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Stability analysis of solutions for fractional Langevin equation involving Hadamard-Caputo derivatives with nonlocal integral and nonperiodic boundary conditions

Amita Devi[1] | Anoop Kumar*[1] | Thabet Abdeljawad*[2,3,4] | Aziz Khan[2]

[1]Department of Mathematics and Statistics. Central University of Punjab, 151001, Bathinda,Punjab, India.
[2]Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, 11586 Riyadh, Saudi Arabia
[3]Department of Medical Research,, China Medical University, 40402, Taichung, Taiwan.
[4]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

Correspondence
*Corresponding authors name, Email: anoopmath85@gmail.com, Email: tabdeljawad@psu.edu.sa

Summary

In this manuscript, we study fractional Langevin equations(FLEs) involving Hadamard-Caputo's derivative of distinct orders associated with nonlocal integral and nonperiodic boundary conditions. The Hyres-Ulam (HU) stability, existence and uniqueness(EU) of solutions are established for proposed equations. Our prospective is based on the Hadamard-Caputo's derivatives and implementation of Krasnoselskii's fixed point theorem and Banach contraction mapping principle. An application is offered to smooth the understanding of the theoretical results.

KEYWORDS:
Hadamard-Caputo's derivative, fixed point theorems, Fractional Langevin equation, Hyres-Ulam stability, Fixed point theorems, nonlocal conditions.

MSC CLASSIFICATION
26A33; 34A08; 45M10.

1 | INTRODUCTION

Fractional derivatives generalize the notion of integer order derivatives and integrals to arbitrary order. The theory of fractional differential equations(FDEs) is an increasing filed of research that has extensive applications in numerous scientific areas. Recently, it has been observed that FDEs characterizes many irregular phenomena in various areas such as viscoelasticity, chemistry, physics, biology, information processing system networking, picture processing, control hypothesis, fluid dynamics, speculation, ground water problems , aerodynamics and hydrodynamics. In addition, FDEs gives the excellent description of anomalous diffusion processes in some complex media and memory properties of several materials. During the last few decades, FDEs gained considerable attention due to their flexible and realistic behavior. For more application of FDEs, reader can see [1,2,3,4]. The Langevin equation expound the progress of some physical phenomena in changing environment. The Brownian motion of a particles is firstly expressed by the ordinary form of Langevin equation [5]. The ordinary form of Langevin equations provides the insufficient results for some fractal disorder regions. Therefore to describe the dynamical behavior of some fractal and inherited properties FLEs was introduced by Kubo [6,7]. Recently, FLEs with various type of boundary conditions has been studied extensively [8,9,10,11,12,13,14,15]. Devi et al. discussed the stability and existence result for the FLEs involving integral conditions via fixed point technique [16]. Fazli and Nieto analysis the FLEs involving nonlocal conditions by using technique of mixed monotone mappings. [17]. Baghani et al. studied the EU of results for the coupled system of FLEs associated with anti periodic conditions by using lower and upper solution mathod [18]. The HU stability defines that an exact solution exists very near to the approximate solution for FDEs. The HU stability and EU of results for FLEs is discussed by Wang and Li. [19]. Mohammed et al. discussed the EU and stability in the sense of Ulam-Hyres for coupled system of FLEs with nonperiodic boundary conditions involving

Caputo-Hadamard derivative[20]. Devi et al. analyses the HU stability and existence for positive solution for general FDEs via fixed point technique[21]. More about stability theory, can see[22,23,24,25,26,27,28].

Inspired from above work, in this manuscript we discuss the HU stability and existence and uniqueness results for FLEs with Hadamarad-Caputo fractional derivatives involving nonperiodic and fractional differential conditions:

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}^{\sigma}({}^{\mathcal{CH}}\mathcal{D}^{\epsilon} + \mu)z(t) = \Psi^*(t, z(t)), \qquad t \in [1, e], \\ z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 0, \quad {}^{\mathcal{CH}}\mathcal{D}^{\epsilon} z(1) = I^{\delta} z(\gamma), \quad {}^{\mathcal{CH}}\mathcal{D}^{\epsilon} z(e) + \kappa z(e) = 0, \end{cases} \tag{1}$$

where $\kappa \neq \mu$ and ${}^{\mathcal{CH}}\mathcal{D}^{\sigma}, {}^{\mathcal{CH}}\mathcal{D}^{\epsilon}$ denotes the Hadamard-Caputo derivatives of fractional order σ and ϵ respectively $\mu, \delta > 0, 2 < \epsilon \leq 3,\ 0 < \sigma \leq 2,\ 1 < \gamma < e, \kappa \in \mathbb{R}$. The article is outlined as follows. Some essential lemmas and basic preliminaries results are serves in section 2. 3^{rd} section contains main result. HU stability is reported in the next section. In last section, the obtained results are illustrated by an application.

