Similarity solutions for magnetogasdynamic cylindrical shock waves in a rotating non-ideal gas

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Abstract. The present paper demonstrates the analysis of cylindrical shock waves in a rotating isothermal flow of a non-ideal gas with the impact of the axial magnetic field. We obtain some special class of similarity solutions to the considered problem by using the Lie group of transformations. We assume that the density is uniform in the undisturbed medium, whereas the axial and azimuthal components of the fluid velocity and magnetic field are supposed to vary. By employing the invariant surface conditions, we obtain the generators of the Lie group of transformations. As per the choice of arbitrary constants arising in the expressions for the generators, we obtain four cases of possible solutions. Among all the cases, the similarity solutions are obtained only in three cases. The first and second cases relate to the power and exponential law shock path, respectively, while the third case shows a special case of the power-law shock path. We solve the case of the power-law shock path numerically. Behind the shock front, the distributions of flow variables are analyzed graphically to elucidate the effects of variation in values of the non-ideal parameter, Alfven-Mach number, ambient azimuthal velocity exponent and adiabatic exponent. All the computational work has been performed by using the software package "MATLAB".

1. Introduction

During the past decades, the study of shock waves has received much consideration in the literature because of its applications in several fields, namely plasma physics, astrophysics, nuclear science, space science, aerodynamics, and geophysics, etc. Even medical science is being no longer untouched by the applications of shock waves. Nowadays, the shock waves are being used in kidney stone treatment. In brief, the shock wave can be characterized as a disturbance that propagates at a velocity which is much higher than the sound speed. Shock waves are well-known in the interstellar medium because of their huge variety in supersonic motions and energetic events such as photo-ionized gas, collisions between fast-moving clumps of interstellar gas, star and galaxy formation, the evolution of planets, supernova explosions, stellar winds, etc. To analyze and understand the evolution of various nebulae and internal motion in stars is the problem of great interest in astrophysics. In the past few years, the analysis of shock wave in rotating supersonic and transonic astrophysical fluid flows and black hole accretion has grabbed the remarkable attention of researchers and scientists. The rotations of planets and stars significantly affect the process occurring in their outer atmospheres due to which the question related to

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the explosions in rotating gas atmospheres is of definite astrophysical interest. The cylindrical shock wave propagation in a gaseous substance produced by the rotation of a solid body was studied by Chaturani [1]. The spherical shock waves moving in a non-uniform rotating interplanetary medium with increasing energy was studied by Nath et al. [2]. Vishwakarma and Nath [3] investigated the propagation of cylindrically symmetric shock waves in a rotating dusty gas by considering the effects of heat conduction and radiation heat flux and obtained the similarity solutions to the considered problem. In the context of the study of shock waves in rotating ideal gas, the works of Vishwakarma and Nath [4], Vishwakarma and Vishwakarma [5], Hishida et. al [6], Ganguly and Jana [7], Levin and Skopina [8] and Nath [9] are worth mentioning.

Because of the extreme temperature, the gas does not follow the ideal gas law. So, one must consider the effects of a non-ideal gas. As the shock waves' strength increases, the effect of non-ideal gas becomes more significant and must be incorporated in the experimental and theoretical investigations. Moreover, Zhao et al. [10] reinforced the fact that shock waves show richer behavior in non-ideal gas than that predicted by the ideal gas model. Chauhan et al. [11] described the behavior of the shock wave in an ideal relaxing gas with dust particles. Many researchers analyzed the propagation of the shock waves in non-ideal gas. Ranga Rao and Purohit [12] discussed the self-similar flows of a non-ideal gas in their work. A complete classification of shock waves in van der Waals fluid was discussed by Zhao et al. [10]. Some other remarkable literature about the shock wave propagation in rotating non-ideal gas can be seen in [13]-[15].

The magnetic field has vital roles in the dynamics of the related medium. Shock waves with the effects of the magnetic field might be useful in the description of phenomena in astrophysics. The magnetic field extends over the universe and is very useful in the study of oceanography, hypersonic aerodynamics and atmospheric sciences, etc. The shock propagation with the effect of the magnetic field forms a problem of great interest to scientists and researchers in several branches of science, particularly, in the study of the coronal heating problem. In the last few decades, various studies have been done to investigate the problem of strong shock waves in ideal and non-ideal magnetogasdynamics. Pullin et al. [17] studied the convergence of cylindrically symmetric shock waves within ideal magnetohydrodynamics by employing a finite volume, shock-capturing numerical technique. Singh and Arora [18] used the Lie group method to study the problem of cylindrical shock wave propagation in non-ideal magnetogasdynamics. Towards the study of strong shock waves in magnetogasdynamics, the works of Arora [16], Singh and Arora [19], Singh et al. [20], Radha and Sharma [21], and Hunter and Ali [23] are worth mentioning.

Problems of physical interest are often illustrated as the mathematical models in terms of non-linear partial differential equations. Many complex physical phenomena related to several scientific applications, namely astrophysics, fluid mechanics, plasma physics, nuclear physics, chemical physics, space plasma can be well modeled by non-linear PDEs. Thus, to study and to find out the numerical or analytical solutions of the non-linear PDEs is of great importance. One of the most effective methods to find similarity solutions of non-linear PDEs is the Lie group method. This method is based on the study of their invariance with respect to a one-parameter Lie group of transformations. Once we get to know the Lie group of transformations, under which the PDEs remain invariant, we can construct a solution that is invariant under the transformations. The basic concept of the Lie group of transformations can be found in [24, 25]. The latest literature on the Lie groups and its applications in several fields can be seen in the works of Logan and Perez

[26], Hydon [27], Singh and Arora [18], Singh et al. [20], Sharma and Arora [28] and Jena [29].

