Research on Definite Solution Problem of Partial Differential Equations under Two Kinds of Elastic Coefficients of External Boundary Conditions*

Xiaoxu Dong [1,†], Kaitao Cui [2], Wenjing Li [1], Zheng Zeng [1], Qun Liu [1], Shunchu Li [1]

[1] School of Science, Xihua University, Chengdu, 610039, China
[2] The Second Research Institute of Civil Aviation Administration of China, Chengdu, 610000, China

dongxiaoxu1028@163.com; cuikaitao@caacsri.com.com; lwj2986345062@163.com; zeng3830@163.com; liuqun6661211@163.com; lishunchu@163.com

Abstract: This paper studies the definite solution problems of partial differential equation (PDE) under two kinds of elastic coefficients (exponential function type, polynomial function type) of external boundary conditions. Then the definite solution problems are solved by Laplace transformation, the method of undetermined coefficients and Gaver-Stehfest numerical inversion equation. Firstly, the definite solution problems of linear PDE are transformed into the boundary value problems of linear differential equations in Laplace Space by Laplace transformation. Secondly, the solutions in Laplace space to the boundary value problems of linear differential equations are obtained through using the method of undetermined coefficients. Finally, the solution in real space to the definite solution of PDE under three kinds of elastic coefficients of external boundary conditions by using the Gaver-Stehfest numerical inversion equation. According to the above solution steps, this paper gives two different examples and obtains images of the solutions to the definite solution problems of PDE under two kinds of elastic coefficients of external boundary conditions. The introduction of two kinds of elastic coefficients of external boundary conditions not only expands the scope of the research on the definite solution of PDE, but also improves the matching degree between the theoretical model and the actual problems.

Keywords: Definite solution problems; Elastic outer boundary condition; Elastic coefficients; Exponential function type, Polynomial function type

1 Introduction

Partial differential equations and their systems as models for describing real-world phenomena often appear in the field of physical sciences and engineering. In practical applications, the solutions to the partial differential equation that satisfies some additional conditions, such as initial conditions and/or boundary conditions are a common concern. Therefore, the study of solutions to definite solution problems of partial differential equations

---

* This research was supported by Research on the coupling model of fracturing flowback in shale gas reservoir development, Talent introduction project of Xihua University (Z202068).

† Corresponding author.

Email address: dongxiaoxu1028@163.com (Xiaoxu Dong).

plays an important role in the study of physical phenomena. Domestic and foreign scholars have done a lot of research in this area. Nowadays, in order to better solve practical problems, the elastic outer boundary conditions derived from the concept of elasticity are gradually applied to the definite solution of PDE.

Elasticity reflects the sensitivity of changes in one variable to changes in another one. In 1920, Marshall [1] proposed firstly the concept of demand elasticity. After that, Woods and Sauro [2] gave formulas for elasticity and elasticity coefficient. Madenci and Dorduncu et al. [3] obtained the numerical solutions of linear and nonlinear partial differential equations (PDEs) by using differential operators. The boundary conditions of model they studied are Dirichlet and Neumann-characteristic boundary conditions. Yang and Goodyer et al. [4] introduced a new software tool for solving efficient solutions of parabolic linear and nonlinear partial differential equations. It is suitable for two different sample problems, which illustrate the flexibility and robustness of the tool. Abraham-Shrauner [5] used the Power Index Method to identify the possible exact analytical nonlinear solutions of any of two characteristics of hyperbolic functions or any of three characteristics of Jacobian elliptic functions of nonlinear partial differential equations, which are invariant under the independent variables translations. Benamou and Froese et al. [6] applied the classical “direct” technique involving Lie symmetries of partial differential equations to the boundary value problems of generalized Burgers equations with time-dependent viscosity coefficient and solved the particular sub-cases. However, this method has its limitations. Polyanin and Zhurov [7] have derived the nonlinear differential-difference equations of motion for viscous incompressible fluid with finite relaxation rate and given the corresponding exact solutions to these equations. The results obtained can be used to solve some hydrodynamic problems for the differential-difference model of viscous fluid. Lee and Manteuffel [8] proposed a natural framework for combining a Newton linearization and a FOSLL discretization approach for nonlinear partial differential equations. Yang and Deng et al. [9] proposed the Riccati-Bernoulli sub-ODE method to construct the exact traveling wave solutions, solitary wave solutions and spike wave solutions of nonlinear partial differential equations and given the Backlund transformation of the Riccati-Bernoulli equation. This method provides a powerful and simple mathematical tool for solving some nonlinear partial differential equations in mathematical physics. Quarteroni et al. [10] summarized the basic idea of domain decomposition method. This method is an iterative method for solving linear or nonlinear systems. Grande et al. [11] introduced and analyzed a new higher-order finite element method for elliptic partial differential equations on stationary smooth surfaces. The results of the numerical experiments demonstrate the high-order convergence of the method.