2 | PRELIMINARIES

We define the term $\mathcal{AC}^{m}_{\gamma}[a, b]$ which contains all absolutely-continuous functions ψ^* and having γ^{m-1}-derivative absolutely continuous on $[a, b]$ $\left(\gamma^{m-1}\psi^* \in \mathcal{AC}([a, b], \mathbb{R})\right)$.

Definition 2.1.[2] Let the function $\psi^* : [1, \infty) \longrightarrow \mathbb{R}$, is integrable and continuous then Hadamard fractional integral of order $p \in \mathbb{R}$ is defined as

$$I^{p}\psi^*(t) = \frac{1}{\Gamma(p)} \int_{1}^{t} \left(\log \frac{t}{s}\right)^{p-1} \frac{\psi^*(s)}{s} ds, \qquad p > 0, \tag{2}$$

provided the integral exists.

Definition 2.2.[2] Let $p \geq 0,\ \ n = p+1$ and $\psi^* : [1, \infty) \longrightarrow \mathbb{R}$. The pth-order Hadamard fractional derivative of ψ is defined by

$${}^{\mathcal{H}}\mathcal{D}^{p}\psi^*(t) = \frac{1}{\Gamma(n-p)} \left(t\frac{d}{dt}\right) \int_{1}^{t} \left(\log \frac{t}{s}\right)^{n-p-1} \frac{\psi^*(s)}{s} ds, \tag{3}$$

provided the right-hand side exists.

Definition 2.3.[31] Let $p \geq 0,\ \ n = p+1$ and the pth-order Caputo-Hadamard fractional derivative of $\psi^* \in \mathcal{AC}^{m}_{\gamma}[a, b]$, where $0 \leq a < \chi < b < \infty$ is defined by

$${}^{\mathcal{CH}}\mathcal{D}^{p}\psi^*(t) = \frac{1}{\Gamma(n-p)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{n-p-1} \left[s\frac{d}{ds}\psi^*(s)\right] \frac{ds}{s}, \tag{4}$$

provided the right-hand side exists.

Lemma 2.4.[32] Let $p > 0,\ \ n = [R(p)] + 1$ and $R(q) > 0$. Then

$$I^{p}\left(\log \chi\right)^{q-1} = \frac{\Gamma q}{\Gamma(q+p)} \left(\log \chi\right)^{q+p-1},$$

and

$${}^{\mathcal{CH}}\mathcal{D}^{p}\left(\log \chi\right)^{q-1} = \frac{\Gamma q}{\Gamma(q-p)} \left(\log \chi\right)^{q-p-1}.$$

Lemma 2.5[32] Let $p > 0$ and $\chi \in C[1, \infty) \cap \mathbb{L}^1[1, \infty)$. Then, FDEs with Caputo-Hadamard derivative

$${}^{\mathcal{CH}}\mathcal{D}^{p}\chi(t) = 0,$$

has a solution

$$\chi(t) = \sum_{i=0}^{m-1} a_i \left(\log t\right)^{i},$$

also satisfies

$$^{\mathcal{CH}}\mathcal{D}^{p} I^{q}\chi(t) = I^{q-p} \quad q > p,$$
$$I^{p}\, {}^{\mathcal{CH}}\mathcal{D}^{p}\chi(t) = \chi(t) + \sum_{i=0}^{m-1} a_i\big(\log t\big)^{i},$$

where $a_i \in \mathbb{R}, i = 0, 1, 2, ..., m$ and $m - 1 < p < m$.

Lemma 2.6. The solution of the given problem (1) in the integral form is

$$\begin{aligned} z(t) = {} & \frac{1}{\Gamma(\sigma+\epsilon)}\int_1^t \left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1}\Psi^*(s,z(s))\frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)}\int_1^t \left(\log\frac{t}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} \\ & + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)+(\kappa-\mu)}\left[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)}\int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} - \frac{1}{\Gamma(\sigma)}\int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1}\Psi^*(s,z(s))\frac{ds}{s}\right. \\ & \left. - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)}\int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\Psi^*(s,z(s))\frac{ds}{s} - \frac{\Gamma(\epsilon+1)+(\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s}\right] \\ & + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s}. \end{aligned} \tag{5}$$

Proof. Applying the Hadamard fractional integral operater I^{σ} to (1) we get

$$({}^{\mathcal{CH}}\mathcal{D}^{\epsilon} + \mu)z(t) = I^{\sigma}\Psi^*(t, z(t)) + c_1\log t + c_0,$$

or

$$^{\mathcal{CH}}\mathcal{D}^{\epsilon} z(t) = I^{\sigma}\Psi^*(t, z(t)) - \mu z(t) + c_1\log t + c_0. \tag{6}$$