In the present article, by employing the Lie group method, we obtain similarity solutions to the problem of one-dimensional cylindrical shock wave propagation in a rotating non-ideal gas with the effect of the axial magnetic field. Here, the flow is considered to be isothermal rather than adiabatic. The assumption of isothermal flow is physically realistic when the radiation heat transfer effects exist implicitly as described in [30, 31, 32]. We determine the infinitesimal generators of the Lie group of transformations, having the arbitrary constants. The various choices of these constants give four cases of possible solutions. Among all the possibilities, only three cases admit the similarity solutions. In the first and third cases, the shock path follows the power law, whereas in the second case shock path follows the exponential law. The first case with the power-law shock path is worked out in detail. Distributions of the flow variables behind the shock front are analyzed graphically to elucidate the effects of variation in the values of the non-ideal parameter, Alfven-Mach number, ambient azimuthal velocity exponent and adiabatic exponent. The present work extends the work of Nath and Singh [33] by taking a non-ideal gas, whereas they have considered ideal gas in their work. Thus, our work is more realistic corresponding to the physical phenomena. To the best of the authors' knowledge, no one has solved the problem under consideration by the Lie group method, which makes this work different from earlier work done.

2. **Equations of motion with R-H conditions**

The system of fundamental equations governing unsteady, one-dimensional and cylindrically symmetric isothermal flow of a rotating non-ideal gas under the effect of axial magnetic field is given by [33]:

$$
\begin{aligned}
&\rho_t + u\rho_r + \rho u_r + \frac{\rho u}{r} = 0,\\
&u_t + uu_r + \frac{1}{\rho}(p_r + \mu h h_r) - \frac{v^2}{r} = 0,\\
&h_t + uh_r + hu_r + \frac{hu}{r} = 0,\\
&v_t + uv_r + \frac{uv}{r} = 0,\\
&w_t + uw_r = 0,\\
&T_r = 0,
\end{aligned}
\tag{2.1}
$$

where t and r are the independent variables describing the time and distance from the axis of symmetry, respectively; ρ, p, h, u, v and w are the gasdynamical quantities representing the density, pressure, axial magnetic field, radial component, azimuthal component and axial component of the fluid velocity $\vec{q}$ in the cylindrical coordinates (r, θ, z), respectively; μ and T are the magnetic permeability and temperature, respectively. The letter subscripts denote the partial derivatives with respect to the indicated variable. We consider the following equation of state and internal energy (e) for the non-ideal gas:

$$
p = \frac{\rho R T}{1 - b\rho}, \qquad e = \frac{p(1 - b\rho)}{(\gamma - 1)\rho}, \tag{2.2}
$$

where γ, b and R represent the adiabatic exponent, the van der Waals excluded gas volume and the gas constant, respectively. Let the shock front $r = \overline{R}(t)$ propagate with velocity $D = d\overline{R}/dt$ in the medium specified by

$$\rho_0 = \text{constant}, \quad p_0 = \text{constant}, \quad h_0 = h_0(r), \quad u_0 = 0, \quad v_0 = v_0(r), \quad w_0 = w_0(r), \tag{2.3}$$

where suffix '0' indicates the flow just ahead of the shock front; h_0, v_0 and w_0 are the functions of r, and ρ_0 and p_0 are the appropriate constants. If u, v, w, ρ, p and h denote the values of the flow variables just behind the shock front, then, the Rankine Hugoniot (R-H) conditions for the strong shock are given as follows [33]:

$$\begin{aligned} & u = (1-\beta)D, \qquad \rho = \frac{\rho_0}{\beta}, \\ & p = \left[1 - \beta + \left(1 - \frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right]\rho_0 D^2, \\ & h = \frac{h_0}{\beta}, \qquad v = v_0, \qquad w = w_0, \end{aligned} \tag{2.4}$$

where $M_A = \sqrt{\frac{\rho_0 D^2}{\mu {h_0}^2}}$ is the Alfven-Mach number. The density ratio $\beta = \frac{\rho_0}{\rho} (0 < \beta < 1)$ is obtained by the following cubic equation:

$$\beta^3 - \beta^2\left[\frac{2\overline{b} + \gamma(1 - M_A^{-2}) - 1}{(\gamma+1)}\right] + \beta\left[\frac{\gamma - 2 + \overline{b}}{(\gamma+1)}\right]M_A^{-2} + \frac{\overline{b}M_A^{-2}}{(\gamma+1)} = 0, \tag{2.5}$$

where $\overline{b} = b\rho_0$ is the non-ideal parameter. Also, we have

$$v = \hat{B}r, \tag{2.6}$$

where $\hat{B}$ is the angular velocity of the non-ideal medium at the distance r from the axis of symmetry. The components of the vorticity vector $\vec{\Omega} = \frac{1}{2}\text{Curl}\vec{q}$ are given by

$$\Omega_r = 0, \qquad \Omega_\theta = -\frac{1}{2}\frac{\partial w}{\partial r}, \qquad \Omega_z = \frac{1}{2r}\frac{\partial (rv)}{\partial r}, \tag{2.7}$$

where $\vec{\Omega} = \Omega_r \hat{e_r} + \Omega_\theta \hat{e_\theta} + \Omega_z \hat{e_z}$. Across the shock front, the jump conditions for the components of vorticity vector are given as follows [8, 9]:

$$\Omega_\theta = \frac{\Omega_{\theta_0}}{\beta}, \qquad \Omega_z = \frac{\Omega_{z_0}}{\beta}. \tag{2.8}$$

3. Similarity Analysis

We consider that for the system of PDEs (2.1), there exists a Lie group of transformation for which the governing system of PDEs reduces into the system of ODEs. We derive the symmetry group of the system (2.1) in order to obtain the similarity solutions such that the system (2.1) remains invariant under this group of transformations. The one-parameter

(ϵ) Lie group of transformation [24, 25, 26] is given as follows:

$$\begin{aligned}
&t^* = t + \epsilon\phi(u,v,w,\rho,p,h,r,t), && r^* = r + \epsilon\psi(u,v,w,\rho,p,h,r,t),\\
&u^* = u + \epsilon U(u,v,w,\rho,p,h,r,t), && v^* = v + \epsilon V(u,v,w,\rho,p,h,r,t),\\
&w^* = w + \epsilon W(u,v,w,\rho,p,h,r,t), && \rho^* = \rho + \epsilon S(u,v,w,\rho,p,h,r,t),\\
&p^* = p + \epsilon P(u,v,w,\rho,p,h,r,t), && h^* = h + \epsilon H(u,v,w,\rho,p,h,r,t),
\end{aligned} \tag{3.1}$$

where $\phi, \psi, U, V, W, S, P$ and H are the generators, which are to be found out in such a manner that the system (2.1) and the shock conditions (2.4) remain invariant under the Lie group of transformations (3.1). Here, we assume the entity ϵ is small enough such that its square and higher power terms can be neglected. The motivation of such a group is that it allows us to reduce the number of independent variables by one in the given system of PDEs (2.1) and, therefore, to transform it into the system of ODEs.