The study of reservoir seepage model is an important application of definite solution problems of differential equation. In recent years, the seepage theory has been continuously improved, but the theoretical and applied research on seepage problems under elastic boundary condition is still in primary stage. The study of seepage mechanics under elastic boundary condition has received more and more attention and has become a new direction in the development of modern seepage mechanics. Li et al [12-14] studied oil and gas flow characteristics in the reservoir with the elastic outer boundary, the fractal homogeneous reservoir with the elastic outer boundary and two-region composite reservoir seepage model under elasticity of the outer boundary. Zheng et al [15] studied the dual media shale gas reservoir with the elastic outer boundary. The Laplace transform is applied in the solution process of these four models. We find that the elastic coefficients were treated as constants during the Laplace transform of the model. However, in real elastic boundary reservoirs, the elastic coefficient is a function of location and time. Therefore, we study the solution of the definite solution problems of PDE whose elastic coefficients are functions.

Based on the above research, this paper studies the definite solution problems of linear partial differential equations (PDE) under two kinds of elastic coefficients (exponential function type, polynomial function type) of external boundary conditions. In section 2, we propose the definite solution problems of PDE under two kinds of elastic coefficients of external boundary conditions and solve definite solution problems by Laplace transformation, the method of undetermined coefficients and Gaver-Stehfest numerical inversion equation. In section 3, examples of the definite solution problems of PDE under two kinds of elastic coefficients of external boundary conditions are solved.

2 Solving the definite solution problem of PDE under two kinds of elastic coefficients of external boundary conditions

In this section, we first propose the following definite solution problem of linear PDE:

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2}+p(x)\dfrac{\partial y}{\partial x}+q(x)y(x,t)=r(x)\dfrac{\partial y}{\partial t} \quad (a\le x\le b, t>0) \\ y(x,0)=0 \\ \left[Ey(x,t)+F\dfrac{\partial y}{\partial x}\right]_{x=a}=g(t) \\ \left[\varepsilon(x,t)y(x,t)+H\dfrac{\partial y}{\partial x}\right]_{x=b}=0 \end{cases} \tag{1}$$

where D, E, F, H, a, b are constants, $E^2+F^2\neq 0$, $p(x)\in C^1[a,b]$, $q(x),r(x)\in C^2[a,b]$, $g(t)\in C^1[0,\infty]$ and $\varepsilon(x,t)$ is elastic coefficient.

Next, we study the definite solution problem (1) of PDEs under the following two kinds of elastic coefficient of external boundary conditions:

(1) Elastic coefficient of exponential function type

$$\varepsilon(x,t)=e^{h(x)t+f(x)} \tag{2}$$

(2) Elastic coefficient of polynomial function type

$$\varepsilon(x,t)=\sum_{k=0}^{n}a_k(x)t^k \tag{3}$$

Laplace transformation is performed on the above definite solution problem (1) of PDE, and $\bar{y}(x,s)=\int_0^\infty e^{-st}y(x,t)dt$, $\bar{g}(s)=\int_0^\infty e^{-st}g(t)dt$, then the definite solution problem (1) is transformed into the following boundary value problems:

(1) Elastic coefficient of exponential function type

$$\begin{cases}\dfrac{d^2\bar{y}}{dx^2}+p(x)\dfrac{d\bar{y}}{dx}+\left[q(x)-r(x)s\right]\bar{y}(x,s)=0 \quad (a\le x\le b)\\ \left[E\bar{y}(x,s)+F\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=a}=\bar{g}(s)\\ \left[e^{f(x)}\bar{y}(x,s-h(x))+H\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=b}=0\end{cases} \tag{4}$$

(2) Elastic coefficient of polynomial function type

$$\begin{cases}\dfrac{\partial^2\bar{y}}{\partial x^2}+p(x)\dfrac{\partial\bar{y}}{\partial x}+\left[q(x)-r(x)s\right]\bar{y}(x,s)=0 \quad (a\le x\le b)\\ \left[E\bar{y}(x,s)+F\dfrac{\partial\bar{y}(x,s)}{dx}\right]_{x=a}=\bar{g}(s)\\ \left[\sum_{k=0}^{n}(-1)^k a_k(x)\dfrac{\partial^k\bar{y}(x,s)}{\partial s^k}+H\dfrac{\partial\bar{y}(x,s)}{\partial x}\right]_{x=b}=0\end{cases} \tag{5}$$

where s is Laplace space variable.