Again applying the Hadamard fractional integral operater integral operater I^{ϵ} to (6), we obtain

$$z(t) = I^{\sigma+\epsilon}\Psi^*(t, z(t)) - \mu I^{\epsilon} z(t) + c_0\frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)} + c_1\frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)} + c_2(\log t)^2 + c_3(\log t) + c_4, \tag{7}$$

where c_0, c_1, c_2, c_3, and c_4 are some real constants.
Using boundary conditions $z(0) = 0, z'(0) = 0, z''(0) = 0$, gives that $c_2 = 0, c_3 = 0$, and $c_4 = 0$.
$\Rightarrow$

$$z(t) = I^{\sigma+\epsilon}\Psi^*(t, z(t)) - \mu I^{\epsilon} z(t) + c_0\frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)} + c_1\frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)}. \tag{8}$$

Now from boundary conditions ${}^{\mathcal{CH}}\mathcal{D}^{\epsilon} z(1) = I^{\delta} z(\kappa)$ and ${}^{\mathcal{CH}}\mathcal{D}^{\epsilon} z(e) + \epsilon z(e) = 0$, we get

$$c_0 = I^{\delta} z(\text{fl}). \tag{9}$$

$$\begin{aligned} c_1 = {} & \frac{\Gamma(\epsilon+2)}{\Gamma(\epsilon+2)+(\kappa-\mu)}\left[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)}\int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} - \frac{1}{\Gamma(\sigma)}\int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1}\Psi^*(s,z(s))\frac{ds}{s}\right. \\ & \left. - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)}\int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\Psi^*(s,z(s))\frac{ds}{s} - \frac{\Gamma(\epsilon+1)+(\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s}\right], \end{aligned} \tag{10}$$

putting the values of c_0 and c_1 in (8), we have the solution of (1) in the integral form (5). This accomplishes the proof.

Theorem 2.7.[32] Let $\mathcal{S}$ is a nonempty convex, closed and bounded subset of a Banach space E. Let A_1, A_2 be the operators from $\mathcal{S}$ to E such that:
(i) $A_1\chi + A_2 y \in \mathcal{S}$ whenever $\chi, y \in \mathcal{S}$;
(ii) A_1 is continuous and compact;
(iii) A_2 is a contraction map.
Then there exists $z \in \mathcal{S}$ such that $z = A_1 z + A_2 z$.

3 | MAIN RESULTS

Let $\mathcal{V} = \mathcal{C}[1, e]$ is the Banach space of all continuous functions from $[1, e]$ to $\mathbb{R}$ with norm given by $\|\mathcal{V}\| = \sup_{t\in[1,e]} |z(t)|$. Let us define an operator $\mathcal{H} : \mathcal{V} \longrightarrow \mathcal{V}$, such that

$$
\begin{aligned}
(\mathcal{H}z)(t) = {} & \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s))\frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} \\
& + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)+(\kappa-\mu)} \left[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1} \Psi^*(s, z(s))\frac{ds}{s}\right. \\
& \left. - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s))\frac{ds}{s} - \frac{\Gamma(\epsilon+1)+(\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s}\right] \\
& + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s}.
\end{aligned} \tag{11}
$$

It is percived that the problem (1) has a solution iff operator $\mathcal{H}$ has a fixed point.

3.1 | Existence result

In this section, we establish the existence of solutions by using the fixed point technique and following assumptions:

- $(\mathcal{R}_1)$ $\Psi^*(t, z(t)) : [1, e] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.
- $(\mathcal{R}_2)$ $|\Psi^*(t, z(t))| \le \hbar(t)\ \ \forall\,(t, z) \in [1, e] \times \mathbb{R}$ and $\hbar(t) \in \mathcal{C}([1, e], \mathbb{R}^+)$.
- $(\mathcal{R}_3)$ $|\Psi^*(t, z) - \Psi^*(t, z_1)| \le \mathcal{L}(|z - z_1|)\ \ \forall\ t \in [1, e],\ z, z_1 \in \mathcal{V},\ L > 0.$
- $(\mathcal{R}_4)$ Take $\mathcal{M} = \sup_{t\in[1,e]} |\Psi^*(t, 0)| < \infty.$

Also we use the following notations:

$$
\begin{aligned}
\Lambda_1 = {} & \left\{ \frac{1}{\Gamma(\sigma+\epsilon+1)} + \frac{|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)} \right. \\
& \left. + \frac{1}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)} \right\},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\Lambda_2 = {} & \left\{ \frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)} + \frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)} \right. \\
& \left. + \frac{|\mu|}{\Gamma(\epsilon+1)} + \frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)} \right\},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\Lambda_3 = {} & \left\{ \frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)} \right. \\
& + \frac{\mathcal{L}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)} + \frac{\mathcal{L}|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)} \\
& \left. + \frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)} + \frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)} \right\}.
\end{aligned} \tag{14}
$$

Theorem 3.1. Let us assume $(\mathcal{R}_1)$, $(\mathcal{R}_2)$and $(\mathcal{R}_3)$ holds. Then the problem (1) has atleast one solution on $[1, e]$ if

$$
\Lambda_3 < 1, \tag{15}
$$

where Λ_3 is defined in (14).