Further, we introduce the symbols $r_1 = t, r_2 = r, u_1 = \rho, u_2 = u, u_3 = h, u_4 = v, u_5 = w, u_6 = p$, and $p^i_j = \frac{\partial u_i}{\partial r_j}$, where $i = 1,2,3,4,5,6$ and $j = 1,2$. The system of equations (2.1) can be written as

$$G_c(r_j, u_i, p^i_j) = 0, \qquad c = 1,2,3,4,5,6. \tag{3.2}$$

The above system is said to be constantly conformally invariant under the Lie group of transformations (3.1), if there exist constants $\alpha_{ca}(c, a = 1,2,3,4,5,6)$, such that

$$ZG_c = \alpha_{ca} G_a, \tag{3.3}$$

for all smooth surfaces $u_i = u_i(r_j)$, where the Lie derivative Z is defined by

$$Z = \xi^j_r \frac{\partial}{\partial r_j} + \xi^i_u \frac{\partial}{\partial u_i} + \xi^i_{p_j} \frac{\partial}{\partial p^i_j}, \tag{3.4}$$

with $\xi^1_r = \phi, \xi^2_r = \psi, \xi^1_u = S, \xi^2_u = U, \xi^3_u = H, \xi^4_u = V, \xi^5_u = W, \xi^6_u = P$, and

$$\xi^i_{p_j} = \frac{\partial \xi^i_u}{\partial r_j} + \frac{\partial \xi^i_u}{\partial u_c} p^c_j - \frac{\partial \xi^l_r}{\partial r_j} p^i_l - \frac{\partial \xi^l_r}{\partial u_n} p^i_l p^n_j \quad (n = 1,2,3,4,5,6,\ l = 1,2) \tag{3.5}$$

being the generalized derivative transformation. Using Eq. (3.4) in Eq. (3.3), we obtain

$$\frac{\partial G_c}{\partial r_j}\xi^j_r + \frac{\partial G_c}{\partial u_i}\xi^i_u + \frac{\partial G_c}{\partial p^i_j}\xi^i_{p_j} = \alpha_{ca} G_a. \tag{3.6}$$

After putting the value of $\xi^i_{p_j}$ from (3.5) into Eq. (3.6), we get a polynomial in p^i_j. We equate the coefficients of p^i_j and $p^i_l p^n_j$ on both sides, and obtain a system of first-order linear PDEs in terms of the generators $\psi, \phi, S, U, P, H, V$ and W. The obtained system is known as the system of determining equations, which we solve further to determine the generators of Lie group of transformations (3.1). From Eqs. $(2.1)_1$ and $(3.6)(c = 1)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&S_\rho - \phi_t - u\phi_r = \alpha_{11}, \quad S_u - \rho\phi_r = \alpha_{12}, \quad S_h = \alpha_{13}, \quad S_p = 0,\\
&S_v = \alpha_{14}, \quad S_w = \alpha_{15}, \quad -\psi_t + uS_\rho - u\psi_r + \rho U_\rho + U = u\alpha_{11} - \frac{p}{\rho}\alpha_{16},\\
&uS_u + \rho U_u + S - \rho\psi_r = \rho\alpha_{11} + u\alpha_{12} + h\alpha_{13}, \quad uS_h + \rho U_h = \frac{\mu h}{\rho}\alpha_{12} + u\alpha_{13},\\
&uS_v + \rho U_v = u\alpha_{14}, \quad uS_w + \rho U_w = u\alpha_{15}, \quad uS_p + \rho U_p = \frac{\alpha_{12}}{\rho} + (1 - b\rho)\alpha_{16},\\
&S_t + uS_r + \rho U_r + \frac{U\rho}{r} + \frac{uS}{r} - \frac{u\rho\psi}{r^2} = \frac{u\rho}{r}\alpha_{11} - \frac{v^2}{r}\alpha_{12} + \frac{uh}{r}\alpha_{13} + \frac{uv}{r}\alpha_{14}.
\end{aligned}
\tag{3.7}
$$

From Eqs. $(2.1)_2$ and $(3.6)(c = 2)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&U_\rho = \alpha_{21}, \qquad U_u - \phi_t - u\phi_r = \alpha_{22}, \qquad U_h - \frac{\mu h}{\rho}\phi_r = \alpha_{23}, \quad U_v = \alpha_{24},\\
&U_w = \alpha_{25}, \qquad U_p - \frac{\phi_r}{\rho} = 0, \qquad uU_\rho + \frac{1}{\rho}P_\rho + \frac{\mu h}{\rho}H_\rho = u\alpha_{21} - \frac{p}{\rho}\alpha_{26},\\
&uU_u - \psi_t - u\psi_r + \frac{1}{\rho}P_u + \frac{\mu h}{\rho}H_u + U = \rho\alpha_{21} + u\alpha_{22} + h\alpha_{23},\\
&uU_h + \frac{1}{\rho}P_h + \frac{\mu h}{\rho}H_h - \frac{\mu h}{\rho}\psi_r + \frac{\mu}{\rho}H - \frac{\mu h}{\rho^2}S = \frac{\mu h}{\rho}\alpha_{22} + u\alpha_{23},\\
&uU_v + \frac{1}{\rho}P_v + \frac{\mu h}{\rho}H_v = u\alpha_{24}, \qquad uU_w + \frac{1}{\rho}P_w + \frac{\mu h}{\rho}H_w = u\alpha_{25},\\
&uU_p + \frac{1}{\rho}P_p - \frac{1}{\rho}\psi_r + \frac{\mu h}{\rho}H_p - \frac{1}{\rho^2}S = \frac{1}{\rho}\alpha_{22} + (1 - b\rho)\alpha_{26},\\
&U_t + uU_r + \frac{1}{\rho}(P_r + \mu h H_r) - \frac{2vV}{r} + \frac{v^2}{r^2}\psi = \frac{u\rho}{r}\alpha_{21} - \frac{v^2}{r}\alpha_{22} + \frac{uh}{r}\alpha_{23} + \frac{uv}{r}\alpha_{24}.
\end{aligned}
\tag{3.8}
$$

