Lower and upper bounds of Dirichlet eigenvalues for Grushin type degenerate elliptic operators in weighted divergence form with a potential

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Abstract

In this paper, we consider the estimates of Dirichlet eigenvalues for Grushin type degenerate elliptic operator in weighted divergence form with a potential $-\mathrm{div}_G(A\nabla_G)+\langle A\nabla_G\phi, \nabla_G\rangle - V$. Using the method of Fourier transformation, we get precise lower bound estimates for the eigenvalues. Then, through the way of trail function, we obtain Yang-type inequalities which give upper bounds of eigenvalues.

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1 Introduction

The so-called Dirichlet eigenvalue problem is stated as

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the classical Laplace operator, and Ω is a bounded domain in $\mathbb{R}^n$. There is a sequence of discrete eigenvalues for this problem which can be ordered as

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots, \quad \text{and} \quad \lambda_k \to +\infty \ (k \to +\infty).$$

Here each eigenvalue is repeated according to its multiplicity.

The problem of estimates for eigenvalues has attracted many mathematician's attention over the years. In 1912, Weyl [35] obtained the asymptotic formula

$$\lambda_k \sim C_n \left(\frac{k}{|\Omega|_n}\right)^{\frac{2}{n}}, \quad k \to \infty,$$

where $C_n = 4\pi^2 (B_n)^{-\frac{2}{n}}$ with B_n being the volume of the unit ball in $\mathbb{R}^n$, and $|\Omega_n|$ is the Lebesgue measure of Ω. In 1961, Pólya [32] proved the inequality

$$\lambda_k \ge C_n \left(\frac{k}{|\Omega|_n}\right)^{\frac{2}{n}}, \quad k = 1, 2, \cdots,$$

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where Ω is a tiling domain in $\mathbb{R}^n$. In 1980, Lieb [26] proved that there exists a constant C_n', such that

$$\lambda_k \geq C_n' \left(\frac{k}{|\Omega|_n}\right)^{\frac{2}{n}}, \quad k = 1, 2, \cdots$$

for any domain $\Omega \subset \mathbb{R}^n$. However, C_n' is different from the constant C_n. In 1983, Li and Yau [25] proved that

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} |\Omega|_n^{-\frac{2}{n}} \tag{1.2}$$

for $k = 1, 2, \cdots$. In 2006, Meals [27] obtained that

$$\sum_{i=1}^{k} \lambda_i \geq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} |\Omega|_n^{-\frac{2}{n}} + c_n \frac{|\Omega|_n}{I(\Omega)},$$

where c_n depends only on the dimension n and

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called the moment of inertia of Ω.

As to the upper bounds for eigenvalues, some famous inequalities are concluded as follows. In 1956, Payne-Póyla and Weinberger [30] proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i$$

for $\Omega \subset \mathbb{R}^2$, which is called the PPW inequality. In 1980, Hile and Protter [22] proved inequality

$$\frac{nk}{4} \leq \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i},$$

which is referred as HP inequality. In 1991, Yang [37] obtained

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i \tag{1.3}$$

and

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_k, \tag{1.4}$$

which is called Yang's first inequality and Yang's second inequality, respectively.

The inequalities for classical Laplacian in $\mathbb{R}^n$ have also been extended to more general operators and more general manifolds. Here we mention the work in [1–4, 6–18, 21, 24, 28, 31, 33, 34] as well as the references therein. Especially, using the method of Fourier transformation, Chen and Luo [5] considered the lower bounds of Dirichlet eigenvalues for Grushin type operator

$$\begin{cases} \sum\limits_{i=1}^{n} X_i^2 u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where

$$X_l = \frac{\partial}{\partial x_l}, l = 1, \cdots, n-2, \quad X_{n-1} = x_i^p \frac{\partial}{\partial x_{n-1}}, \quad X_n = x_j^q \frac{\partial}{\partial x_n}, \quad i, j \in \{1, 2, \cdots, n-2\}.$$

They proved that there exist constants C' and C'' such that

$$\sum_{j=1}^{k} \lambda_j \geq C' k^{1+\frac{2}{v}} - C''(p,q)k \tag{1.6}$$

for all $k \geq 1$, where $v = n + p + q$ is the Métivier index. Inequality (1.6) implies that

$$\lambda_k \geq C' k^{-\frac{2}{v}} - C''(p,q).$$

Niu and Zhang [29] studied the upper bounds of Dirichlet eigenvalues for the degenerate elliptic operator

$$\sum_{i=1}^{h} X_i^2 + \sum_{i=h+1}^{n} Y_i^2 \tag{1.7}$$

on Grushin type vector fields

$$X_i = \frac{\partial}{\partial x_i}, i = 1, \cdots, h, Y_j = |x|^{\alpha} \frac{\partial}{\partial y_j}, j = h+1, \cdots, n,$$

$\alpha \geq 1, \alpha \in R$. They obtained an inequality for eigenvalues of PPW type as

$$\lambda_{k+1} - \lambda_k \leq \frac{4n}{kh^2} \max\{1, d^{4\alpha}\} \sum_{i=1}^{k} \lambda_i \tag{1.8}$$

for all $k \geq 1$, where d is the diameter of Ω_x, the projection of Ω in $(x_1, x_2, \cdots, x_h)$ space.

The estimates of eigenvalues for operators on manifolds with density have been investigated with increasing interest in recent years [17,36]. Xia and Xu [36] considered the eigenvalue estimate problem of drifting Laplacian $\Delta_\phi = -\Delta + \langle \nabla\phi, \nabla \rangle$ on Riemannian manifold $(M, g, e^{-\phi}dv)$, where ϕ is a real-valued positive function defined on M. They got a Yang type inequality for eigenvalues as

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(4\lambda_i + 4\phi_0 \lambda_i^{\frac{1}{2}} + n^2 H_0^2 + \phi_0^2),$$

where $H_0 = \sup_\Omega |H|$, H is the mean curvature vector and $\phi_0 = \sup_{\overline{\Omega}} |\nabla \phi|$.

If we endow the space $\mathbb{R}^n$ with density $e^{-\phi}$, the triple $(\mathbb{R}^n, \langle,\rangle, d\mu = e^{-\phi}d\nu)$ is a smooth metric measure space, where $d\nu$ is the Riemannian volume element related to $\langle,\rangle$. Let Ω be a bounded domain in $(\mathbb{R}^n, \langle,\rangle, d\mu)$ with non-characteristic smooth boundary. We can define the Grushin type degenerate elliptic operator in weighted divergence form as

$$\mathscr{L}_\phi = -\mathrm{div}_{G,\phi}(A\nabla_G),$$

where $\mathrm{div}_{G,\phi} X = e^{\phi} \mathrm{div}_G (e^{-\phi} X)$ is the weighted divergence of Grushin type vector fields X, A is a smooth symmetric positive definite matrix, and

$$\nabla_G u = (X_1 u, \cdots, X_h u, X_{h+1} u, \cdots, X_n u), \quad u \in C^1(\mathbb{R}^n; \mathbb{R});$$

$$\mathrm{div}_G\varphi = X_1\varphi_1 + \cdots + X_h\varphi_h + X_{h+1}\varphi_{h+1} + \cdots + X_n\varphi_n, \quad \varphi = (\varphi_1, \cdots, \varphi_n) \in C^1(\mathbb{R}^n, \mathbb{R}^n);$$
$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \cdots, h;$$
$$X_j = x_l^{p_j}\frac{\partial}{\partial x_j}, \quad j = h+1, \cdots, n, \quad l \in \{1, 2, \cdots, h\}, \quad p_j \in Z^+.$$

In this paper, we consider the Dirichelet eigenvalue problem

$$\begin{cases} \mathscr{L}_\phi u - Vu = \lambda\rho u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where V is a nonnegative potential function, ρ is a positive function continuous on $\overline{\Omega}$. Through integration by part, we have

$$\int_\Omega f(\mathscr{L}_\phi - V)g d\mu = \int_\Omega g(\mathscr{L}_\phi - V)f d\mu$$

where f, g are smooth functions defined on Ω and $f|_{\partial\Omega} = g|_{\partial\Omega} = 0$. So the operator $\mathscr{L}_\phi - V$ is a positive defined self-adjoint operator in $L^2(\Omega)$. Thus this Dirichlet eigenvalue problem (1.9) has discrete eigenvalues [20] which can be ordered as

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots, \quad \text{and} \quad \lambda_k \to +\infty,$$

where each eigenvalue is repeated according to its multiplicity.

