $0 < a \leq a(k) \leq A$, $0 < b \leq b(k) \leq B$.

Let

$$\Omega_2 = \{\mathbf{u} \in X : P_1 < u(k) < Q_1\}, \tag{15}$$

which is an open set in X, where

$$Q_1 = Q + \tau,\ Q = (\frac{B}{a})^{\frac{1}{\alpha-\beta}}, \tag{16}$$

$$P_1 = P - \tau,\ P = (\frac{b}{A})^{\frac{1}{\alpha-\beta}}, \tag{17}$$

where $\tau > 0$ small enough such that $(\frac{b}{A})^{\frac{1}{\alpha-\beta}} - \tau > 0$. Obviously, P_1 and Q_1 are well defined. By $\alpha > \beta$, $0 < a \leq a(k) \leq A$ and $0 < b \leq b(k) \leq B$, we obtain

$$0 < P_1 < P \leq (\frac{a(k)}{b(k)})^{\frac{1}{\alpha-\beta}} \leq Q < Q_1$$

uniformly for $k \in Z$.

By virtue of (16) and (17) we obtain

$$a(k)P_1^{\alpha-\beta} - b(k) \leq AP_1^{\alpha-\beta} - b < 0$$

and

$$a(k)Q_1^{\alpha-\beta} - b(k) \geq aQ_1^{\alpha-\beta} - B > 0.$$

Therefore,

$$a(k)P_1^{\alpha-\beta} - b(k) < 0, \quad a(k)Q_1^{\alpha-\beta} - b(k) > 0,$$

uniformly for $k \in Z$.

The remaining proof is similar to the proof of Theorem 1.1, and so we omit it. Furthermore, we deduce that (1) has at least one positive T-periodic solution in $\overline{\Omega}_2$.

Case 2: If the coefficient functions $a(k), b(k)$ are negative T-periodic functions, we have that $-A \leq a(k) \leq -a < 0$, $-B \leq b(k) \leq -b < 0$. Let $\widetilde{a}(k) = -a(k), \widetilde{b}(k) = -b(k)$, then, we can see that

$$0 < a \leq \widetilde{a}(k) \leq A, \quad 0 < b \leq \widetilde{b}(k) \leq B.$$

It is obvious that

$$-\widetilde{a}(k)P_1^{\alpha-\beta} + \widetilde{b}(k) > 0, \quad -\widetilde{a}(k)Q_1^{\alpha-\beta} + \widetilde{b}(k) < 0,$$

uniformly for $k \in Z$.

The remaining proof is similar to the proof of Theorem 1.1, and so we omit it. Furthermore, we conclude that (1) has at least one positive T-periodic solution in $\overline{\Omega}_2$.

5. Proof of Theorem 1.4

Proof. Summing the equation (1) from 2 to $T+1$, we obtain that

$$\sum_{i=2}^{T+1}[\Delta^4 u(i-2) - a(i)u^{\alpha}(i) + b(i)u^{\beta}(i)] = 0.$$

In view of $\operatorname{Dom} L = \{\mathbf{u}|\mathbf{u} \in X, \Delta u(k+T) = \Delta u(k), \Delta^2 u(k+T) = \Delta^2 u(k)$ and $\Delta^3 u(k+T) = \Delta^3 u(k)\}$. Hence,

$$\sum_{i=2}^{T+1}[-a(i)u^{\alpha}(i) + b(i)u^{\beta}(i)] = 0. \tag{18}$$

If $a(k) > 0$, $b(k) \leq 0$, it follows from (18) that (1) does not have any positive solutions. Other cases are similar.

□

6. Example

Example 6.1. The difference equation

$$\Delta^4 u(k-2) - (\frac{1}{100}\sin(\frac{2k\pi}{T}) + \frac{1}{50})u(k) + (\frac{1}{200}|\cos(\frac{2k\pi}{T})| + \frac{1}{100})u^3(k) = 0 \tag{19}$$

is one of the form (1), where $a = \frac{1}{100}, A = \frac{3}{100}, b = \frac{1}{100}$, $B = \frac{3}{200}, \alpha = 1$ and $\beta = 3$. Letting $\rho = \frac{\sqrt{6}}{6}$, hence we obtain

$$0 < T^4 \leq \frac{6400}{\kappa(31 + 6\sqrt{2})},$$

where $\kappa = 1 + \frac{\sqrt{2}}{6}$. Therefore, we can prove that (19) has at least one positive T-periodic solution in $\overline{\Omega}_1$, where $\Omega_1 = \{\mathbf{u} \in X| \ \frac{\sqrt{2}}{6} < u(k) < \frac{6\sqrt{3}+\sqrt{6}}{6}\}$.

Example 6.2. The difference equation

$$\Delta^4 u(k-2) + (\frac{1}{5000}\cos(\frac{2k\pi}{T}) + \frac{1}{2500})u^3(k) - (\frac{1}{2000}\sin(\frac{2k\pi}{T}) + \frac{1}{500})u(k) = 0 \tag{20}$$

is one of the form (1), where $-a = -\frac{1}{5000}, -A = -\frac{3}{5000}, -b = -\frac{3}{2000}, -B = -\frac{1}{400}, \alpha = 3$ and $\beta = 1$. Letting $\tau = \frac{\sqrt{10}}{4}$, hence we obtain

$$0 < T^4 \leq \frac{128000}{\iota(83 + 12\sqrt{5})},$$

where $\iota = 1 + \frac{\sqrt{5}}{10}$. Therefore, we can prove that (20) has at least one positive T-periodic solution in $\overline{\Omega}_2$, where $\Omega_2 = \{\mathbf{u} \in X| \ \frac{\sqrt{10}}{4} < u(k) < \frac{10\sqrt{10}+\sqrt{10}}{4}\}$.

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