From Eqs. $(2.1)_3$ and $(3.6)(c = 3)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&H_\rho = \alpha_{31}, \qquad H_u - h\phi_r = \alpha_{32}, \qquad H_h - \phi_t - u\phi_r = \alpha_{33}, \qquad H_v = \alpha_{34},\\
&H_w = \alpha_{35}, \qquad H_p = 0, \qquad uH_\rho + hU_\rho = u\alpha_{31} - \frac{p}{\rho}\alpha_{36}, \qquad uH_v + hU_v = u\alpha_{34},\\
&uH_u + hU_u + H - h\psi_r = \rho\alpha_{31} + u\alpha_{32} + h\alpha_{33}, \qquad uH_w + hU_w = u\alpha_{35},\\
&-\psi_t + uH_h - u\psi_r + hU_h + U = \frac{\mu h}{\rho}\alpha_{32} + u\alpha_{33}, \quad uH_p + hU_p = \frac{1}{\rho}\alpha_{32} + (1 - b\rho)\alpha_{36},\\
&H_t + uH_r + hU_r + \frac{uH}{r} + \frac{hU}{r} - \frac{uh}{r^2}\psi = \frac{u\rho}{r}\alpha_{31} - \frac{v^2}{r}\alpha_{32} + \frac{uh}{r}\alpha_{33} + \frac{uv}{r}\alpha_{34}.
\end{aligned}
\tag{3.9}
$$

From Eqs. $(2.1)_4$ and $(3.6)(c=4)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&V_\rho = \alpha_{41}, \qquad V_u = \alpha_{42}, \qquad V_h = \alpha_{43}, \qquad V_u - \phi_t - u\phi_r = \alpha_{44}, \qquad V_w = \alpha_{45},\\
&V_p = 0, \qquad uV_\rho = u\alpha_{41} - \frac{p}{\rho}\alpha_{46}, \qquad uV_u = \rho\alpha_{41} + u\alpha_{42} + h\alpha_{43}, \qquad uV_w = u\alpha_{45},\\
&uV_h = \frac{\mu h}{\rho}\alpha_{42} + u\alpha_{43}, \quad -\psi_t + uV_v - u\psi_r + U = u\alpha_{44}, \qquad uV_p = \frac{1}{\rho\alpha_{42}} + (1-b\rho)\alpha_{46},\\
&V_t + uV_r + \frac{vU}{r} + \frac{uV}{r} - \frac{uv}{r^2}\psi = \frac{u\rho}{r}\alpha_{41} - \frac{v^2}{r}\alpha_{42} + \frac{uh}{r}\alpha_{43} + \frac{uv}{r}\alpha_{44}.
\end{aligned}
\tag{3.10}
$$

From Eqs. $(2.1)_5$ and $(3.6)(c=5)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&W_\rho = \alpha_{51}, \qquad W_u = \alpha_{52}, \qquad W_h = \alpha_{53}, \qquad W_v = \alpha_{54}, \qquad W_w - \phi_t - u\phi_r = \alpha_{55},\\
&W_p = 0, \qquad uW_\rho = u\alpha_{51} - \frac{p}{\rho}\alpha_{56}, \qquad uW_u = \rho\alpha_{51} + u\alpha_{52} + h\alpha_{53}, \qquad uW_v = u\alpha_{54},\\
&uW_h = \frac{\mu h}{\rho}\alpha_{52} + u\alpha_{53}, \qquad -\psi_t + uW_w - u\psi_r + U = u\alpha_{55},\\
&uW_p = \frac{1}{\rho\alpha_{52}} + (1-b\rho)\alpha_{56}, \qquad W_t + uW_r = \frac{u\rho}{r}\alpha_{51} - \frac{v^2}{r}\alpha_{52} + \frac{uh}{r}\alpha_{53} + \frac{uv}{r}\alpha_{54}.
\end{aligned}
\tag{3.11}
$$

From Eqs. $(2.1)_6$ and $(3.6)(c=6)$, the determining equations are obtained as follows:

$$
\begin{aligned}
&\frac{p}{\rho}\phi_r = \alpha_{61}, \qquad \alpha_{62} = 0, \qquad \alpha_{63} = 0, \qquad \alpha_{64} = 0, \qquad \alpha_{65} = 0,\\
&(1-b\rho)\phi_r = 0, \qquad (1-b\rho)P_\rho - \frac{p}{\rho}S_\rho + \frac{p}{\rho}\psi_r - \frac{P}{\rho} + \frac{p}{\rho^2}S = u\alpha_{61} - \frac{p}{\rho}\alpha_{66},\\
&(1-b\rho)P_u - \frac{p}{\rho}S_u = \rho\alpha_{61} + u\alpha_{62} + h\alpha_{63}, \qquad (1-b\rho)P_h - \frac{p}{\rho}S_h = \frac{\mu h}{\rho}\alpha_{62} + u\alpha_{63},\\
&(1-b\rho)P_v - \frac{p}{\rho}S_v = u\alpha_{64}, \qquad w - \frac{p}{\rho}S_w = u\alpha_{65},\\
&(1-b\rho)(P_p - \psi_r) - \frac{p}{\rho}S_p + (1-bS) = \frac{1}{\rho}\alpha_{62} + (1-b\rho)\alpha_{66},\\
&(1-b\rho)P_r - \frac{p}{\rho}S_r = \frac{u\rho}{r}\alpha_{61} - \frac{v^2}{r}\alpha_{62} + \frac{uh}{r}\alpha_{63} + \frac{uv}{r}\alpha_{64}.
\end{aligned}
\tag{3.12}
$$

After solving the systems of determining equations (3.7)-(3.12) simultaneously, the generators of the Lie group of transformations (3.1) are obtained as follows:

$$
\begin{aligned}
&\phi = kt + d, && \psi = (\alpha_{22} + 2k)r,\\
&S = 0, && U = (\alpha_{22} + k)u,\\
&P = 2(\alpha_{22} + k)p, && H = (\alpha_{22} + k)h,\\
&V = (\alpha_{22} + k)v, && W = (\alpha_{55} + k)w,
\end{aligned}
\tag{3.13}
$$

where $\alpha_{22}, \alpha_{55}, k$ and d are the arbitrary constants.