Through direct calculation, we have

$$\mathscr{L}_\phi = -\mathrm{div}_{G,\phi}(A\nabla_G) = -\mathrm{div}_G(A\nabla_G) + \langle A\nabla_G\phi, \nabla_G\rangle.$$

Obviously, when A is the identity matrix I, $\rho = 1$, $V = 0$, $h = n-2$, and ϕ is a constant, problem (1.9) is just problem (1.5). So the problem we study in this paper is a generalization of problem (1.5). When $A = I, V = 0$, the operator $\Delta_{G,\phi} = -\Delta_G + \langle\nabla_G\phi, \nabla_G\rangle$ is just the drifting Lapacian (or witten Laplacian) of Grushin type vector fields, where $\Delta_G = \sum_{i=1}^n X_i^2$.

Because of the degenerate property of Grushin vector fields, the order of action between different vector fields can not be exchanged. So it is difficult to generalize the methods and results in the classical case to the degenerate case. In order to overcome this difficulty, we follow the similar way of Chen in [5] when dealing with the problem of lower bounds estimation. When dealing with the estimation of the upper bound, we use the method of trail function which is extensively used in the study of universal upper bounds. But due to the degeneration of vector fields, it is much more complex in the calculation and the results will include some constants depending on the property of the domain Ω which is different from the classical case.

The paper is organized as follow. In Section 2, we present some basic definitions and the main results. In Section 3, the proofs of main results will be given.

2 Preliminaries and main results

For $n \ge 2$, the following space

$$H_X^1(\Omega) = \{u \in L^2(\Omega) | X_j u \in L^2(\Omega), j = 1, 2, \cdots, n\}$$

is a Hilbert space with norm $\|u\|_{H_X(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla_G u\|_{L^2(\Omega)}^2$. The subspace $H_{X,0}^1(\Omega)$ is defined as a closure of C_0^∞ in $H_X^1(\Omega)$, which is also a Hilbert space [28].

Definition 2.1 *([23]) For $J = (j_1, \cdots, j_k)$ with $1 \le j \le m$, We denote $|J| = k$. We say that $X = \{X_1, X_2, \cdots, X_m\}$ satisfies the Hörmander's condition on Ω if there exist a positive integer Q, such that for any $k \le Q$, X together with all $k-th$ repeated commutators*

$$X_J = [X_{j1}, [X_{j2}, [X_{j3}, \cdots [X_{jk-1}, X_{jk}] \cdots]]]$$

span the tangent space at each point of Ω. We call Q the Hörmander index of X on Ω, which is defined as the smallest positive integer for the Hörmander's condition above being satisfied.

Definition 2.2 *([5, 38]) Let $V_j(x)$ be the subspace of the tangent space $T_x(\Omega)$ which is spanned by the vector fields X_J with $|J| \le j$. Let $v_j(x)$ be the dimension of $V_j(x)$ of each $x \in \Omega$. Namely $v_j(x) = \mathrm{dimV_j(x)}$. Set $v(x) = \sum\limits_{j=1}^{Q} j(v_j(x) - v_{j-1}(x))$ with $v_0(x) = 0$. The generalized Métivier's index is defined as*

$$v = \max_{x \in \Omega} v(x).$$

The Métivier's index is also called the homogeneous dimension of Ω related to the sub-elliptic metric induced by the vector fields X.

In this paper, we consider the following Grushin type vector field on Ω satisfying $\Omega \bigcap \{x_i = 0\} \ne \emptyset$, $i = 1, 2 \cdots, h$.

$$\begin{aligned}
&X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, h, \\
&X_j = x_l^{p_j} \frac{\partial}{\partial x_j}, \quad j = h+1, \cdots, n, \quad l \in \{1, 2, \cdots, h\}, \quad p_j \in Z^+.
\end{aligned}$$

Through direct calculations, it is easy to verify that the Hörmander's index of the vector field $X = \{X_1, \cdots, X_n\}$ is $Q = 1 + \max\{p_j | j = h+1, \cdots, n\}$. So X is a finitely degenerate system of vector fields on Ω. Next, we will calculate the Métivier's index of X in the case $h = n-2$ as an example. In order to simplify the calculation, we suppose

$$X_{n-1} = x_1^{p_{n-1}} \frac{\partial}{\partial x_{n-1}}, X_n = x_2^{p_n} \frac{\partial}{\partial x_n},$$

and $p_{n-1} \le p_n$ on the base of no loss of generality. It is easy to verify that

$$Z_1 = \underbrace{[X_1, [X_1, [X_1, \cdots, [X_1, X_{n-1}]]]]}_{p_{n-1}} = p_{n-1}! \frac{\partial}{\partial x_{n-1}}$$

and

$$\begin{aligned}
&Z_2 = \underbrace{[X_2, [X_2, [X_2, \cdots, [X_2, X_n]]]]}_{p_{n-1}} = \frac{p_n!}{(p_n - p_{n-1})!} x_2^{(p_n - p_{n-1})} \frac{\partial}{\partial x_n}, \\
&Z_3 = \underbrace{[X_2, [X_2, [X_2, \cdots, [X_2, X_n]]]]}_{p_n} = p_n! \frac{\partial}{\partial x_n}.
\end{aligned}$$

Obviously, when $x_1 = 0, x_2 = 0$,

$$V_1(x) = \mathrm{span}\{X_1, X_2, \cdots, X_{n-1}, X_n\}, \qquad v_1 = \mathrm{dimV_1 = n - 2};$$

$$
\begin{aligned}
V_2(x) &= \operatorname{span}\{X_1,\cdots,X_n,[X_1,X_{n-1}],[X_2,X_n]\}, \quad v_2=\mathrm{dimV_1}=\mathrm{n}-2;\\
&\vdots\\
V_{1+p_{n-1}}(x) &= \operatorname{span}\{X_1,X_2,\cdots,X_n,Z_1,Z_2\}, \qquad v_{p_{n-1}}=\mathrm{dimV_{p_{n-1}}}=\mathrm{n}-1;\\
&\vdots\\
V_{1+p_n}(x) &= \operatorname{span}\{X_1,X_2,\cdots,X_n,Z_1,Z_3\}, \qquad v_{p_n}=\mathrm{dimV_{p_n}}=\mathrm{n}.
\end{aligned}
$$

Then $v(x)=p_n+p_{n-1}+n$. When $x_1\neq 0$ or $x_2\neq 0$, it is easy to verify that $v(x)<p_n+p_{n-1}+n$. So the generalized Métivier's index is $p_n+p_{n-1}+n$. Thus we can deduce that the generalized Métivier's index of the vector fields considered in this paper is $v=n+\sum_{j=h+1}^{n}p_j$.

In this paper, we get lower bounds of Dirichlet eigenvalues for the weighted divergence operator.