Proof Let us take $\bar{r} \geq \frac{\Lambda_1 \|h\|}{(1-\Lambda_2)}$ and $\mathcal{B}_{\bar{r}} = \{z \in \mathcal{V} : \|z\| \leq \bar{r}\}$ is a closed ball.

Let the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{B}_{\bar{r}}$ are defined as

$$(\mathcal{H}_1 z)(t) = \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s)) \frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\epsilon-1} z(s) \frac{ds}{s},$$

$$\begin{aligned}
(\mathcal{H}_2 z)(t) = {} & \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s)) \frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\epsilon-1} z(s) \frac{ds}{s} \\
& + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2) + (\kappa-\mu)} \left[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\epsilon-1} z(s) \frac{ds}{s} - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \Psi^*(s, z(s)) \frac{ds}{s} \right. \\
& \left. - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s)) \frac{ds}{s} - \frac{\Gamma(\epsilon+1) + (\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log \frac{\gamma}{s}\right)^{\delta-1} z(s) \frac{ds}{s} \right] \\
& + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log \frac{\gamma}{s}\right)^{\delta-1} z(s) \frac{ds}{s}.
\end{aligned}$$

Noticed that $\mathcal{H}z = \mathcal{H}_1 z + \mathcal{H}_2 z$, $z \in \mathcal{B}_r$ on $[1, e]$.

We will establish the conditions related to Theorem 2.1 in below mentioned three steps:

1. To prove $\mathcal{H}z = \mathcal{H}_1 z + \mathcal{H}_2 z_1 \in \mathcal{B}_{\bar{r}}$.
   For any $z, z_1 \in \mathcal{B}_{\bar{r}}$

$$\begin{aligned}
\|\mathcal{H}_1 z + \mathcal{H}_2 z_1\| = {} & \sup_{t \in [1,e]} |\mathcal{H}_1 z(t) + \mathcal{H}_2 z_1(t)| \\
= {} & \sup_{t \in [1,e]} \left\{ \left| \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z(s)) \frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\epsilon-1} z(s) \frac{ds}{s} \right. \right. \\
& + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2) + (\kappa-\mu)} \left[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\epsilon-1} z_1(s) \frac{ds}{s} \right. \\
& - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \Psi^*(s, z_1(s)) \frac{ds}{s} - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s, z_1(s)) \frac{ds}{s} \\
& \left. \left. \left. - \frac{\Gamma(\epsilon+1) + (\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log \frac{\gamma}{s}\right)^{\delta-1} z_1(s) \frac{ds}{s} \right] + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log \frac{\gamma}{s}\right)^{\delta-1} z_1(s) \frac{ds}{s} \right| \right\} \\
\leq {} & \sup_{t \in [1,e]} \left\{ \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\sigma+\epsilon-1} \left|\Psi^*(s, z(s))\right| \frac{ds}{s} + \frac{|\mu|}{\Gamma(\epsilon)} \int_1^t \left(\log \frac{t}{s}\right)^{\epsilon-1} \left|z(s)\right| \frac{ds}{s} \right. \\
& + \frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2) + (\kappa-\mu)|} \left[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\epsilon-1} \left|z_1(s)\right| \frac{ds}{s} \right. \\
& + \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma-1} \left|\Psi^*(s, z_1(s))\right| \frac{ds}{s} + \frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log \frac{e}{s}\right)^{\sigma+\epsilon-1} \left|\Psi^*(s, z_1(s))\right| \frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z_1(s)\Big|\frac{ds}{s}\Bigg]\\
&+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z_1(s)\Big|\frac{ds}{s}\Bigg\}\\
\leq\ &\|h\|\sup_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1}\frac{ds}{s}\\
&+\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\Bigg[\frac{1}{\Gamma(\sigma)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{ds}{s}+\frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\frac{ds}{s}\Bigg]\Bigg\}\\
&+\|z\|\sup_{t\in[1,e]}\Bigg\{\frac{|\mu|}{\Gamma(\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\epsilon-1}\frac{ds}{s}\Bigg\}\\
&+\|z_1\|\sup_{t\in[1,e]}\Bigg\{\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\Bigg[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\epsilon-1}\frac{ds}{s}\\
&+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\frac{ds}{s}\Bigg]+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\frac{ds}{s}\Bigg\}\\
\leq\ &\|h\|\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon+1)}+\frac{|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)}+\frac{1}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)}\Bigg\}\\
&+r\Bigg\{\frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)}+\frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)}\\
&+\frac{|\mu|}{\Gamma(\epsilon+1)}+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)}\Bigg\}\\
\leq\ &\|h\|\Lambda_1+\Lambda_2\bar r\leq\bar r,
\end{aligned}
$$

which shows that $\mathcal{H}z=\mathcal{H}_1z+\mathcal{H}_2z_1\in\mathcal{B}_{\bar r}$.