4. **Similarity solutions**

As per the choice of arbitrary constants in the expressions for the infinitesimal generators, given by Eq. (3.13), possible four cases of solutions are obtained, and they are described as follows:

Case 1: $k \neq 0$ **and** $(\alpha_{22} + 2k) \neq 0$**.**

Here, we generate new variables $\tilde{r}$ and $\tilde{t}$ given as below:

$$\tilde{r} = r, \qquad \tilde{t} = t + \frac{d}{k}. \tag{4.1}$$

Now, the new set of generators in terms of $\tilde{r}$ and $\tilde{t}$ after dropping the tilde sign are obtained from Eq. (3.13) as

$$\begin{aligned} &\phi = kt, && \psi = (\alpha_{22} + 2k)r, \\ &S = 0, && U = (\alpha_{22} + k)u, \\ &P = 2(\alpha_{22} + k)p, && H = (\alpha_{22} + k)h, \\ &V = (\alpha_{22} + k)v, && W = (\alpha_{55} + k)w. \end{aligned} \tag{4.2}$$

The invariant surface conditions [26] are given as follows:

$$\begin{aligned} &\phi\rho_t + \psi\rho_r = S, \qquad \phi u_t + \psi u_r = U, \qquad \phi p_t + \psi p_r = P, \\ &\phi h_t + \psi h_r = H, \qquad \phi v_t + \psi v_r = V, \qquad \phi w_t + \psi w_r = W. \end{aligned} \tag{4.3}$$

On solving Eqs. (4.3) and (4.2), we obtain the following forms of the flow variables:

$$\begin{aligned} &\rho = \hat{S}(y), && u = t^{\delta-1}\hat{U}(y), && p = t^{2(\delta-1)}\hat{P}(y), \\ &h = t^{\delta-1}\hat{H}(y), && v = t^{\delta-1}\hat{V}(y), && w = t^{(\alpha_{55}+k)/k}\hat{W}(y), \end{aligned} \tag{4.4}$$

where $\delta = \frac{(\alpha_{22}+2k)}{k}$. The functions $\hat{S}, \hat{U}, \hat{P}, \hat{H}, \hat{V}$ and $\hat{W}$ depend only on the similarity variable y, which is given by

$$y = \frac{r}{t^{\delta}}. \tag{4.5}$$

Let $y = 1$ denotes the basic position of the shock front, then, the shock path $\overline{R}(t)$ and the shock velocity D are given by

$$\overline{R}(t) = t^{\delta}, \qquad D = \delta t^{\delta-1}. \tag{4.6}$$

It is obvious from the above equation that the shock path obeys the power law. At $y = 1$ the flow variables from Eq. (4.4) are given as follows:

$$\begin{aligned} &\rho|_{y=1} = \hat{S}(1), && u|_{y=1} = t^{\delta-1}\hat{U}(1), && p|_{y=1} = t^{2(\delta-1)}\hat{P}(1), \\ &h|_{y=1} = t^{\delta-1}\hat{H}(1), && v|_{y=1} = t^{\delta-1}\hat{V}(1), && w|_{y=1} = t^{(\alpha_{55}+k)/k}\hat{W}(1). \end{aligned} \tag{4.7}$$

The invariance of the shock conditions (2.4), yields the following forms of ρ_0, h_0, v_0 and w_0:

$$\rho_0 = \rho^*, \qquad h_0 = h^* r^{-\theta}, \qquad v_0 = v^* r^{\lambda_1}, \qquad w_0 = w^* r^{\lambda_2}, \tag{4.8}$$

where

$$\lambda_1 = -\theta = \frac{\delta - 1}{\delta} = \frac{\alpha_{22} + k}{\alpha_{22} + 2k}, \quad \lambda_2 = \frac{\alpha_{55} + k}{\alpha_{22} + 2k}. \tag{4.9}$$

Here, $\rho^*, h^*, v^*, w^*, \theta, \lambda_1$ and λ_2 are constants, and the following relation holds:

$$\lambda_1 = -\theta = \frac{\rho^* {v^*}^2}{\mu {h^*}^2}. \tag{4.10}$$

The following equation gives the value of angular velocity in the undisturbed medium:

$$\hat{B}_0 = v^* r^{\lambda_1 - 1}, \tag{4.11}$$

where $\lambda_1 = 1$ shows that angular velocity is constant. The components of the vorticity vector in the undisturbed medium vary as follows:

$$\Omega_{r_0} = 0, \qquad \Omega_{\theta_0} = -\frac{w^* \lambda_2}{2} r^{\lambda_2 - 1}, \qquad \Omega_{z_0} = \frac{(\lambda_1 + 1) v^*}{2} r^{\lambda_1 - 1}. \tag{4.12}$$

In this case, the condition for M_A to be constant is given as

$$2\delta + 2\delta\theta - 2 = 0. \tag{4.13}$$

Now, from Eqs. (2.4), (4.7) and (4.8), the boundary conditions for the functions $\hat{S}, \hat{U}, \hat{P}, \hat{H}, \hat{V}$ and $\hat{W}$ are given as:

$$\begin{aligned}
&\hat{S}(1) = \frac{\rho^*}{\beta}, \qquad \hat{U}(1) = (1-\beta)\delta, \\
&\hat{P}(1) = \left[1 - \beta + \left(1 - \frac{1}{\beta^2}\right) \frac{M_A^{-2}}{2}\right] \rho^* \delta^2, \\
&\hat{H}(1) = \frac{\delta\sqrt{\rho^*}}{\sqrt{\mu} M_A \beta}, \qquad \hat{V}(1) = v^*, \qquad \hat{W} = w^*.
\end{aligned} \tag{4.14}$$