Theorem 2.1 *Let Ω be a bounded domain with smooth non-characteristic boundary, and $\Omega\bigcap\{x_i=0\}\neq\emptyset$ for any $i=1,2\cdots,h$, A be a symmetric and positive definite matrix satisfying $\zeta_1 I\leq A\leq\zeta_2 I$, $\rho(x)$ be a positive continuous function defined on $\bar{\Omega}$, and $V(x)$ be a nonnegative potential function. Let u_i be the eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem*

$$
\begin{cases}
(\mathscr{L}_\phi-V)u_i=\lambda\rho u_i \quad \text{in } \Omega,\\
\displaystyle\int_\Omega \rho u_i u_j d\mu=\delta_{ij},\\
u_i|_{\partial\Omega}=0.
\end{cases}
\tag{2.1}
$$

Then we have

$$
\sum_{j=1}^{k}\frac{\sigma}{\zeta_1}(\lambda_j+V_0\sigma)\geq \bar{C}k^{1+\frac{2}{\nu}}-\tilde{C}\sigma^2 k,
$$

where

$$
\bar{C}=\frac{2^n n^{-\frac{2+v}{2}}\left[(2\pi)^{-n}\sigma^2|\Omega_n|\right]^{\frac{-2}{v}}}{3+3(n-h)\max\{C_{j1}|j=h+1,\cdots,n\}}\left[\frac{\tau^2 v}{w_{n-1}\prod_{j=h+1}^{n}(p_j+1)}\right]^{1+\frac{2}{v}}
$$

and

$$
\tilde{C}=\frac{(n-h)\max\{C_{j1}C_{j2}|j=h+1,\cdots,n\}}{1+(n-h)\max\{C_{j1}|j=h+1,\cdots,n\}},
$$

C_{j1},C_{j2} are constants in (3.1) below, $v=n+\sum_{j=1}^{n}p_j$ is the generalized Métivier's index, $\tau=(\sup_{\mathrm{x}\in\bar{\Omega}}\rho)^{-1},\sigma=(\inf_{\mathrm{x}\in\bar{\Omega}}\rho)^{-1},\mathrm{V_0}=\sup_{\mathrm{x}\in\bar{\Omega}}\mathrm{V(x)}$, $w_{n-1}(x)$ is the area of the unit sphere in $\mathbb{R}^n$, and $|\Omega|_n$ is the volume of Ω.

Remark 2.1 (1) Considering the monotonicity of sequence $\{\lambda_j\}_{j=1}^{\infty}$, we can deduce that the $k-$th Dirichlet eigenvalue λ_k satisfies

$$
\lambda_k\geq\frac{\zeta_1\bar{C}}{\sigma}k^{\frac{2}{v}}-\tilde{C}\sigma\zeta_1-V_0\sigma,\quad \text{for all } k\geq 1.
$$

(2) When $A=I,V=0,\rho=1,p_j=0,j=h+1,\cdots,n$, the operator $-\mathrm{div}_G(A\nabla_G)+\langle\nabla_G\phi,\nabla_G\rangle-V$ is just the classical Laplace operator, the result in Theorem 2.1 is just the same to Li-Yau's [25].

(3) When $A = I, V = 0, \rho = 1, h = n-2$, the result in Theorem 2.1 is just the same to the result (1.6) in [5].

(4) We can find that the result has no relation with the density $e^{-\phi}$.

Especially, if the potential function $V(x)$ satisfies the Hardy type inequality

$$\int_\Omega V f^2 d\mu(x) \leq \int_\Omega |\nabla_G f|^2 d\mu(x),$$

then we have the following theorem:

Theorem 2.2 *Let Ω be a bounded domain with non-characteristic smooth boundary, and $\Omega \bigcap \{x_i = 0\} \neq \emptyset$ for any $i = 1, 2 \cdots, h$, A be a symmetric and positive definite matrix satisfying $\zeta_1 I \leq A \leq \zeta_2 I$ and $\zeta_1 > 1$, $\rho(x)$ be a positive continuous function defined on $\bar{\Omega}$, and $V(x) \geq 0$ be a potential function satisfying*

$$\int_\Omega V f^2 d\mu(x) \leq \int_\Omega |\nabla_G f|^2 d\mu(x)$$

for all $f \in H^1_{X,0}(\Omega)$. Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1). Then we have

$$\sum_{j=1}^k \frac{\sigma}{\zeta_1 - 1} \lambda_j \geq \bar{C} k^{1+\frac{2}{v}} - \tilde{C}\sigma^2 k,$$

where $\bar{C}, \tilde{C}$ are the same to the constants in Theorem 2.1.

Remark 2.2 (1) Considering the monotonicity of sequence $\{\lambda_j\}_{j=1}^\infty$, we have

$$\lambda_k \geq \frac{\zeta_1 - 1}{\sigma} \bar{C} k^{\frac{2}{v}} - \sigma(\zeta_1 - 1)\tilde{C}. \tag{2.2}$$

(2) When $A = I, V = 0, \rho = 1$, the result in Theorem 2.2 is just the same with Chen's [9].

Next, we present two theorems about the upper bounds of eigenvalues for problem (2.1).

Theorem 2.3 *Let Ω be a bounded domain with smooth non-characteristic boundary, and $\Omega \bigcap \{x_i = 0\} \neq \emptyset$ for any $i = 1, 2 \cdots, h$, A be a symmetric and positive definite matrix satisfying $\zeta_1 I \leq A \leq \zeta_2 I$, $\rho(x)$ be a positive continuous function defined on $\bar{\Omega}$, ϕ be a smooth function satisfying $|\nabla_G \phi| \leq C_0$ and $V(x)$ be nonnegative potential function. Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1). Then we have*

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq A \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left[P(\lambda_i + V_0\sigma) + C_0(\sigma\zeta_1)^{\frac{1}{2}} (\lambda_i + V_0\sigma)^{\frac{1}{2}} + \frac{1}{4}\zeta_1 C_0^2 \sigma \right],$$

where

$$A = \frac{4\zeta_2\sigma^2}{h^2\tau^2\zeta_1} \left(h + \sum_{j=h+1}^n d^{2p_j} \right)$$

and $P = \max\{1, d^{2p_{h+1}}, d^{2p_{h+2}}, \cdots, d^{2p_n}\}$, d is the diameter of Ω_x, the projection of Ω in the $(x_1, x_2, \cdots, x_h)$ space .

Remark 2.3 (1) Under the condition of Theorem 2.3, Let $V = 0$ and ϕ be a constant. Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4P\zeta_2\sigma^2\left(h+\sum\limits_{j=h+1}^{n} d^{2p_j}\right)}{h^2\tau^2\zeta_1}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

(2) It is easy to find that our method can also be used to get the upper bound of Dirichlet eigenvalues for the operator (1.7) in [29]. Through similar calculation, We can get

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\left[h+(n-h)d^{2\alpha}\right]}{h^2}\max\{1,d^{2\alpha}\}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i,$$

which is sharper than the result in (1.8).

(3) When $A = I, V = 0, \rho = 1, p_j = 0, (j = h+1, \cdots, n)$, and ϕ is a constant, the operator $-\mathrm{div}_G(A\nabla_G) + \langle\nabla_G\phi, \nabla_G\rangle - V$ is just the classical Laplace operator. The result in Theorem 2.3 is just the same to Yang's [37].

(4) Under the condition of Theorem 2.3, we have

$$\lambda_{k+1} \le \frac{1}{2k}\left\{2\sum_{i=1}^{k}\lambda_i + A\sum_{i=1}^{k}B + \left[\left(2\sum_{i=1}^{k}\lambda_i + A\sum_{i=1}^{k}B\right)^2 - 4k\left(\sum_{i=1}^{k}\lambda_i^2 + A\sum_{i=1}^{k}\lambda_i B\right)\right]^{\frac{1}{2}}\right\}, \tag{2.3}$$

where

$$B = P(\lambda_i + V_0\sigma) + C_0(\sigma\zeta_1)^{\frac{1}{2}}(\lambda_i + V_0\sigma)^{\frac{1}{2}} + \frac{1}{4}\zeta_1 C_0^2\sigma.$$

The inequality (2.3) is a kind of the Yang's second inequality. Through this inequality, we can find that λ_{k+1} can be up bounded by the first k eigenvalues and the upper bounds depend on the domain.