2. Now we prove $\mathcal{H}_2$ is contraction map. For $z, z_1\in\mathcal{V}$,

$$
\begin{aligned}
\|\mathcal{H}_2z-\mathcal{H}_2z_1\| =\ &\sup_{t\in[1,e]}\Big|\mathcal{H}_2z(t)-\mathcal{H}_2z_1(t)\Big|\\
\leq\ &\sup_{t\in[1,e]}\Bigg\{\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\Bigg[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\epsilon-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\sigma)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\Bigg]\\
&+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Bigg\{ \frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)} \\
&\quad + \frac{\mathcal{L}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)} + \frac{\mathcal{L}|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)} \\
&\quad + \frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)} + \frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)} \Bigg\} \|z-z_1\| \\
&\leq \Lambda_3 \|z-z_1\|, \qquad \forall \;\; t\in[1,e],
\end{aligned}
$$

as $\Lambda_3 < 1$. Therefore $\mathcal{H}_2$ is contraction map.

3. In view of assumption $(\mathcal{R}_1)$, Ψ^* is continuous function on $t \in [1, e]$, therefore the operator $\mathcal{H}_1$ is continuous.
   To prove $\mathcal{H}_1$ is bounded uniformly on $\mathcal{B}_{\bar{r}}$.

$$
\begin{aligned}
\|\mathcal{H}_1 z\| &= \sup_{t\in[1,e]} |\mathcal{H}_1 z| \\
&\leq \sup_{t\in[1,e]} \Bigg\{ \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\sigma+\epsilon-1} \Big|\Psi^*(s,z(s))\Big| \frac{ds}{s} + \frac{|\mu|}{\Gamma(\epsilon)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\epsilon-1} \Big|z(s)\Big| \frac{ds}{s} \Bigg\}, \\
&\leq \|\hbar\| \sup_{t\in[1,e]} \Bigg\{ \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\sigma+\epsilon-1} \frac{ds}{s} \Bigg\} + \|z\| \sup_{t\in[1,e]} \Bigg\{ \frac{|\mu|}{\Gamma(\epsilon)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\epsilon-1} \frac{ds}{s} \Bigg\}, \\
&\leq \frac{\|\hbar\|}{\Gamma(\sigma+\epsilon+1)} + \frac{|\mu|\bar{r}}{\Gamma(\epsilon+1)} \leq \infty.
\end{aligned}
$$

Hence $\mathcal{H}_1$ is uniformly bounded.
Now we prove compactness of operator $\mathcal{H}_1$.
By Lagrange mean value theorem, we have

$$
\begin{aligned}
\Big|\Big(\log t_2\Big)^{\sigma+\epsilon-1} - \Big(\log\frac{t_1}{s}\Big)^{\sigma+\epsilon-1}\Big| &\leq b_1 | t_2 - t_2, \\
\Bigg|\Big(\log\frac{t_2}{s}\Big)^{\sigma+\epsilon-1} - \Big(\log\frac{t_1}{s}\Big)^{\sigma+\epsilon-1}\Bigg| &\leq b_2 | t_2 - t_2|,
\end{aligned}
$$

where b_1 and b_2 are independent of t.

$$
\begin{aligned}
\Big|\mathcal{H}_1 z(t_2) - \mathcal{H}_1 z(t_1)\Big| &\leq \Bigg| \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\sigma+\epsilon-1} \Psi^*(s,z(s)) \frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\epsilon-1} z(s) \frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^{t_1} \Big(\log\frac{t_1}{s}\Big)^{\sigma+\epsilon-1} \Psi^*(s,z(s)) \frac{ds}{s} + \frac{\mu}{\Gamma(\epsilon)} \int_1^{t_1} \Big(\log\frac{t_1}{s}\Big)^{\epsilon-1} z(s) \frac{ds}{s} \Bigg| \\
&\leq \Bigg| \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^{t_1} \Bigg(\Big(\log\frac{t_2}{s}\Big)^{\sigma+\epsilon-1} - \Big(\log\frac{t_1}{s}\Big)^{\sigma+\epsilon-1} \Bigg) \Psi^*(s,z(s)) \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\sigma+\epsilon)} \int_{t_1}^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\sigma+\epsilon-1} \Psi^*(s,z(s)) \frac{ds}{s} \\
&\quad - \frac{\mu}{\Gamma(\epsilon)} \int_1^{t_1} \Bigg(\Big(\log\frac{t_2}{s}\Big)^{\epsilon-1} - \Big(\log\frac{t_2}{s}\Big)^{\epsilon-1} \Bigg) z(s) \frac{ds}{s} + \frac{\mu}{\Gamma(\epsilon)} \int_{t_1}^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\epsilon-1} z(s) \frac{ds}{s} \Bigg|
\end{aligned}
$$