Theorem 1: Under the premise of existence and uniqueness of solution to boundary value problem (5), the solution to the definite solution problem (4) with elastic coefficient of exponential function type of external boundary condition is

$$\bar{y}(x,s)=\bar{g}(s)\cdot\frac{G_1(x,s)}{EG_1(a,s)+F} \tag{6}$$

where

$$G_1(x,s)=\frac{e^{f(b)}\Psi_{0,0}(x,b,s,h(b))+H\Psi_{0,1}(x,b,s,0)}{e^{f(b)}\Psi_{1,0}(a,b,s,h(b))+H\Psi_{1,1}(a,b,s,0)} \tag{7}$$

$$\begin{aligned}\Psi_{m,n}(x,\xi,\eta(s),\tau)=&\frac{d^m y_1(x,\eta(s))}{dx^m}\frac{d^n y_2(\xi,\eta(s-\tau))}{d\eta^n}\\&-\frac{d^n y_1(\xi,\eta(s-\tau))}{d\eta^n}\frac{d^m y_2(x,\eta(s))}{dx^m}\quad (m,n\in Z^+)\end{aligned} \tag{8}$$

and $y_1(x,s)$, $y_2(x,s)$ are two linearly independent solutions to governing equation $\dfrac{d^2\bar{y}}{dx^2}+p(x)\dfrac{d\bar{y}}{dx}+\left[q(x)-r(x)s\right]\bar{y}(x,s)=0$ of the definite solution problem (4).

Proof: The general solution to governing equation of the definite solution problem (4) is

$$\bar{y}(x,s)=A_1y_1(x,s)+A_2y_2(x,s) \tag{9}$$

Substituting equation (9) into the boundary conditions of the definite solution problem (4), then

$$A_1\left[Ey_1(x,s)+F\frac{dy_1(x,s)}{dx}\right]_{x=a}+A_2\left[Ey_2(x,s)+F\frac{dy_2(x,s)}{dx}\right]_{x=a}=\bar{g}(s) \tag{10}$$

$$A_1\left[e^{f(x)}y_1(x,s-h(x))+H\frac{dy_1(x,s)}{dx}\right]_{x=b}+A_2\left[e^{f(x)}y_2(x,s-h(x))+H\frac{dy_2(x,s)}{dx}\right]_{x=b}=0 \tag{11}$$

Equations (10) and (11) are a system of linear equations for A_1 and A_2, whose coefficient determinant is

$$\begin{aligned}\Delta&=\begin{vmatrix}\left[Ey_1(x,s)+F\frac{dy_1(x,s)}{dx}\right]_{x=a} & \left[Ey_2(x,s)+F\frac{dy_2(x,s)}{dx}\right]_{x=a}\\ \left[e^{f(x)}y_1(x,s-h(x))+H\frac{dy_1(x,s)}{dx}\right]_{x=b} & \left[e^{f(x)}y_2(x,s-h(x))+H\frac{dy_2(x,s)}{dx}\right]_{x=b}\end{vmatrix}\\ &=E\left[e^{f(b)}\Psi_{0,0}(a,b,s,h(b))+H\Psi_{0,1}(a,b,s,0)\right]\\ &\quad+F\left[e^{f(b)}\Psi_{1,0}(a,b,s,h(b))+H\Psi_{1,1}(a,b,s,0)\right]\end{aligned} \tag{12}$$

Using Cramer's law, then

$$\begin{aligned}A_1&=\frac{1}{\Delta}\begin{vmatrix}\bar{g}(s) & \left[Ey_2(x,s)+F\frac{dy_2(x,s)}{dx}\right]_{x=a}\\ 0 & \left[e^{f(x)}y_2(x,s-h(x))+H\frac{dy_2(x,s)}{dx}\right]_{x=b}\end{vmatrix}\\ &=\frac{\bar{g}(s)}{\Delta}\left[e^{f(x)}y_2(x,s-h(x))+H\frac{dy_2(x,s)}{dx}\right]_{x=b}\end{aligned} \tag{13}$$

$$\begin{aligned}A_2&=\frac{1}{\Delta}\begin{vmatrix}\left[Ey_1(x,s)+F\frac{dy_1(x,s)}{dx}\right]_{x=a} & \bar{g}(s)\\ \left[e^{f(x)}y_1(x,s-h(x))+H\frac{dy_1(x,s)}{dx}\right]_{x=b} & 0\end{vmatrix}\\ &=-\frac{\bar{g}(s)}{\Delta}\left[e^{f(x)}y_1(x,s-h(x))+H\frac{dy_1(x,s)}{dx}\right]_{x=b}\end{aligned} \tag{14}$$

Substituting equations (13) and (14) into equation (9), then the solution to the definite solution problem (5) is equation (6).