Theorem 2.4 *Let Ω be a bounded domain with smooth non-characteristic boundary, and $\Omega\bigcap\{x_i = 0\} \neq \emptyset$ for any $i = 1, 2\cdots, h$, A be a symmetric and positive definite matrix satisfying $\zeta_1 I \le A \le \zeta_2 I$ and $\zeta_1 > 1$, $\rho(x)$ be a positive continuous function defined on $\bar{\Omega}$,ϕ be a smooth function on Ω satisfying $|\nabla_G\phi| \le C_0$, and $V(x) \ge 0$ be a potential function satisfying*

$$\int_\Omega Vf^2 d\mu(x) \le \int_\Omega |\nabla_G f|^2 d\mu(x)$$

for all $f \in H^1_{X,0}(\Omega)$. Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1). Then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le A'\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[P\lambda_i + C_0(\zeta_1-1)^{\frac{1}{2}}(\sigma\lambda_i)^{\frac{1}{2}} + \frac{1}{4}C_0^2\sigma(\zeta_1-1)\right],$$

where

$$A' = \frac{4\sigma^2\zeta_2}{h^2\tau^2(\zeta_1-1)}\left(h+\sum_{j=h+1}^{n} d^{2p_j}\right).$$

Remark 2.4 Through direct calculation, we have the second Yang's inequality

$$\lambda_{k+1} \le \frac{1}{2k}\left\{2\sum_{i=1}^{k}\lambda_i + A\sum_{i=1}^{k}B' + \left[\left(2\sum_{i=1}^{k}\lambda_i + A\sum_{i=1}^{k}B'\right)^2 - 4k\left(\sum_{i=1}^{k}\lambda_i^2 + A\sum_{i=1}^{k}\lambda_i B'\right)\right]^{\frac{1}{2}}\right\},$$

where

$$B' = P\lambda_i + C_0(\zeta_1-1)^{\frac{1}{2}}(\sigma\lambda_i)^{\frac{1}{2}} + \tfrac{1}{4}C_0^2\sigma(\zeta_1-1).$$

3 The proof of main results

Lemma 3.1 *([19]) Suppose the system of vector fields $X_1, X_2, \cdots, X_m$ satisfy the Hömander's condition on Ω with index $Q \ge 1$. Then the following subelliptic estimate*

$$\||\nabla|^{\frac{1}{Q}}f\|_{L^2(\Omega)}^2 \le \tilde{C}_1(Q)\left(\|\nabla_X f\|_{L^2(\Omega)}^2 + \bar{C}_1(Q)\|f\|_{L^2(\Omega)}^2\right)$$

holds for all $f \in C_0^\infty(\Omega)$, where $\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$ and $|\nabla|^{\frac{1}{Q}}$ is a pesudo-differential operator with the symbol $|\xi|^{\frac{1}{Q}}$, $\tilde{C}_1(Q) > 0$ and $\bar{C}_1(Q) \ge 0$.

Lemma 3.2 *Suppose Ω is a bounded domain and $f \in C_0^\infty(\Omega)$. Then there exist constants C_3, C_4 such that*

$$\sum_{i=1}^{h}\int_\Omega (\partial_{x_i}f)^2\,d\mu + \sum_{j=h+1}^{n}\int_\Omega \left(|\partial_{x_j}|^{\frac{1}{p_j+1}}f\right)^2 d\mu \le C_3\int_\Omega |\nabla_G f|^2 d\mu + C_4\int_\Omega f^2 d\mu, f \in C_0^\infty(\Omega).$$

Proof. It is obviously that

$$\sum_{i=1}^{h}\int_\Omega (\partial_{x_i}f)^2 d\mu(x) \le \int_\Omega |\nabla_G f|^2 d\mu(x).$$

Let $\tilde{\nabla} = (\partial_{x_1}, \cdots, \partial_{x_h}, x_l^{p_j}\partial_{x_j})$ be defined on Ω_j for any $h+1 \le j \le n$, where Ω_j is the projection of Ω on the direction $x' = (x_1, \cdots, x_h, x_j)$. Considering the Plancherel's formula and Lemma 3.1, we have

$$\begin{aligned}\int_{\Omega_j}\left(|\partial_{x_j}|^{\frac{1}{p_j+1}}f\right)^2 d\mu(x) &= \int_{\mathbb{R}^n}\left(|\xi_j|^{\frac{1}{p_j+1}}\hat{f}\right)^2 d\mu(x) \le \int_{\mathbb{R}^n}\left(|\tilde{\xi}|^{\frac{1}{p_j+1}}\hat{f}\right)^2 d\mu(x)\\ &= \int_{\mathbb{R}^n}\left(|\tilde{\nabla}|^{\frac{1}{p_j+1}}f\right)^2 d\mu(x) = \int_{\Omega_j}\left(|\tilde{\nabla}|^{\frac{1}{p_j+1}}f\right)^2 d\mu(x)\\ &\le C_{j1}\left[\int_{\Omega_j}|\tilde{\nabla}f|^2 d\mu(x) + C_{j2}\int_{\Omega_j}f^2 d\mu(x)\right],\end{aligned} \tag{3.1}$$

where $|\partial_{x_j}|^{\frac{1}{p_j+1}}$ are pesudo-differential operators with the symbol $|\xi_j|^{\frac{1}{p_j+1}}$ for any $h+1 \le j \le n$, and $\tilde{\xi} = (\xi_1, \xi_2, \cdots, \xi_h, \xi_j)$, C_{j1}, C_{j2} are constants depending on the Hömander index $Q_j = 1 + p_j$.

Then through direct calculation, we have

$$\sum_{i=1}^{h}\int_\Omega (\partial_{x_i}f)^2 d\mu(x) + \sum_{j=h+1}^{n}\int_{\Omega_j}\left(|\partial_{x_j}|^{\frac{1}{p_j+1}}f\right)^2 d\mu(x)$$

$$
\begin{aligned}
&\le \int_\Omega |\nabla_G f|^2 d\mu(x) + \sum_{j=h+1}^{n} C_{j1}\left[\int_{\Omega_j} |\tilde{\nabla} f|^2 d\mu(x) + C_{j2}\int_{\Omega_j} f^2 d\mu(x)\right]\\
&= \int_\Omega |\nabla_G f|^2 d\mu(x) + \sum_{j=h+1}^{n} \left[C_{j1}\int_{\Omega_j} |\tilde{\nabla} f|^2 d\mu(x) + C_{j1}C_{j2}\int_{\Omega_j} f^2 d\mu(x)\right]\\
&\le \int_\Omega |\nabla_G f|^2 d\mu(x) + \bar{C_3}\int_\Omega |\nabla_G f|^2 d\mu(x) + C_4\int_\Omega f^2 d\mu(x)\\
&= C_3\int_{\Omega_j} |\nabla_G f|^2 d\mu(x) + C_4\int_\Omega f^2 d\mu(x),
\end{aligned} \tag{3.2}
$$

where $\bar{C_3} = (n-h)\max\{C_{j1}|j = h+1,\cdots,n\}, C_4 = (n-h)\max\{C_{j1}C_{j2}|j = h+1,\cdots,n\}, C_3 = 1+\bar{C_3}$. $\square$

Lemma 3.3 *Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1), $\sigma = (\inf_{\bar{\Omega}}\rho)^{-1}, \tau = (\sup_{\bar{\Omega}}\rho)^{-1}$. Set $\Psi(x,y) = \sum_{j=1}^{k} u_j(x)u_j(y)$. Then we have*

$$
\tau^2 k \le \int_\Omega \int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 d\mu(z)d\mu(y) \le \sigma^2 k
$$

and

$$
\int_\Omega |\hat{\Psi}(z,y)|^2 d\mu(y) \le (2\pi)^{-n}\sigma^2|\Omega|_n,
$$

where $\hat{\Psi}(z,y)$ is the partial Fourier transformation of $\Psi(x,y)$ in the $x-$variable,

$$
\hat{\Psi}(z,y) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \Psi(x,y)e^{-ix\cdot z}d\mu(x).
$$