$$\leq \|\hbar\| \frac{1}{\Gamma(\sigma+\epsilon)}\left\{ \int_1^{t_1} \left| \left(\log\frac{t_2}{s}\right)^{\sigma+\epsilon-1} - \left(\log\frac{t_1}{s}\right)^{\sigma+-1} \right| \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\sigma+\epsilon-1} \frac{ds}{s} \right\}$$

$$+ \frac{\mu\|z\|}{\Gamma(\epsilon)}\left\{ \int_1^{t_1} \left| \left(\log\frac{t_2}{s}\right)^{\epsilon-1} - \left(\log\frac{t_2}{s}\right)^{\epsilon-1} \right| \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\epsilon-1} \frac{ds}{s} \right\}$$

$$|\mathcal{H}_1 z(t_2) - \mathcal{H}_1 z(t_1)| \longrightarrow 0 \quad as \quad (t_2 - t_1) \longrightarrow 0,$$

Therefore, $\mathcal{H}_1$ is an equicontinuous. Thus, by Arzela Ascoli theorem we canclude that $\mathcal{H}_1$ compact on $\mathcal{B}_{\bar{r}}$. As from $(1) - (3)$ steps, all the conditions of Theorem 2.7 are satisfied. Thus, there exist a point $z \in \mathcal{B}_{\bar{r}}$ such that $z = \mathcal{H}_1 z + \mathcal{H}_2 z$. Hence, given Equ. (1) has at least one solution on $[1, e]$.

3.2 | Uniqueness of Solutions

Theorem 3.2.. Assuming that assumptions $(\mathcal{R}_1), (\mathcal{R}_2), (\mathcal{R}_3)$and $(\mathcal{R}_4)$are satisfied. Then the problem (1) has unique solution on $[1, e]$ if

$$\Delta^* = \mathcal{L}\Lambda_1 + \Lambda_2 \leq 1, \tag{16}$$

where Λ_1 and Λ_2 are defined in (12) and (13) respectively .

Proof. Let $\mathcal{B}_{\mathcal{R}} = \{z \in \mathcal{V} : ||\mathcal{V}|| \leq \mathcal{R}\}$ is a covex, bounded and closed subset of $\mathcal{V}$ where

$$\mathcal{R} > \frac{\mathcal{M}\Lambda_1}{1 - \Lambda_1 - \Lambda_2},$$

and To show $\mathcal{H}\mathcal{B}_{\mathcal{R}}$ is bounded i.e., $\mathcal{H}\mathcal{B}_{\mathcal{R}} \subseteq \mathcal{B}_{\mathcal{R}}$ and $\mathcal{H}$ is contractive.

$$\begin{aligned}
\left|\Psi^*(s, z_1(s)\right| &= \left|\Psi^*(y, z_1(s) - \Psi^*(s,0) + \Psi^*(s,0)\right| \\
&\leq \left|\Psi^*(s, z_1(s) - \Psi^*(s,0)\right| + \left|\Psi^*(s,0)\right| \\
&\leq \left(\mathcal{L}\left|z_1(s)\right| + \left|\Psi^*(s,0)\right| \\
&\leq \mathcal{L}\mathcal{R} + \mathcal{M},
\end{aligned}$$

then

$$\begin{aligned}
\left|\mathcal{H}z(t)\right| \leq \sup_{t\in[1,e]} &\left\{ \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1} \left|\Psi^*(s, z(s))\right| \frac{ds}{s} + \frac{|\mu|}{\Gamma(\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\epsilon-1} \left|z(s)\right| \frac{ds}{s} \right. \\
&+ \frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2) + (\kappa-\mu)|} \left[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} \left|z(s)\right| \frac{ds}{s} \right. \\
&+ \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1} \left|\Psi^*(s, z(s))\right| \frac{ds}{s} + \frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1} \left|\Psi^*(s, z(s))\right| \frac{ds}{s} \\
&\left. + \frac{|\Gamma(\epsilon+1) + (\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} \left|z(s)\right| \frac{ds}{s} \right] \\
&\left. + \frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} \left|z(s)\right| \frac{ds}{s} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathcal{L}\mathcal{R}+\mathcal{M}\right)\sup_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1}\frac{ds}{s}\\
&\quad+\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\left[\frac{1}{\Gamma(\sigma)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma-1}\frac{ds}{s}+\frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\frac{ds}{s}\right]\Bigg\}\\
&\quad+\|z\|\sup_{t\in[1,e]}\Bigg\{\frac{|\mu|}{\Gamma(\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\epsilon-1}\frac{ds}{s}+\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\left[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\epsilon-1}\frac{ds}{s}\right.\\
&\quad\left.+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\frac{ds}{s}\right]+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\frac{ds}{s}\Bigg\}\\
&\leq \left(\mathcal{L}\mathcal{R}+\mathcal{M}\right)\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon+1)}+\frac{|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)}\\
&\quad+\frac{1}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)}\Bigg\}\\
&\quad+\mathcal{R}\Bigg\{\frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)}+\frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)}\\
&\quad+\frac{|\mu|}{\Gamma(\epsilon+1)}+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)}\Bigg\}\\
&\leq \left(\mathcal{L}\mathcal{R}+\mathcal{M}\right)\Lambda_1+\Lambda_2\mathcal{R}\leq\mathcal{R},
\end{aligned}
$$

which implies that $\|\mathcal{H}z\|\leq \boldsymbol{R}$.