Here, $\lambda_1 = \lambda_2$ is necessary to use for obtaining the similarity solutions. Using Eqs. (4.4), (4.6) and (4.8), we can re-write the flow variables as given below:

$$\begin{aligned}
&\rho = \rho_0 \overline{S}(y), \qquad u = D\overline{U}(y), \qquad p = \rho_0 D^2 \overline{P}(y), \\
&\sqrt{\mu} h = \sqrt{\rho_0} D\overline{H}(y), \qquad v = D\overline{V}(y), \qquad w = D\overline{W}(y),
\end{aligned} \tag{4.15}$$

where $\overline{S} = \frac{\hat{S}(y)}{\rho^*}, \overline{U} = \frac{\hat{U}(y)}{\delta}, \overline{P} = \frac{\hat{P}(y)}{\rho^* \delta^2}, \overline{H} = \frac{\sqrt{\mu}\hat{H}(y)}{\sqrt{\rho^*}\delta}, \overline{V} = \frac{\hat{V}(y)}{\delta}$ and $\overline{W} = \frac{\hat{W}(y)}{\delta}$. Substituting Eq. (4.15) into Eq. (2.1), and using Eqs. (4.5) and (4.6), we have the following system of ODEs in terms of $\overline{S}, \overline{U}, \overline{P}, \overline{H}, \overline{V}$ and $\overline{W}$ which on suppressing the bar sign becomes:

$$\begin{aligned}
&(U - y)S' + SU' + \frac{SU}{y} = 0, \\
&\frac{(\delta - 1)}{\delta} US + (U - y)SU' + P' + HH' - \frac{V^2 S}{y} = 0, \\
&\frac{(\delta - 1)}{\delta} H + (U - y)H' + HU' + \frac{HU}{y} = 0, \\
&\frac{(\delta - 1)}{\delta} V + (U - y)V' + \frac{VU}{y} = 0, \\
&\frac{(\delta - 1)}{\delta} W + (U - y)W' = 0, \\
&(1 - \bar{b}S)SP' - PS' = 0,
\end{aligned} \tag{4.16}$$

where the prime represents the differentiation with respect to the similarity variable y. The non-dimensional components of the vorticity vector, $\chi_r = \frac{\Omega_r}{D/R}, \chi_\theta = \frac{\Omega_\theta}{D/R}, \chi_z = \frac{\Omega_z}{D/R}$, are given from Eqs. (2.7) and (4.15) as follows:

$$\begin{aligned}\chi_r &= 0,\\ \chi_\theta &= \frac{1}{2}\frac{\lambda_1 W}{(U-y)},\\ \chi_z &= \frac{V}{2y(U-y)}\left[(U-y) - y\left(\lambda_1 + \frac{U}{y}\right)\right].\end{aligned} \tag{4.17}$$

Using Eqs. (4.14) and (4.15), the boundary conditions are obtained as follows, after suppressing the bar sign:

$$\begin{aligned}&S(1) = \frac{1}{\beta}, \qquad U(1) = 1-\beta,\\ &P(1) = \left[1-\beta+\left(1-\frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right],\\ &H(1) = \frac{1}{M_A\beta}, \qquad V(1) = \frac{v^*}{\delta}, \qquad W(1) = \frac{w^*}{\delta}.\end{aligned} \tag{4.18}$$

The system of equations (4.16) and Eq. (4.17) along with the boundary conditions (4.18) can be solved numerically to obtain the solutions for the present case.

Case 2: $k = 0$ and $\alpha_{22} \neq 0$.

This case leads to the following forms of the generators in Eq. (3.13):

$$\begin{aligned}&\phi = d, \qquad \psi = \alpha_{22} r, \qquad S = 0, \qquad U = \alpha_{22} u,\\ &P = 2\alpha_{22} p, \qquad H = \alpha_{22} h, \qquad V = \alpha_{22} v, \qquad W = \alpha_{55} w.\end{aligned} \tag{4.19}$$

Integrating Eqs. (4.3) together with Eqs. (4.19), we obtain the flow variables as follows:

$$\begin{aligned}&\rho = \hat{S}(y), \qquad u = e^{\delta t}\hat{U}(y), \qquad p = e^{2\delta t}\hat{P}(y),\\ &h = e^{\delta t}\hat{H}(y), \qquad v = e^{\delta t}\hat{V}(y), \qquad w = e^{\frac{\alpha_{55}}{d}t}\hat{W}(y),\end{aligned} \tag{4.20}$$

where $\delta = \frac{\alpha_{22}}{d}$. The similarity variable y, shock path $\overline{R}(t)$ and shock velocity D are described as

$$y = \frac{r}{e^{\delta t}}, \qquad \overline{R}(t) = e^{\delta t}, \qquad D = \delta e^{\delta t}. \tag{4.21}$$

From the above equation, we observe that the shock path is exponentially varying. For the present case, the forms of ρ_0, h_0, v_0 and w_0 are given by

$$\rho_0 = \rho^*, \qquad h_0 = h^* r^{-\theta}, \qquad v_0 = v^* r^{\lambda_1}, \qquad w_0 = w^* r^{\lambda_2}, \tag{4.22}$$

where

$$\theta = -1, \quad \lambda_1 = 1, \quad \lambda_2 = \frac{\alpha_{55}}{\alpha_{22}}. \tag{4.23}$$

In this case, M_A remains constant for all values of δ. At $y = 1$, the flow variables are written as follows:

$$\begin{aligned}&\rho|_{y=1} = \hat{S}(1), \qquad u|_{y=1} = e^{\delta t}\hat{U}(1), \qquad p|_{y=1} = e^{2\delta t}\hat{P}(1),\\ &h|_{y=1} = e^{\delta t}\hat{H}(1), \qquad v|_{y=1} = e^{\delta t}\hat{V}(1), \qquad w|_{y=1} = e^{\frac{\alpha_{55}}{d}t}\hat{W}(1).\end{aligned} \tag{4.24}$$