Theorem 2: Under the premise of existence and uniqueness of solution to boundary value problem (5), the solution to the definite solution problem (5) with elastic coefficient of polynomial function type of external boundary condition is

$$\bar{y}(x,s)=\bar{g}(s)\cdot\frac{G_2(x,s)}{EG_2(a,s)+F} \tag{15}$$

where

$$G_2(x,s)=\frac{\sum_{k=0}^{n}(-1)^k a_k(b)\varphi_{0,0}^{k}(x,b,s)+H\varphi_{0,1}^{0}(x,b,s)}{\sum_{k=0}^{n}(-1)^k a_k(b)\varphi_{1,0}^{k}(a,b,s)+H\varphi_{1,1}^{0}(a,b,s)} \tag{16}$$

$$\begin{aligned}\varphi_{p,q}^{0}(x,\xi,\eta(s))=&\frac{\partial^p y_1(x,\eta(s))}{\partial x^p}\frac{\partial^q y_2(\xi,\eta(s))}{\partial \xi^q}\\&-\frac{\partial^q y_1(\xi,\eta(s))}{\partial \xi^q}\frac{\partial^p y_2(x,\eta(s))}{\partial x^p}\quad (p,q\in Z^+)\end{aligned} \tag{17}$$

$$\begin{aligned}\varphi_{p,0}^{m}(x,\xi,\eta(s))=&\frac{\partial^p y_1(x,\eta(s))}{\partial x^p}\frac{\partial^m y_2(\xi,\eta(s))}{\partial s^m}\\&-\frac{\partial^m y_1(\xi,\eta(s))}{\partial s^m}\frac{\partial^p y_2(x,\eta(s))}{\partial x^p}\quad (p,m\in Z^+)\end{aligned} \tag{18}$$

and $y_1(x,s)$, $y_2(x,s)$ are two linearly independent solutions to governing equation $\frac{d^2\bar{y}}{dx^2}+p(x)\frac{d\bar{y}}{dx}+\left[q(x)-r(x)s\right]\bar{y}(x,s)=0$ of the definite solution problem (5).

Proof: The general solution to governing equation of the definite solution problem (5) is

$$\bar{y}(x,s)=B_1y_1(x,s)+B_2y_2(x,s) \tag{19}$$

Substituting equation (19) into the boundary conditions of the definite solution problem (5), then

$$B_1\left[Ey_1(x,s)+F\frac{\partial y_1(x,s)}{\partial x}\right]_{x=a}+B_2\left[Ey_2(x,s)+F\frac{\partial y_2(x,s)}{\partial x}\right]_{x=a}=\bar{g}(s) \tag{20}$$

$$\begin{aligned}&B_1\left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_1(x,s)}{\partial s^k}+H\frac{\partial y_1(x,s)}{\partial x}\right]_{x=b}\\&\quad+B_2\left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_2(x,s)}{\partial s^k}+H\frac{\partial y_2(x,s)}{\partial x}\right]_{x=b}=0\end{aligned} \tag{21}$$

Equations (20) and (21) are a system of linear equations for B_1 and B_2, whose coefficient determinant is

$$\begin{aligned}\Delta&=\begin{vmatrix}\left[Ey_1(x,s)+F\frac{\partial y_1(x,s)}{\partial x}\right]_{x=a} & \left[Ey_2(x,s)+F\frac{\partial y_2(x,s)}{\partial x}\right]_{x=a}\\ \left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_1}{\partial s^k}+H\frac{\partial y_1(x,s)}{\partial x}\right]_{x=b} & \left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_2}{\partial s^k}+H\frac{\partial y_2(x,s)}{\partial x}\right]_{x=b}\end{vmatrix}\\&=E\left[\sum_{k=0}^{n}(-1)^k a_k(b)\varphi_{0,0}^{k}(a,b,s)+H\varphi_{0,1}^{0}(a,b,s)\right]\\&\quad+F\left[\sum_{k=0}^{n}(-1)^k a_k(b)\varphi_{1,0}^{k}(a,b,s)+H\varphi_{1,1}^{0}(a,b,s)\right]\end{aligned} \tag{22}$$

Using Cramer's law, then

$$B_1=\frac{1}{\Delta}\begin{vmatrix}\bar{g}(s) & \left[Ey_2(x,s)+F\frac{\partial y_2(x,s)}{\partial x}\right]_{x=a}\\ 0 & \left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_2(x,s)}{\partial s^k}+H\frac{\partial y_2(x,s)}{\partial x}\right]_{x=b}\end{vmatrix}$$
$$=\frac{\bar{g}(s)}{\Delta}\left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_2(x,s)}{\partial s^k}+H\frac{\partial y_2(x,s)}{\partial x}\right]_{x=b} \tag{23}$$

$$B_2=\frac{1}{\Delta}\begin{vmatrix}\left[Ey_1(x,s)+F\frac{\partial y_1(x,s)}{\partial x}\right]_{x=a} & \bar{g}(s)\\ \left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_1(x,s)}{\partial s^k}+H\frac{\partial y_1(x,s)}{\partial x}\right]_{x=b} & 0\end{vmatrix}$$
$$=-\frac{\bar{g}(s)}{\Delta}\left[\sum_{k=0}^{n}(-1)^k a_k(x)\frac{\partial^k y_1(x,s)}{\partial s^k}+H\frac{\partial y_1(x,s)}{\partial x}\right]_{x=b} \tag{24}$$

Substituting equations (23) and (24) into equation (19), then the solution to the definite solution problem (5) is equation (15).