$$
\mathcal{H}\mathcal{B}_R\subseteq\mathcal{B}_R. \tag{17}
$$

Consider for all $z, z_1\in\mathcal{V}$,

$$
\begin{aligned}
\|\mathcal{H}z-\mathcal{H}z_1\|\leq\ &\sup_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|\mu|}{\Gamma(\epsilon)}\int_1^t\left(\log\frac{t}{s}\right)^{\epsilon-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\\
&+\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\left[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\epsilon-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\right.\\
&+\frac{1}{\Gamma(\sigma)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)}\int_1^e\left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&\left.+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\right]\\
&+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\left(\log\frac{\gamma}{s}\right)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\Bigg\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \Bigg\{ \frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)} \\
&\quad + \frac{\mathcal{L}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)} + \frac{\mathcal{L}|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)} \\
&\quad + \frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)} + \frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)} \Bigg\} \|z-z_1\| \\
&\leq [\mathcal{L}\Lambda_1+\Lambda_2]\|z-z_1\|, \qquad \forall \ t\in[1,e],
\end{aligned}$$

in view of (16), $(\mathcal{L}\Lambda_1+\Lambda_2)<1$. Thus $\mathcal{H}$ is contraction map. Also implying (17), holds. Hence by Banach fixed point theorem $\mathcal{H}$ has a unique fixed point. Therefore unique solution exist for Equ. (1).

4 | STABILITY RESULTS

In this section, we discuss the HU stability of (1). The definition of the HU stability is given below:

Definition 4.1[29] The integral Equ. (11) is said to be HU stable if there exists non negative constant Ω^*, for a constant $\Upsilon^*>0$ with given below conditions hold true:

If,

$$\begin{aligned}
&\Bigg| z(t) - \Bigg(\frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s,z(s))\frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} \\
&\quad + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)+(\kappa-\mu)} \Bigg[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} z(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1} \Psi^*(s,z(s))\frac{ds}{s} - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s,z(s))\frac{ds}{s} \\
&\quad - \frac{\Gamma(\epsilon+1)+(\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s} \Bigg] + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z(s)\frac{ds}{s} \Bigg) \Bigg| \leq \Upsilon^*,
\end{aligned} \tag{18}$$

then $\exists \ z_1(t)$, which is continuous and satisfies the given below equation:

$$\begin{aligned}
z_1(t) &= \frac{1}{\Gamma(\sigma+\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s,z_1(s))\frac{ds}{s} - \frac{\mu}{\Gamma(\epsilon)} \int_1^t \left(\log\frac{t}{s}\right)^{\epsilon-1} z_1(s)\frac{ds}{s} \\
&\quad + \frac{(\log t)^{\epsilon+1}}{\Gamma(\epsilon+2)+(\kappa-\mu)} \Bigg[\frac{\mu(\kappa-\mu)}{\Gamma(\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\epsilon-1} z_1(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\sigma)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma-1} \Psi^*(s,z_1(s))\frac{ds}{s} - \frac{(\kappa-\mu)}{\Gamma(\sigma+\epsilon)} \int_1^e \left(\log\frac{e}{s}\right)^{\sigma+\epsilon-1} \Psi^*(s,z_1(s))\frac{ds}{s} \\
&\quad - \frac{\Gamma(\epsilon+1)+(\kappa-\mu)}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z_1(s)\frac{ds}{s} \Bigg] + \frac{(\log t)^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)} \int_1^\gamma \left(\log\frac{\gamma}{s}\right)^{\delta-1} z_1(s)\frac{ds}{s},
\end{aligned} \tag{19}$$

with

$$|z(t)-z_1(t)| \leq \Omega^*\Upsilon^*. \tag{20}$$

Theorem 4.1 . Assuming that assumptions $(\mathcal{R}_1),(\mathcal{R}_2),(\mathcal{R}_3)$and $(\mathcal{R}_4)$are hold true. Then the Equ. (1) is HU stable.

Proof: Firstly, we have to prove integral Equ. (11) is HU stable.