From Eqs. (2.4), (4.22) and (4.24), we have the following boundary conditions for the functions $\hat{S}, \hat{U}, \hat{P}, \hat{H}, \hat{V}$ and $\hat{W}$:

$$\begin{aligned} &\hat{S}(1) = \frac{\rho^*}{\beta}, \qquad \hat{U}(1) = (1-\beta)\delta, \\ &\hat{P}(1) = \left[1 - \beta + \left(1 - \frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right]\rho^*\delta^2, \\ &\hat{H}(1) = \frac{\delta\sqrt{\rho^*}}{\sqrt{\mu}M_A\beta}, \qquad \hat{V}(1) = v^*, \qquad \hat{W} = w^*, \end{aligned} \tag{4.25}$$

where $\lambda_1 = \lambda_2$ is necessary for obtaining the similarity solutions. From Eqs. (4.20)-(4.22), the flow variables are obtained as follows:

$$\begin{aligned} &\rho = \rho_0\overline{S}(y), \qquad u = D\overline{U}(y), \qquad p = \rho_0 D^2\overline{P}(y), \\ &\sqrt{\mu}h = \sqrt{\rho_0}D\overline{H}(y), \qquad v = D\overline{V}(y), \qquad w = D\overline{W}(y), \end{aligned} \tag{4.26}$$

where $\overline{S} = \frac{\hat{S}(y)}{\rho^*}, \overline{U} = \frac{\hat{U}(y)}{\delta}, \overline{P} = \frac{\hat{P}(y)}{\rho^*\delta^2}, \overline{H} = \frac{\sqrt{\mu}\hat{H}(y)}{\sqrt{\rho^*}\delta}, \overline{V} = \frac{\hat{V}(y)}{\delta}$ and $\overline{W} = \frac{\hat{W}(y)}{\delta}$. Substituting Eqs. (4.26) into Eqs. (2.1), and using Eqs. (4.21), we get the system of ODEs, after dropping the bar sign as follows:

$$\begin{aligned} &(U-y)S' + SU' + \frac{SU}{y} = 0, \\ &US + (U-y)SU' + P' + HH' - \frac{V^2S}{y} = 0, \\ &H + (U-y)H' + HU' + \frac{HU}{y} = 0, \\ &V + (U-y)V' + \frac{VU}{y} = 0, \\ &W + (U-y)W' = 0, \\ &(1-\bar{b}S)SP' - PS' = 0, \end{aligned} \tag{4.27}$$

where the prime represents the differentiation with respect to the independent variable y. The non-dimensional components of the vorticity vector for this case are obtained from Eqs. (2.7) and (4.26) as follows:

$$\begin{aligned} &\chi_r = 0, \\ &\chi_\theta = \frac{1}{2}\frac{W}{(U-y)}, \\ &\chi_z = -\frac{V}{(U-y)}. \end{aligned} \tag{4.28}$$

Using Eqs. (4.25) and (4.26), the boundary conditions after suppressing the bar sign are obtained as follows:

$$\begin{aligned} &S(1) = \frac{1}{\beta}, \qquad U(1) = 1-\beta, \\ &P(1) = \left[1-\beta+\left(1-\frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right], \\ &H(1) = \frac{1}{M_A\beta}, \qquad V(1) = \frac{v^*}{\delta}, \qquad W(1) = \frac{w^*}{\delta}. \end{aligned} \tag{4.29}$$

The system of equations (4.27) and Eq. (4.28) together with boundary conditions (4.29) can be solved numerically to obtain the solutions for this case.

Case 3: $k \neq 0$ **and** $\alpha_{22} = 0$**.**

We define new variable $\tilde{r}$ and $\tilde{t}$ as follows:

$$\tilde{r} = r, \qquad \tilde{t} = t + \frac{d}{k}.$$

So, the new set of generators in terms of $\tilde{r}$ and $\tilde{t}$ after dropping the tilde sign are obtained from Eq. (3.13) as

$$\begin{aligned} &\phi = kt, \qquad \psi = 2kr, \qquad S = 0, \qquad U = ku, \\ &P = 2kp, \qquad H = kh, \qquad V = kv, \qquad W = (\alpha_{55}+k)w. \end{aligned} \tag{4.30}$$

On integrating Eq. (4.3) together with the Eq. (4.30), we obtain the following forms of the flow variables:

$$\begin{aligned} &\rho = \hat{S}(y), \qquad u = t\hat{U}(y), \qquad p = t^2\hat{P}(y), \\ &h = t\hat{H}(y), \qquad v = t\hat{V}(y), \qquad w = t^{\frac{\alpha_{55}+k}{k}}\hat{W}(y), \end{aligned} \tag{4.31}$$

where the similarity variable y, shock path $\overline{R}(t)$ and shock velocity D are given as

$$y = \frac{r}{t^2}, \qquad \overline{R}(t) = t^2, \qquad D = 2t. \tag{4.32}$$

This is a particular case of the power law shock path ($\overline{R}(t) = t^2$) with shock velocity $D = 2t$. For the present case, the forms of ρ_0, h_0, v_0 and w_0 are given by:

$$\rho_0 = \rho^*, \qquad h_0 = h^* r^{-\theta}, \qquad v_0 = v^* r^{\lambda_1}, \qquad w_0 = w^* r^{\lambda_2}, \tag{4.33}$$

where

$$\theta = -\frac{1}{2}, \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{\alpha_{55}+k}{2k}. \tag{4.34}$$

M_A remains constant for all values of δ for this case also. At $y = 1$, the flow variables are given as follows:

$$\begin{aligned} &\rho|_{y=1} = \hat{S}(1), \qquad u|_{y=1} = t\hat{U}(1), \qquad p|_{y=1} = t^2\hat{P}(1), \\ &h|_{y=1} = t\hat{H}(1), \qquad v|_{y=1} = t\hat{V}(1), \qquad w|_{y=1} = t^{\frac{\alpha_{55}+k}{k}}\hat{W}(1). \end{aligned} \tag{4.35}$$