Finally, using the Gaver-Stehfest numerical inversion equation [16] to the equations (6) and (15), the real-space solutions to the definite solution problem (1) under two kinds of elastic coefficients of external boundary conditions are:

(1) Elastic coefficient of exponential function type

$$y(x,t)=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\bar{y}\left(x,\frac{j\ln 2}{t}\right)$$
$$=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\left\{\bar{g}\left(\frac{j\ln 2}{t}\right)\frac{G_1\left(x,\frac{j\ln 2}{t}\right)}{EG_1\left(a,\frac{j\ln 2}{t}\right)+F}\right\} \tag{25}$$

(2) Elastic coefficient of polynomial function type

$$y(x,t)=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\bar{y}\left(x,\frac{j\ln 2}{t}\right)$$
$$=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\left\{\bar{g}\left(\frac{j\ln 2}{t}\right)\frac{G_2\left(x,\frac{j\ln 2}{t}\right)}{EG_2\left(a,\frac{j\ln 2}{t}\right)+F}\right\} \tag{26}$$

where

$$V_j=(-1)^{\frac{N}{2}+j}\sum_{k=\left[\frac{j+1}{2}\right]}^{\min\left(j,\frac{N}{2}\right)}\frac{k^{\frac{N}{2}}(2k+1)!}{\left(\frac{N}{2}-k+1\right)!k!(k+1)!(j-k+1)!(2k-j+1)!}$$

3 Case studies

Case 1: The following definite solution problem with elastic coefficient of exponential function type of external boundary condition is solved.

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2}+\dfrac{1}{x}\dfrac{\partial y}{\partial x}-\dfrac{v^2}{x^2}y=u^2\dfrac{\partial y}{\partial t} \quad (a\le x\le b, t>0) \\ y(x,0)=0 \\ \left[Ey(x,t)+F\dfrac{\partial y}{\partial x}\right]_{x=a}=D \\ \left[e^{(\gamma x+\varsigma)t+\alpha x+\beta}y(x,t)+H\dfrac{\partial y}{\partial x}\right]_{x=b}=0 \end{cases} \tag{27}$$

Laplace transformation is performed on the above definite solution problem (27) of PDE, and $\bar{y}(x,s)=\int_0^{\infty}e^{-st}y(x,t)dt$, then the definite solution problem (27) is transformed into the following boundary value problem:

$$\begin{cases} \dfrac{d^2\bar{y}}{dx^2}+\dfrac{1}{x}\dfrac{d\bar{y}}{dx}-\left(\mu^2 s+\dfrac{v^2}{x^2}\right)\bar{y}(x,s)=0 \quad (a\le x\le b) \\ \left[E\bar{y}(x,s)+F\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=a}=\dfrac{D}{s} \\ \left[e^{\alpha x+\beta}\bar{y}(x,s-\gamma x-\varsigma)+H\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=b}=0 \end{cases} \tag{28}$$

$I_v(\mu\sqrt{s}x)$ and $K_v(\mu\sqrt{s}x)$ are two linearly independent solutions of differential equations $\dfrac{d^2\bar{y}}{dx^2}+\dfrac{1}{x}\dfrac{d\bar{y}}{dx}-\left(\mu^2 s+\dfrac{v^2}{x^2}\right)\bar{y}(x,s)=0$. Here $I_v(\bullet)$ and $K_v(\bullet)$ are respectively the first and the second class of modified Bessel functions of order v [17].

According to Theorem 1, the solution to the definite solution problem (28) with elastic coefficient of exponential function type of external boundary condition is

$$\bar{y}(x,s)=\frac{D}{s}\cdot\frac{G_1(x,s)}{EG_1(a,s)+F} \tag{29}$$

where

$$G_1(x,s)=\frac{e^{\alpha b+\beta}\Psi_{0,0}\left(x,b,\mu\sqrt{s},\gamma b+\varsigma\right)+H\Psi_{0,1}\left(x,b,\mu\sqrt{s},0\right)}{e^{\alpha b+\beta}\Psi_{1,0}\left(a,b,\mu\sqrt{s},\gamma b+\varsigma\right)+H\Psi_{1,1}\left(a,b,\mu\sqrt{s},0\right)} \tag{30}$$