Consider

$$\begin{aligned}
&|z(t)-z_1(t)|\\
&\le \sup_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\sigma+\epsilon)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\sigma+\epsilon-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|\mu|}{\Gamma(\epsilon)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\epsilon-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\\
&+\frac{|(\log t)|^{\epsilon+1}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|}\Bigg[\frac{|\mu||(\kappa-\mu)|}{\Gamma(\epsilon)}\int_1^e\Big(\log\frac{e}{s}\Big)^{\epsilon-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\sigma)}\int_1^e\Big(\log\frac{e}{s}\Big)^{\sigma-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|(\kappa-\mu)|}{\Gamma(\sigma+\epsilon)}\int_1^e\Big(\log\frac{e}{s}\Big)^{\sigma+\epsilon-1}\Big|\Psi^*(s,z(s))-\Psi^*(s,z_1(s))\Big|\frac{ds}{s}\\
&+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\Big(\log\frac{\gamma}{s}\Big)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\Bigg]\\
&+\frac{|(\log t)|^{\epsilon}}{\Gamma(\epsilon+1)\Gamma(\delta)}\int_1^\gamma\Big(\log\frac{\gamma}{s}\Big)^{\delta-1}\Big|z(s)-z_1(s)\Big|\frac{ds}{s}\Bigg\}\\
&\le\Bigg\{\frac{|\kappa-\mu||\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)}\\
&+\frac{\mathcal{L}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+1)}+\frac{\mathcal{L}|\kappa-\mu|}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\sigma+\epsilon+1)}\\
&+\frac{|\Gamma(\epsilon+1)+(\kappa-\mu)|(\log\gamma)^{\delta}}{|\Gamma(\epsilon+2)+(\kappa-\mu)|\Gamma(\epsilon+1)\Gamma(\delta+1)}+\frac{(\log\gamma)^{\delta}}{\Gamma(\delta+1)\Gamma(\epsilon+1)}\Bigg\}\|z_1-z_2\|\\
&\le[\mathcal{L}\Lambda_1+\Lambda_2]\|z-z_1\|,\qquad \forall\ t\in[1,e].
\end{aligned}\tag{21}$$

Consequently, by using (21) the HU stability is proved for the Equ. (11). Hence, the FDEs (1) is HU stable.

5 | APPLICATION

Let us investigate the following FLEs:

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}^{\sigma}({}^{\mathcal{CH}}\mathcal{D}^{\epsilon}+\mu)z(t)=\Psi^*(t,z(t)), & t\in[1,e],\\ z(0)=0,\quad z'(0)=0,\quad z''(0)=0,\quad {}^{\mathcal{CH}}\mathcal{D}^{\epsilon}z(1)=I^{\delta}z(\gamma), & {}^{\mathcal{CH}}\mathcal{D}^{\epsilon}z(e)+\kappa z(e)=0,\end{cases}\tag{22}$$

where $\sigma=\frac{3}{4},\quad \epsilon=\frac{9}{4},\quad \kappa=\frac{1}{4}\quad \mu=\frac{1}{20},\quad \gamma=\frac{e}{2},\quad \delta=\frac{1}{5}$

$|\Psi^*(t,z(t))|=\frac{1}{t+100}\left(\frac{|z|}{(5+|z|)}\right)$,
the assumptions $(\mathcal{R}_1),(\mathcal{R}_2),(\mathcal{R}_3)$ and $(\mathcal{R}_4)$ hold true for $\Psi^*(t,z(t))$.
Also,

$$|\Psi^*(t,z(t))|\le \hbar(t),\tag{23}$$

where $\hbar(t)=\frac{1}{(t+100)}$,

$$|\Psi^*(t, z(t)) - \Psi^*(t, z_1(t))| \leq \frac{1}{100}|z - z_1|.$$

Hence, $\Lambda_1 \approx 0.298828$, $\Lambda_2 \approx 0.398094$ and $\Lambda_3 \approx 0.379803 < 1$. all Thus the Theorem 3.1 implies there exist atleast one solution of Equ.(22)
Also $\mathcal{L}\Lambda_1 + \Lambda_2 = 0.401012 < 1$, then Theorem 3.2 shows that there exist a unique solution of the Equ. (22). In the easy way, We can also find the conditions for HU stability. We canclude that, the Equ. (22) is HU stable.

6 | CONCLUSION

The Langevin FDEs with Hadamard-Caputo's derivative gained less attention as compare to Langevin FDEs with Caputo's derivatives. Although latest interests in the analysis of FDEs with differential boundary condition and to the best of our knowledge, there is no paper concerning this type of problem.The present work let out some notable results about nonlinear CaputoâĂŞHadamard Langevin equations involving nonlocal condition. In this investigation, we analysed the EU of solutions for FLE assosiated with nonperiodic and nonlocal condtions involving differential operator by using fixed point theorems. In the same way under some conditions and assumptions, HU stability results for the solution also investigated. The obtained results are demonstrated by an application that explain the reliability of results.

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AUTHOR'S CONTRIBUTIONS

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