Proof. Using the Plancherel's formula and the orthonormality of u_j, we have

$$
\begin{aligned}
\int_\Omega\int_{R^n} |\hat{\Psi}(z,y)|^2 d\mu(z)d\mu(y) &= \int_\Omega\int_{\mathbb{R}^n} |\Psi(x,y)|^2 d\mu(x)d\mu(y) = \int_\Omega\int_\Omega |\Psi(x,y)|^2 d\mu(x)d\mu(y)\\
&= \int_\Omega\int_\Omega \frac{1}{\rho(x)}|\sqrt{\rho(x)}\Psi(x,y)|^2 d\mu(x)d\mu(y) \le \sigma\int_\Omega \sum_{j=1}^{k}|u_j(y)|^2 d\mu(y)\\
&\le \sigma^2\int_\Omega \sum_{j=1}^{k}|\sqrt{\rho(y)}u_j(y)|^2 d\mu(y) = \sigma^2 k.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_\Omega\int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 d\mu(z)d\mu(y) &= \int_\Omega\int_\Omega \frac{1}{\rho(x)}|\sqrt{\rho(x)}\Psi(x,y)|^2 d\mu(x)d\mu(y)\\
&\ge \tau\int_\Omega \sum_{j=1}^{k}|u_j(y)|^2 d\mu(y)\\
&\ge \tau^2\int_\Omega \sum_{j=1}^{k}|\sqrt{\rho(y)}u_j(y)|^2 d\mu(y) = \tau^2 k.
\end{aligned}
$$

On the other hand, considering the definition of Fourier transformation of $\Psi(x,y)$ in the $x-$variable, we have

$$
\int_\Omega |\hat{\Psi}(z,y)|^2 d\mu(y) = \int_\Omega (2\pi)^{-n}\left|\int_{\mathbb{R}^n} \Psi(x,y)e^{-ix\cdot z}d\mu(x)\right|^2 d\mu(y)
$$

$$= \int_\Omega (2\pi)^{-n} \left| \int_\Omega \Psi(x,y) e^{-ix\cdot z} d\mu(x) \right|^2 d\mu(y). \tag{3.0}$$

Recall that the sequence $\{u_j\}_{j=1}^\infty$ is a standard orthogonal basis in $L^2(\Omega)$ which implies that a function $e^{-ix\cdot z}$ can be written in the form

$$e^{-ix\cdot z} = \sum_{j=1}^\infty a_j(z) \sqrt{\rho(x)} u_j(x),$$

where $a_j(z) = \int_\Omega e^{-ix\cdot z} \sqrt{\rho(x)} u_j(x) d\mu(x)$. Considering (3.0), we can obtain that

$$\begin{aligned}
\int_\Omega |\hat{\Psi}(z,y)|^2 d\mu(y) &= \int_\Omega (2\pi)^{-n} \left| \int_\Omega \Psi(x,y) e^{-ix\cdot z} d\mu(x) \right|^2 d\mu(y) \\
&= (2\pi)^{-n} \int_\Omega \left| \int_\Omega \sum_{j=1}^k \sum_{l=1}^\infty a_l(z) \sqrt{\rho(x)} u_l(x) u_j(x) u_j(y) d\mu(x) \right|^2 d\mu(y) \\
&\le (2\pi)^{-n} \sigma \int_\Omega \left| \sum_{j=1}^k a_j(z) u_j(y) \right|^2 d\mu(y) \\
&\le (2\pi)^{-n} \sigma^2 \int_\Omega \left| \sum_{j=1}^k \sqrt{\rho(y)} a_j(z) u_j(y) \right|^2 d\mu(y) \\
&= (2\pi)^{-n} \sigma^2 \sum_{j=1}^k |a_j(z)|^2 \le (2\pi)^{-n} \sigma^2 \sum_{j=1}^\infty |a_j(z)|^2 \\
&= (2\pi)^{-n} \sigma^2 \int_\Omega |e^{-ix\cdot z}|^2 dx = (2\pi)^{-n} \sigma^2 |\Omega|_n.
\end{aligned}$$

□

Lemma 3.4 *Suppose $f, g \in C_0^2(\Omega)$ and Ω be a bounded domain in $\mathbb{R}^n$. Then*

$$\int_\Omega f \mathscr{L}_\phi g d\mu = \int_\Omega \langle A\nabla_G g, \nabla_G f \rangle d\mu. \tag{3.1}$$

Proof. We have

$$\begin{aligned}
\int_\Omega f \mathrm{div_G}(A\nabla_G g) d\mu(x) &= \int_\Omega f \mathrm{div_G}(A\nabla_G g) e^{-\phi} d\nu(x) \\
&= -\int_\Omega \langle A\nabla_G g, \nabla_G (f e^{-\phi}) \rangle d\nu(x) \\
&= -\int_\Omega \left(\langle A\nabla_G g, \nabla_G f \rangle - f \langle A\nabla_G g, \nabla_G \phi \rangle \right) e^{-\phi} d\nu(x) \\
&= -\int_\Omega \left(\langle A\nabla_G g, \nabla_G f \rangle - f \langle A\nabla_G g, \nabla_G \phi \rangle \right) d\mu(x)
\end{aligned}$$

Then

$$\int_\Omega \langle A\nabla_G g, \nabla_G f \rangle d\mu = \int_\Omega f \left(-\mathrm{div_G}(A\nabla_G g) + \langle A\nabla_G g, \nabla_G \phi \rangle \right) d\mu.$$

Then we have finished the proof. □

Lemma 3.5 *Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1), $\zeta_1 I \leq A \leq \zeta_2 I$, $\sigma = (\inf_{\bar{\Omega}} \rho)^{-1}$, $\tau = (\sup_{\bar{\Omega}} \rho)^{-1}$, $V_0 = \sup_{\bar{\Omega}} V(x)$. Then we have*

$$\int_{\Omega}\int_{\Omega} |\nabla_G \Psi(x,y)|^2 d\mu(x)d\mu(y) \leq \sum_{j=1}^{k} \frac{\sigma}{\zeta_1}(\lambda_j + V_0\sigma).$$

Proof. Considering the orthogonality of $\{u_i\}_{i=1}^{\infty}$, we have

$$\begin{aligned}
\int_{\Omega}\int_{\Omega} |\nabla_G \Psi(x,y)|^2 d\mu(x)d\mu(y) &= \int_{\Omega}\int_{\Omega} \sum_{l=1}^{n} \left| X_l \sum_{j=1}^{k} u_j(x)u_j(y) \right|^2 d\mu(x)d\mu(y) \\
&\leq \sigma \int_{\Omega}\int_{\Omega} \sum_{l=1}^{n} \left| X_l \sum_{j=1}^{k} u_j(x)\sqrt{\rho(y)}u_j(y) \right|^2 d\mu(x)d\mu(y) \\
&= \sigma \sum_{j=1}^{k} \int_{\Omega} |\nabla_G u_j(x)|^2 d\mu(x). \qquad (3.2)
\end{aligned}$$

Because of $\zeta_1 I < A < \zeta_2 I$ and Lemma 3.4, we have

$$\begin{aligned}
&\int_{\Omega} |\nabla_G u_j(x)|^2 d\mu(x) \leq \zeta_1^{-1} \int_{\Omega} \langle A\nabla_G u_j(x), \nabla_G u_j(x)\rangle d\mu(x) = \zeta_1^{-1} \int_{\Omega} u_j(x)\mathscr{L}_\phi u_j(x) d\mu(x) \\
&= \zeta_1^{-1}\left[\int_{\Omega} u_j(x)(\mathscr{L}_\phi - V)u_j(x)d\mu(x) + \int_{\Omega} V u_j^2 d\mu(x)\right] \leq \zeta_1^{-1}(\lambda_j + V_0\sigma) \qquad (3.3)
\end{aligned}$$

Substituting (3.3) into (3.2), we can finish the proof. □

Lemma 3.6 *Let f be a real-valued function defined on R^n with $0 \leq f \leq M_1$, and for $p_{h+1}, \cdots p_n \in Z^+$,*

$$\int_{R^n} \left(\sum_{i=1}^{h} z_i^2 + \sum_{j=h+1}^{n} |z_j|^{\frac{2}{p_j+1}} \right) f(z) d\mu(z) \leq M_2. \qquad (3.4)$$