$$\Psi_{0,0}(x,\xi,\eta(s),\tau)=\psi_{v,v}(x,\xi,\eta(s),\tau) \tag{31}$$

$$\begin{aligned}\Psi_{0,1}(x,\xi,\eta(s),\tau)&=\frac{\partial\Psi_{0,0}(x,\xi,\eta(s),\tau)}{\partial\xi}=\frac{\partial\psi_{v,v}(x,\xi,\eta(s),\tau)}{\partial\xi}\\&=\frac{v}{\xi}\psi_{v,v}(x,\xi,\eta(s),\tau)-\eta(s-\tau)\psi_{v,v+1}(x,\xi,\eta(s),\tau)\end{aligned} \tag{32}$$

$$\Psi_{1,0}\left(x,\xi,\eta(s),\tau\right)=\frac{\partial\Psi_{0,0}\left(x,\xi,\eta(s),\tau\right)}{\partial x}=\frac{\partial\psi_{v,v}\left(x,\xi,\eta(s),\tau\right)}{\partial x}$$
$$=\frac{v}{x}\psi_{v,v}\left(x,\xi,\eta(s),\tau\right)+\eta(s)\psi_{v+1,v}\left(x,\xi,\eta(s),\tau\right) \tag{33}$$

$$\Psi_{1,1}\left(x,\xi,\eta(s),\tau\right)=\frac{\partial\ \Psi_{0,0}\left(x,\xi,\eta(s),\tau\right)}{\partial x\partial\xi}=\frac{\partial^2\psi_{v,v}\left(x,\xi,\eta(s),\tau\right)}{\partial x\partial x\partial\xi}$$
$$=\frac{v^2}{x\xi}\psi_{v,v}\left(x,\xi,\eta(s),\tau\right)-\frac{v\eta(s-\tau)}{x}\psi_{v,v+1}\left(x,\xi,\eta(s),\tau\right) \tag{34}$$
$$+\frac{v\eta(s)}{\xi}\psi_{v+1,v}\left(x,\xi,\eta(s),\tau\right)-\eta(s)\eta(s-\tau)\psi_{v+1,v+1}\left(x,\xi,\eta(s),\tau\right)$$

$$\psi_{m,n}\left(x,\xi,\eta(s),\tau\right)=I_m\left(x\eta(s)\right)K_n\left(\xi\eta(s-\tau)\right)+(-1)^{m+n-1}I_n\left(\xi\eta(s-\tau)\right)K_m\left(x\eta(s)\right) \tag{35}$$

Using the Gaver-Stehfest numerical inversion equation [16] to the equation (29), the real-space solutions to the definite solution problem (27) is:

$$y(x,t)=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\bar{y}\left(x,\frac{j\ln 2}{t}\right)$$
$$=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\left\{\frac{Dj}{j\ln 2}\frac{G_1\left(x,\frac{j\ln 2}{t}\right)}{EG_1\left(a,\frac{j\ln 2}{t}\right)+F}\right\} \tag{36}$$

The image of the solution of the definite solution problem (27) is shown in Figure 1.

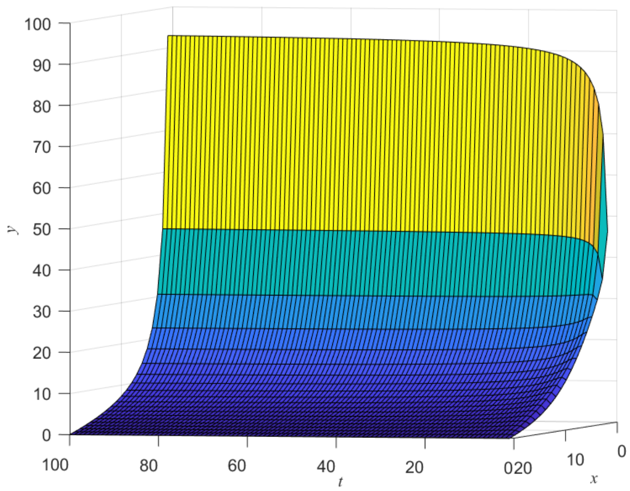

Figure 1 Image of the solution of the definite solution problem (27)
($a=1$, $b=20$, $\mu=0.1$, $v=1$, $\alpha=2$, $\beta=2$, $\gamma=0.1$, $\varsigma=-2$, $D=10$, $E=3$, $F=2$, $H=5$, $N=10$)

Case 2: The following definite solution problem with elastic coefficient of polynomial function type of external boundary condition is solved.