Then

$$\int_{R^n} f(z)d\mu(z) \leq D M_1^{\frac{2}{n+2+\sum_{j=h+1}^{n} p_j}} M_2^{\frac{n+\sum_{j=h+1}^{n} p_j}{n+2+\sum_{j=h+1}^{n} p_j}}, \qquad (3.5)$$

where

$$D = \frac{\prod_{j=h+1}^{n}(p_j+1)w_{n-1}}{n + \sum_{j=h+1}^{n} p_j} \left[\frac{3}{2^n}\right]^{\frac{n+\sum_{j=h+1}^{n} p_j}{n+2+\sum_{j=h+1}^{n} p_j}} n^{\frac{n+\sum_{j=h+1}^{n} p_j}{2}}$$

and w_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. It is easy to verify the correctness of this lemma following the way of Chen's in [5]. □

Proof of Theorem 2.1. Using Plancherel's formula and Lemma 3.2, we have

$$\int_{\mathbb{R}^n}\int_{\Omega} \left(\sum_{i=1}^{h} z_i^2 + \sum_{j=h+1}^{n} |z_n|^{\frac{2}{p_j+1}} \right) |\hat{\Psi}(z,y)|^2 d\mu(y)d\mu(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{h}|\partial_{x_i}\Psi(x,y)|^2 + \sum_{j=h+1}^{n}\left||\partial_{x_j}|^{\frac{1}{p_j+1}}\Psi(x,y)\right|^2\right)d\mu(y)d\mu(x)\\
&= \int_{\Omega}\int_{\Omega}\left(\sum_{i=1}^{h}|\partial_{x_i}\Psi(x,y)|^2 + \sum_{j=h+1}^{n}\left||\partial_{x_j}|^{\frac{1}{p_j+1}}\Psi(x,y)\right|^2\right)d\mu(y)d\mu(x)\\
&\le C_3\int_{\Omega}\int_{\Omega}|\nabla_G\Psi(x,y)|^2 d\mu(x)d\mu(y) + C_4\int_{\Omega}\int_{\Omega}|\Psi(x,y)|^2 d\mu(x)d\mu(y). \qquad (3.6)
\end{aligned}
$$

Substituting the results in Lemma 3.3 and Lemma 3.5 into inequality (3.6), we can obtain that

$$
\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{h}z_i^2 + \sum_{j=h+1}^{n}|z_n|^{\frac{2}{p_j+1}}\right)|\hat{\Psi}(z,y)|^2 d\mu(y)d\mu(z) \le C_3\sum_{j=1}^{k}\frac{\sigma}{\zeta_1}(\lambda_j + V_0\sigma) + C_4\sigma^2 k.
$$

Then we set

$$
f(z) = \int_{\Omega}|\hat{\Psi}(z,y)|^2 d\mu(y), M_1 = (2\pi)^{-n}\sigma^2|\Omega_n|, \qquad (3.7)
$$

and

$$
M_2 = C_3\sum_{j=1}^{k}\frac{\sigma}{\zeta_1}(\lambda_j + V_0\sigma) + C_4\sigma^2 k. \qquad (3.8)
$$

Substituting (3.7), (3.8) into (3.5), we have

$$
\begin{aligned}
\tau^2 k \le & \frac{\prod_{j=h+1}^{n}(p_j+1)w_{n-1}}{n+\sum_{j=h+1}^{n}p_j}\left((2\pi)^{-n}\sigma^2|\Omega_n|\right)^{\frac{2}{n+2+\sum_{j=h+1}^{n}p_j}}\left[\frac{3n^{\frac{n+2+\sum_{j=h+1}^{n}p_j}{2}}}{2^n}\right]^{\frac{n+\sum_{j=h+1}^{n}p_j}{n+2+\sum_{j=h+1}^{n}p_j}}\times\\
&\left[C_3\sum_{j=1}^{k}\frac{\sigma}{\zeta_1}(\lambda_j + V_0\sigma) + C_4\sigma^2 k\right]^{\frac{n+\sum_{j=h+1}^{n}p_j}{n+2+\sum_{j=h+1}^{n}p_j}}
\end{aligned}
$$

for any $k \ge 1$. Thus

$$
\sum_{j=1}^{k}\frac{\sigma}{\zeta_1}(\lambda_j + V_0\sigma) \ge \bar{C}k^{1+\frac{2}{n+\sum_{j=1}^{n}p_j}} - \tilde{C}\sigma^2 k,
$$

where

$$
\bar{C} = \frac{2^n}{3C_3}n^{-\frac{n+2+\sum_{j=1}^{n}p_j}{2}}\left[\frac{\tau^2\left(n+\sum_{j=h+1}^{n}p_j\right)}{\prod_{j=h+1}^{n}(p_j+1)w_{n-1}}\right]^{1+\frac{2}{n+\sum_{j=1}^{n}p_j}}\left[(2\pi)^{-n}\sigma^2|\Omega_n|\right]^{-\frac{2}{n+\sum_{j=1}^{n}p_j}}.
$$

and $\tilde{C} = C_4/C_3$. □

Proof of Theorem 2.2. Through similar method in the proof of Lemma 3.5, one has

$$
\int_{\Omega}|\nabla_G u_j(x)|^2 d\mu(x) \le \frac{\lambda_j}{\zeta_1 - 1}. \qquad (3.9)
$$

Then substituting (3.9) into (3.2), we obtain that

$$\int_\Omega\int_\Omega |\nabla_G\Psi(x,y)|^2 d\mu(x)d\mu(y) \le \sum_{j=1}^{k} \frac{\sigma}{\zeta_1 - 1}\lambda_j. \tag{3.10}$$

Then combining (3.10), (3.6), and the result in Lemma 3.3, we have

$$\int_{\mathbb{R}^n}\int_\Omega \left(\sum_{i=1}^{h} z_i^2 + \sum_{j=h+1}^{n} |z_n|^{\frac{2}{p_j+1}}\right) |\hat{\Psi}(z,y)|^2 d\mu(y)d\mu(z) \le C_3 \sum_{j=1}^{k} \frac{\sigma}{\zeta_1 - 1}\lambda_j + C_4\sigma^2 k. \tag{3.11}$$

Then we set

$$f(z) = \int_\Omega |\hat{\Psi}(z,y)|^2 d\mu(y), M_1 = (2\pi)^{-n}\sigma^2|\Omega_n|, \tag{3.12}$$

and

$$M_2 = C_3 \sum_{j=1}^{k} \frac{\sigma}{\zeta_1 - 1}\lambda_j + C_4\sigma^2 k. \tag{3.13}$$

Through similar argument in the proof of Theorem 2.1, we have

$$\sum_{j=1}^{k} \frac{\sigma}{\zeta_1 - 1}\lambda_j \ge \bar{C}k^{1+\frac{2}{v}} - \tilde{C}\sigma^2 k.$$

□

Lemma 3.7 *Let u_i be the orthonormal eigenfunction corresponding to the Dirichlet eigenvalues λ_i of problem (2.1). Then the inequality*

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2\langle\nabla_G f, \nabla_G f\rangle d\mu(x) &\le \epsilon \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2\langle A\nabla_G f, \nabla_G f\rangle d\mu(x)\\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega \frac{1}{\rho}(\langle\nabla_G u_i, \nabla_G f\rangle + \frac{1}{2}u_i\Delta_{G,\phi} f)^2 d\mu(x)\end{aligned} \tag{3.14}$$

holds for all $k \ge 1$ and $f \in C^3(\Omega)\cap C^2(\partial\Omega)$, where ϵ is any positive constant.