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2}+\dfrac{1}{x}\dfrac{\partial y}{\partial x}-\dfrac{v^2}{x^2}y=u^2\dfrac{\partial y}{\partial t} \quad (a\le x\le b, t>0) \\ y(x,0)=0 \\ \left[Ey(x,t)+F\dfrac{\partial y}{\partial x}\right]_{x=a}=D \\ \left[\left[(px+q)t+lx^2+kx+c\right]y(x,t)+N\dfrac{\partial y}{\partial x}\right]_{x=b}=0 \end{cases} \tag{37}$$

Laplace transformation is performed on the above definite solution problem (37) of PDE, and $\bar{y}(x,s)=\int_0^\infty e^{-st}y(x,t)dt$, then the definite solution problem (37) is transformed into the following boundary value problem:

$$\begin{cases} \dfrac{d^2\bar{y}}{dx^2}+\dfrac{1}{x}\dfrac{d\bar{y}}{dx}-\left(\mu^2 s+\dfrac{v^2}{x^2}\right)\bar{y}(x,s)=0 \quad (a\le x\le b) \\ \left[E\bar{y}(x,s)+F\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=a}=\dfrac{D}{s} \\ \left[e^{\alpha x+\beta}\bar{y}(x,s-\gamma x-\varsigma)+N\dfrac{d\bar{y}(x,s)}{dx}\right]_{x=b}=0 \\ \left[(lx^2+kx+c)\bar{y}(x,s)-(px+q)\dfrac{\partial\bar{y}(x,s)}{\partial s}+N\dfrac{\partial\bar{y}(x,s)}{\partial x}\right]_{x=b}=0 \end{cases} \tag{38}$$

According to Theorem 2, the solution to the definite solution problem (38) with elastic coefficient of exponential function type of external boundary condition is

$$\bar{y}(x,s)=\frac{D}{s}\cdot\frac{G_2(x,s)}{EG_2(a,s)+F} \tag{39}$$

where

$$G_2(x,s)=\frac{(lb^2+kb+c)\varphi_{0,0}^0(x,b,\mu\sqrt{s})-(pb+q)\varphi_{0,0}^1(x,b,\mu\sqrt{s})+N\varphi_{0,1}^0(x,b,\mu\sqrt{s})}{(lb^2+kb+c)\varphi_{1,0}^0(a,b,\mu\sqrt{s})-(pb+q)\varphi_{1,0}^1(a,b,\mu\sqrt{s})+N\varphi_{1,1}^0(a,b,\mu\sqrt{s})} \tag{40}$$

$$\varphi_{0,0}^0(x,\xi,\eta(s))=\psi_{v,v}(x,\xi,\eta(s),0) \tag{41}$$

$$\varphi_{0,1}^0(x,\xi,\eta(s))=\frac{v}{\xi}\psi_{v,v}(x,\xi,\eta(s),0)-\eta(s)\psi_{v,v+1}(x,\xi,\eta(s),0) \tag{42}$$

$$\varphi_{1,0}^0(x,\xi,\eta(s))=\frac{v}{x}\psi_{v,v}(x,\xi,\eta(s),0)+\eta(s)\psi_{v+1,v}(x,\xi,\eta(s),0) \tag{43}$$

$$\begin{aligned}\varphi_{1,1}^0(x,\xi,\eta(s))=&\frac{v^2}{x\xi}\psi_{v,v}(x,\xi,\eta(s),0)-\frac{v}{x}\eta(s)\psi_{v,v+1}(x,\xi,\eta(s),0)\\&+\frac{v}{\xi}\eta(s)\psi_{v+1,v}(x,\xi,\eta(s),0)-\eta^2(s)\psi_{v+1,v+1}(x,\xi,\eta(s),0)\end{aligned} \tag{44}$$

$$\varphi_{0,0}^1(x,\xi,\eta(s))=\frac{d\eta(s)}{ds}\left[\frac{v}{\eta(s)}\psi_{v,v}(x,\xi,\eta(s),0)-\xi\psi_{v,v+1}(x,\xi,\eta(s),0)\right] \tag{45}$$

$$\varphi_{1,0}^{1}\left(x,\xi,\eta(s)\right)=\frac{d\eta(s)}{ds}\left[\frac{v^{2}}{x\eta(s)}\psi_{v,v}\left(x,\xi,\eta(s),0\right)-\frac{v\xi}{x}\psi_{v,v+1}\left(x,\xi,\eta(s),0\right)\right.$$
$$\left.+v\psi_{v+1,v}\left(x,\xi,\eta(s),0\right)-\eta(s)\xi\psi_{v+1,v+1}\left(x,\xi,\eta(s),0\right)\right] \tag{46}$$

$\psi_{m,n}\left(x,\xi,\eta(s),\tau\right)$ is equation (35).

Using the Gaver-Stehfest numerical inversion equation [16] to the equation(39), the real-space solutions to the definite solution problem (37) is:

$$y(x,t)=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\bar{y}\left(x,\frac{j\ln 2}{t}\right)$$
$$=\frac{\ln 2}{t}\sum_{j=1}^{N}V_j\left\{\frac{Dj}{j\ln 2}\frac{G_2\left(x,\frac{j\ln 2}{t}\right)}{EG_2\left(a,\frac{j\ln 2}{t}\right)+F}\right\} \tag{47}$$

The image of the solution of the definite solution problem (37)is shown in Figure 2.