Proof. We define the trail function

$$\varphi_i = fu_i - \sum_{j=1}^{k} a_{ij}u_j,$$

where $a_{ij} = \int_\Omega \rho f u_i u_j d\mu(x) = a_{ji}$. Then

$$\int_\Omega \rho\varphi_i u_j d\mu(x) = 0, \varphi_i|_{\partial\Omega} = 0, \qquad i,j = 1,\cdots,k.$$

Through direct calculation, we have

$$L\varphi_i \triangleq \left(-\mathrm{div}_G(A\nabla_G(\cdot)) + \langle A\nabla_G\phi, \nabla_G(\cdot)\rangle - V\right)\varphi_i$$

$$= fLu_i + u_iLf + Vfu_i - 2\langle A\nabla_G f, \nabla_G u_i\rangle - \sum_{j=1}^{k} a_{ij}\lambda_j \rho u_j. \tag{3.15}$$

Substituting (3.15) into the well-known Rayleigh-Ritz inequality [3]

$$\lambda_{k+1} \leq \frac{\int_\Omega \varphi_i L\varphi_i d\mu(x)}{\int_\Omega \rho\varphi_i^2 d\mu(x)},$$

we can obtain that

$$(\lambda_{k+1} - \lambda_i)\int_\Omega \rho\varphi_i^2 d\mu(x) \leq \int_\Omega \varphi_i \left(u_iLf + Vfu_i - 2\langle A\nabla_G f, \nabla_G u_i\rangle\right) d\mu(x). \tag{3.16}$$

Set

$$b_{ij} = \int_\Omega (u_iLf + Vfu_i - 2\langle A\nabla_G f, \nabla_G u_i\rangle) u_j d\mu(x).$$

Considering $d\mu = e^{-\phi}d\nu$, we have

$$\begin{aligned}
&\int_\Omega \langle A\nabla_G f, \nabla_G u_i\rangle u_j d\mu(x) = \int_\Omega \langle A\nabla_G f, \nabla_G u_i\rangle u_j e^{-\phi} d\nu = -\int_\Omega u_i \mathrm{div}(u_j e^{-\phi} A\nabla_G f) d\nu \\
&= -\int_\Omega u_i \left[u_j \mathrm{div}(A\nabla_G f) + u_i\langle A\nabla_G f, \nabla_G u_j\rangle - u_j\langle A\nabla_G f, \nabla_G \phi\rangle\right] e^{-\phi} d\nu \\
&= -\int_\Omega u_i \left[u_j \mathrm{div}(A\nabla_G f) + u_i\langle A\nabla_G f, \nabla_G u_j\rangle - u_j\langle A\nabla_G f, \nabla_G \phi\rangle\right] d\mu.
\end{aligned}$$

Then we can find that

$$b_{ij} = -b_{ji}, b_{ij} = (\lambda_i - \lambda_j)a_{ij}.$$

Considering (3.16), we get

$$\begin{aligned}
&(\lambda_{k+1} - \lambda_i)\int_\Omega \rho\varphi_i^2 d\mu(x) \leq \\
&-\int_\Omega fu_i \left[u_i(\mathrm{div}_G(A\nabla_G f) - \langle A\nabla_G \phi, \nabla_G f\rangle) + 2\langle \nabla_G u_i, A\nabla_G f\rangle\right] d\mu(x) + \sum_{j=1}^{k} (\lambda_i - \lambda_j)a_{ij}^2.
\end{aligned} \tag{3.17}$$

Set

$$c_{ij} = \int_\Omega u_j(\langle \nabla_G u_i, \nabla f\rangle + \frac{1}{2}u_i \Delta_{G,\phi} f) d\mu(x).$$

Then through direct calculation, we have

$$c_{ij} = -c_{ji},$$

and

$$\begin{aligned}
&\int_\Omega \varphi_i(\langle \nabla_G u_i, \nabla_G f\rangle + \frac{1}{2}u_i \Delta_{G,\phi} f) d\mu(x) \\
&= \int_\Omega fu_i(\langle \nabla_G u_i, \nabla_G f\rangle + \frac{1}{2}u_i \Delta_{G,\phi} f) d\mu(x) - \sum_{j=1}^{k} a_{ij}c_{ij} \\
&= -\frac{1}{2}\int_\Omega u_i^2 \langle \nabla_G f, \nabla_G f\rangle d\mu(x) - \sum_{j=1}^{k} a_{ij}c_{ij}.
\end{aligned} \tag{3.18}$$

Using Young's inequality and considering (3.17), (3.18), we get

$$
\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega u_i^2\langle\nabla_G f,\nabla_G f\rangle d\mu(x)+2\sum_{j=1}^k a_{ij}c_{ij}\right)\\
=&(\lambda_{k+1}-\lambda_i)^2\int_\Omega(-2)\sqrt{\rho}\phi_i\left[\frac{1}{\sqrt{\rho}}(\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f)-\sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right]d\mu(x)\\
\leq&\epsilon(\lambda_{k+1}-\lambda_i)^3\int_\Omega\rho\phi_i^2 d\mu(x)\\
&+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f\right)-\sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right]^2 d\mu(x)\\
=&\epsilon(\lambda_{k+1}-\lambda_i)^3\int_\Omega\rho\phi_i^2 d\mu(x)\\
&+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\left[\int_\Omega\frac{1}{\rho}\left(\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f\right)^2 d\mu(x)-\sum_{j=1}^k c_{ij}^2\right]\\
\leq&\epsilon(\lambda_{k+1}-\lambda_i)^2\Bigg\{\int_\Omega -fu_i\left[u_i\left(\mathrm{div}_G(A\nabla_G f)-\langle A\nabla_G\phi,\nabla_G f\rangle\right)+2\langle\nabla_G u_i,A\nabla_G f\rangle\right]d\mu(x)\\
&+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2\Bigg\}+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\left[\int_\Omega\frac{1}{\rho}\left(\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f\right)^2 d\mu(x)-\sum_{j=1}^k c_{ij}^2\right],
\end{aligned}
$$

where ϵ is any positive constant. Then summing over i from 1 to k, we have

$$
\begin{aligned}
&\sum_{j=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\langle\nabla_G f,\nabla_G f\rangle d\mu(x)-2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)a_{ij}c_{ij}\\
\leq&\epsilon\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega -fu_i\left[u_i(\mathrm{div}_G(A\nabla_G f)-\langle A\nabla_G\phi,\nabla_G f\rangle)+2\langle\nabla_G u_i,A\nabla_G f\rangle\right]d\mu+\\
&\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\left[\int_\Omega\frac{1}{\rho}(\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f)^2 d\mu-\sum_{i,j=1}^k\epsilon(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2 a_{ij}^2\right]\\
&-\sum_{i,j=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\epsilon}c_{ij}^2.
\end{aligned}
$$

Because of $a_{ij}=a_{ji}, c_{ij}=-c_{ji}$,

$$
\begin{aligned}
&\sum_{j=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\langle\nabla_G f,\nabla_G f\rangle d\mu(x)\\
\leq&\epsilon\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega -fu_i\left[u_i\left(\mathrm{div}_G(A\nabla_G f)-\langle A\nabla_G\phi,\nabla_G f\rangle\right)+2\langle\nabla_G u_i,A\nabla_G f\rangle\right]d\mu(x)\\
&+\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega\frac{1}{\rho}\left[\langle\nabla_G u_i,\nabla_G f\rangle+\frac{1}{2}u_i\Delta_{G,\phi}f\right]^2 d\mu(x). \qquad (3.19)
\end{aligned}
$$

Considering $d\mu = e^{-\phi}d\nu$, we can obtain that

$$\int_\Omega f u_i^2 \mathrm{div}_G(A\nabla_G f)d\mu = \int_\Omega f u_i^2 \mathrm{div}_G(A\nabla_G f)e^{-\phi}d\mu$$
$$= -\int_\Omega \langle A\nabla_G f, \nabla_G f\rangle u_i^2 e^{-\phi}d\nu - \int_\Omega \langle A\nabla_G f, \nabla_G u_i\rangle 2fu_i e^{-\phi}d\nu + \int_\Omega \langle A\nabla_G f, \nabla_G \phi\rangle f u_i^2 e^{-\phi}d\nu$$
$$= -\int_\Omega \langle A\nabla_G f, \nabla_G f\rangle u_i^2 d\mu - \int_\Omega \langle A\nabla_G f, \nabla_G u_i\rangle 2fu_i d\mu + \int_\Omega \langle A\nabla_G f, \nabla_G \phi\rangle f u_i^2 d\mu. \tag{3.20}$$