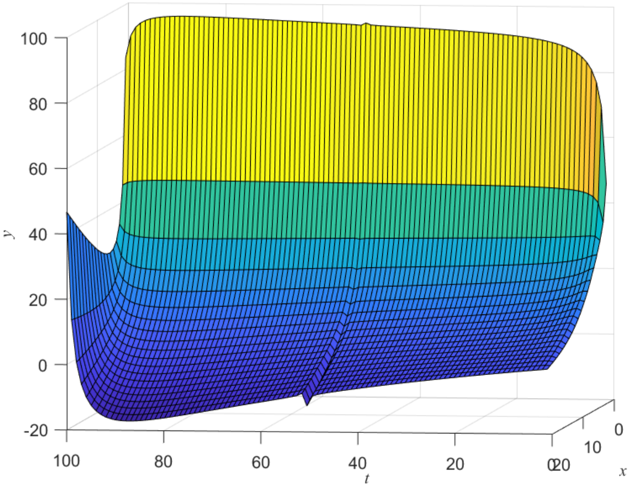

Figure 2 Image of the solution of the definite solution problem (37)

($a=1$, $b=20$, $\mu=0.1$, $v=1$, $k=5$, $l=1.85$, $c=1$, $p=1$, $q=2.15$ $D=10$, $E=3$, $F=2$, $H=5$, $N=10$)

4. Conclusions

1. The definite solution problems are solved by Laplace transformation, the method of undetermined coefficients and Gaver-Stehfest numerical inversion equation, and the form of the real space solution are relatively simple, which is beneficial to the programming of curve analysis. .
2. The elastic outer boundary condition established in this paper regards the elastic coefficient as a function of space variables and time variables, which makes the established theoretical model more suitable for practical problems. With the help of the elastic outer boundary, we can better understand and characterize the generalized state of the outer boundary.

References

1. Marshall A. Principles of economics[M]. London Macmillan and Co. 1920.102-116.
2. Woods JH, Sauro HM. Elasticities in metabolic control analysis: algebraic derivation of simplified expressions. Cabios[J], 1997, 13(2), 123-30.
3. Madenci E, Dorduncu M, Barut A, et al. Numerical solution of linear and nonlinear partial differential equations using the peridynamic differential operator. Numerical Methods for Partial Differential Equations, 2017, 33.
4. Yang FW, Goodyer CE, Hubbard ME, et al. An optimally efficient technique for the solution of systems of nonlinear parabolic partial differential equations. Advances in Engineering Software, 2017, 103.
5. Abraham-Shrauner B. Exact solutions of nonlinear partial differential equations. Discrete and Continuous Dynamical Systems - Series S, 2017, 11(4), 577-582.
6. Benamou JD, Froese BD, Oberman AM. Numerical solution of the optimal transportation problem via viscosity solutions for the Monge-Ampere equation. Journal of Computational Physics, 2012, 260(2), 107-126.
7. Polyanin AD, Zhurov AI. Exact solutions of non-linear differential-difference equations of a viscous fluid with finite relaxation time. International Journal of Non-Linear Mechanics, 2013, 57(4), 116-122.
8. Lee E, Manteuffel TA. FOSLL for nonlinear partial differential equations. Siam Journal on Scientific Computing, 2015, 37(5), S503-S525.
9. Yang XF, Deng ZC, Wei Y. A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application. Advances in Difference Equations, 2015, 2015(1), 117.
10. Quarteroni A, Valli A. Domain Decomposition Methods for Partial Differential Equations[J]. Mathematics of Computation, 2019.
11. Grande J, Reusken A. A Higher Order Finite Element Method for Partial Differential Equations on Surfaces[J]. SIAM Journal on Numerical Analysis, 2016.
12. Li SC, Zhao CC, Zheng PS, Gui QM. Analysis of oil and gas flow characteristics in the reservoir with the elastic outer boundary[J]. Journal of Petroleum science and Engineering, 2018, 175:280-285.
13. Li SC, Guo H, Zheng PS, Dong XX, Zhao CC. The elastic boundary value problem of extended modified Bessel equation and its application in fractal homogeneous reservoir[J]. Computational and Applied Mathematics, 2020, 39(2):1-16.
14. Li SC, He Q, Dong XX, Xia X, Gui QM. The elasticity of the outer boundary and the solution of two-region composite reservoir seepage model. https://doi.org/10.1080/10916466.2022.2048015
15. Zheng PS, Zheng YS, Li SC, Leng LH, Xia X. Analysis of flow characteristics of the dual media shale gas reservoir with the elastic outer boundary[J]. SN Applied Sciences, 2021, 3(10).
16. Stehfest H. Remark on algorithm 368: Numerical inversion of Laplace transforms. Communications of the ACM, 1970, 13(10):624.
17. Liu SS, Liu SD. Special Function [M]. Beijing: China Meteorological Press, 2002.