Substituting (3.20) into (3.19), we can finish the proof of this lemma. □

Proof of Theorem 2.3. Considering $\zeta_1 I \leq A \leq \zeta_2 I$ in (3.14), we have

$$\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 \langle \nabla_G f, \nabla_G f\rangle d\mu(x) \leq \epsilon\zeta_2 \sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 \langle \nabla_G f, \nabla_G f\rangle d\mu(x)$$
$$+ \sum_{i=1}^k \frac{\lambda_{k+1}-\lambda_i}{\epsilon} \int_\Omega \frac{1}{\rho}(\langle \nabla_G u_i, \nabla_G f\rangle) + \frac{1}{2}u_i \Delta_{G,\phi} f)^2 d\mu(x). \tag{3.21}$$

Taking $f = x_\alpha$ and summing over α from 1 to n, then through direct calculation, we can obtain that

$$\sum_{\alpha=1}^n \langle \nabla_G x_\alpha, \nabla_G x_\alpha\rangle = h + \sum_{j=h+1}^n |x_l|^{2p_j} \leq h + \sum_{j=h+1}^n d^{2p_j} \tag{3.22}$$

and

$$\sum_{\alpha=1}^n \Delta_G x_\alpha = 0, \qquad \sum_{\alpha=1}^n \langle \nabla_G \phi, \nabla_G x_\alpha\rangle^2 = |\nabla_G \phi|^2, \tag{3.23}$$

$$\sum_{\alpha=1}^n \langle \nabla_G u_i, \nabla_G x_\alpha\rangle \langle \nabla_G \phi, \nabla_G x_\alpha\rangle = \langle \nabla_G u_i, \nabla_G \phi\rangle, \tag{3.24}$$

$$\sum_{\alpha=1}^n \langle \nabla_G u_i, \nabla_G x_\alpha\rangle^2 = \sum_{j=1}^h |X_j u_i|^2 + \sum_{j=h+1}^n |x_l|^{2p_j}|X_j u_i|^2 \leq P|\nabla_G u_i|^2, \tag{3.25}$$

where $P = \max\{1, d^{2p_{h+1}}, d^{2p_{h+2}}, \cdots, d^{2p_n}\}$, d is the diameter of Ω_x, the projection of Ω in the $(x_1, x_2, \cdots, x_h)$ space.

Substituting (3.22), (3.23), (3.24), (3.25) into (3.21), then we have

$$h\sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 d\mu(x) \leq \epsilon(h + \sum_{j=h+1}^n d^{2p_j})\zeta_2 \sum_{i=1}^k (\lambda_{k+1}-\lambda_i)^2 \int_\Omega u_i^2 d\mu(x)$$
$$+ \frac{\sigma}{\epsilon}\sum_{i=1}^k (\lambda_{k+1}-\lambda_i) \int_\Omega P|\nabla_G u_i|^2 + u_i \langle \nabla_G u_i, \nabla_G \phi\rangle + \frac{1}{4}u_i^2 |\nabla_G \phi|^2 d\mu(x).$$

Then considering the result in Lemma 3.5, we have

$$\int_\Omega P|\nabla_G u_i|^2 d\mu(x) \leq \frac{P(\lambda_i + V_0\sigma)}{\zeta_1},$$

$$\begin{aligned}\left|\int_\Omega u_i\langle\nabla_G u_i,\nabla_G\phi\rangle d\mu(x)\right| \le&\|u_i\|_{L^2}\|\langle\nabla_G u_i,\nabla_G\phi\rangle\|_{L^2}\\ \le&\sigma^{\frac12}\left[\int_\Omega|\nabla_G u_i|^2|\nabla_G\phi|^2 d\mu\right]^{\frac12}\\ \le&C_0\sigma^{\frac12}\left(\frac{\lambda_i+V_0\sigma}{\zeta_1}\right)^{\frac12},\end{aligned}\tag{3.26}$$

and

$$\int_\Omega u_i^2|\nabla_G\phi|^2 d\mu(x)\le C_0^2\sigma\int_\Omega \rho u_i^2 d\mu(x)=C_0^2\sigma.\tag{3.27}$$

Thus

$$\begin{aligned}h\tau\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\le&\epsilon\left(h+\sum_{j=h+1}^n d^{2p_j}\right)\sigma\zeta_2\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\\ &+\frac{\sigma}{\epsilon}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\left[\frac{P(\lambda_i+V_0\sigma)}{\zeta_1}+C_0\sigma^{\frac12}\left(\frac{\lambda_i+V_0\sigma}{\zeta_1}\right)^{\frac12}+\frac14C_0^2\sigma\right].\end{aligned}\tag{3.28}$$

Obviously, the right hand of (3.28) attains its minimum at

$$\epsilon=\left\{\frac{\sum\limits_{i=1}^k(\lambda_{k+1}-\lambda_i)\left[\frac{P(\lambda_i+V_0\sigma)}{\zeta_1}+C_0\sigma^{\frac12}\left(\frac{\lambda_i+V_0\sigma}{\zeta_1}\right)^{\frac12}+\frac14C_0^2\sigma\right]}{\zeta_2\left(h+\sum_{j=h+1}^n d^{2p_j}\right)\sum\limits_{i=1}^k(\lambda_{k+1}-\lambda_i)^2}\right\}^{\frac12}.$$

Then we can finish the proof. □

Proof of Theorem 2.4. Using similar method in the proof of Theorem 2.3, we can obtain that

$$\begin{aligned}h\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2 d\mu(x)\le\epsilon&\left(h+\sum_{j=h+1}^n d^{2p_j}\right)\zeta_2\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2 d\mu(x)\\ &+\frac{\sigma}{\epsilon}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\int_\Omega P|\nabla_G u_i|^2+u_i\langle\nabla_G u_i,\nabla_G\phi\rangle+\frac14u_i^2|\nabla_G\phi|^2 d\mu(x).\end{aligned}\tag{3.29}$$

Substituting the inequality

$$\int_\Omega|\nabla_G u_i(x)|^2 d\mu(x)\le\frac{\lambda_i}{\zeta_1-1}$$

which has been proved in the proof of Theorem 2.2, and the inequalities in (3.26),(3.27) into (3.29), we have

$$\begin{aligned}h\tau\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\le&\epsilon\sigma\zeta_2\left(h+\sum_{j=h+1}^n d^{2p_j}\right)\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\\ &+\frac{\sigma}{\epsilon}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\left[\frac{P\lambda_i}{\zeta_1-1}+C_0\sigma^{\frac12}\left(\frac{\lambda_i}{\zeta_1-1}\right)^{\frac12}+\frac14C_0^2\sigma\right].\end{aligned}\tag{3.30}$$

Taking

$$\epsilon = \left\{ \frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left[\frac{P\lambda_i}{\zeta_1-1} + C_0\sigma^{\frac{1}{2}}\left(\frac{\lambda_i}{\zeta_1-1}\right)^{\frac{1}{2}} + \frac{1}{4}C_0^2\sigma\right]}{\zeta_2\left(h+\sum_{j=h+1}^{n} d^{2p_j}\right)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2} \right\}^{\frac{1}{2}},$$

the right hand of (3.30) attains its minimum. Then we can finish the proof. □

4 Conclusion

We consider the weighted Dirichlet eigenvalue problem of degenerate elliptic operator in divergence form with a potential $-\mathrm{div}_G(A\nabla_G) + \langle A\nabla_G\phi, \nabla_G\rangle - V$. Following the way of Fourier transformation, we obtain the lower bounds for eigenvalues. Through the way of trail function, we get Yang's type inequalities as upper bounds. Especially, we get the corresponding results when the potential function $V(x)$ satisfies the Hardy type inequality $\int_\Omega V f^2 d\mu(x) \leq \int_\Omega |\nabla_G f|^2 d\mu(x)$.

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