From Eqs. (2.4), (4.33) and (4.35), we have the following boundary conditions for the functions $\hat{S}, \hat{U}, \hat{P}, \hat{H}, \hat{V}$ and $\hat{W}$:

$$\begin{aligned}
&\hat{S}(1) = \frac{\rho^*}{\beta}, \qquad \hat{U}(1) = 2(1-\beta), \\
&\hat{P}(1) = 4\left[1 - \beta + \left(1 - \frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right]\rho^*, \\
&\hat{H}(1) = \frac{2\sqrt{\rho^*}}{\sqrt{\mu}M_A\beta}, \qquad \hat{V}(1) = v^*, \qquad \hat{W} = w^*.
\end{aligned} \tag{4.36}$$

where $\lambda_1 = \lambda_2$ is necessary for obtaining the similarity solutions. From Eqs. (4.31)-(4.33), the flow variables are obtained as follows:

$$\begin{aligned}
&\rho = \rho_0 \overline{S}(y), \qquad u = D\overline{U}(y), \qquad p = \rho_0 D^2 \overline{P}(y), \\
&\sqrt{\mu}h = \sqrt{\rho_0}D\overline{H}(y), \qquad v = D\overline{V}(y), \qquad w = D\overline{W}(y),
\end{aligned} \tag{4.37}$$

where $\overline{S} = \frac{\hat{S}(y)}{\rho^*}, \overline{U} = \frac{\hat{U}(y)}{2}, \overline{P} = \frac{\hat{P}(y)}{4\rho^*}, \overline{H} = \frac{\sqrt{\mu}\hat{H}(y)}{2\sqrt{\rho^*}}, \overline{V} = \frac{\hat{V}(y)}{2}$ and $\overline{W} = \frac{\hat{W}(y)}{2}$. Substituting Eqs. (4.37) into Eqs. (2.1), and using Eqs. (4.32), we obtain the following system of ODEs after dropping the bar sign:

$$\begin{aligned}
&(U - y)S' + SU' + \frac{SU}{y} = 0, \\
&\frac{1}{2}US + (U - y)SU' + P' + HH' - \frac{V^2 S}{y} = 0, \\
&\frac{1}{2}H + (U - y)H' + HU' + \frac{HU}{y} = 0, \\
&\frac{1}{2}V + (U - y)V' + \frac{VU}{y} = 0, \\
&\frac{1}{2}W + (U - y)W' = 0, \\
&(1 - \bar{b}S)SP' - PS' = 0,
\end{aligned} \tag{4.38}$$

where the prime represents the differentiation with respect to the independent variable y. As in previous case, we obtain the non-dimensional components of the vorticity vector for this case using Eqs. (2.7) and (4.37) as follows:

$$\begin{aligned}
&\chi_r = 0, \\
&\chi_\theta = \frac{1}{4}\frac{W}{(U - y)}, \\
&\chi_z = \frac{1}{2y}\left[V - \frac{yV}{(U - y)}\left(\frac{1}{2} + \frac{U}{y}\right)\right].
\end{aligned} \tag{4.39}$$

Using Eqs. (4.36) and (4.37), the boundary conditions after suppressing the bar sign are obtained as follows:

$$\begin{aligned} &S(1) = \frac{1}{\beta}, \qquad U(1) = 1-\beta, \\ &P(1) = \left[1-\beta+\left(1-\frac{1}{\beta^2}\right)\frac{M_A^{-2}}{2}\right], \\ &H(1) = \frac{1}{M_A\beta}, \qquad V(1) = \frac{v^*}{2}, \qquad W(1) = \frac{w^*}{2}. \end{aligned} \tag{4.40}$$

The system of Eqs. (4.38) and Eqs. (4.39) together with boundary conditions (4.40) can be integrated numerically to obtain the solutions for this case.

Case 4: $k = 0$ **and** $\alpha_{22} = 0$**.**

In this case, similarity solutions do not exist.

5. **Numerical results and discussion**

In this section, we discuss Case 1 in detail. For obtaining the distributions of the flow variables, namely the density $S(y)$, pressure $P(y)$, magnetic field $H(y)$, radial fluid velocity $U(y)$, azimuthal fluid velocity $V(y)$, axial fluid velocity $W(y)$, azimuthal component $\chi_\theta(y)$ and axial component $\chi_z(y)$ of vorticity vector in Case 1, we integrate the system of ODEs (4.16) and Eqs. (4.17) together with the boundary conditions (4.18) numerically by using the Runge-Kutta method of 4^{th}-order with the help of the software package "MATLAB". We obtain the distributions of the flow variables behind the shock front. For numerical calculations, we take the values of physical parameters as follows:
$\gamma = 4/3, 5/3, \bar{b} = 0, 0.05, 0.1, M_A^{-2} = 0, 0.02, 0.04, \lambda_1 = 0.2, 0.3, 0.4, v^* = 1, w^* = 1$ [33, 34]. The values $\gamma = 4/3$ and $\gamma = 5/3$ stand for relativistic gas and fully ionized gas, respectively, therefore, appropriate for the stellar medium. These two values of γ mark the most general values observed in real stars. According to Rosenau and Frankenthal [35], the effect of the magnetic field on the flow-field behind the shock wave is noteworthy when $M_A^{-2} \geq 0.01$; therefore, the above values of M_A^{-2} are taken for calculation in the present problem. The values $\bar{b} = 0$ and $M_A^{-2} = 0$ correspond to ideal case and non-magnetic case, respectively. The similarity exponent δ is found from Eq. (4.9) as $\delta = \frac{1}{1-\lambda_1}$.

Table 1 shows the values of density ratio β for different values of $\gamma, \bar{b}$ and M_A^{-2}. We have used these values in numerical calculations. From Table 1, we observe that β increases with an increase in the value of any of the parameters $\gamma, \bar{b}$ or M_A^{-2}. Figures 1-4 exhibit the distributions of the non-dimensional flow variables $\frac{S(y)}{S(1)}, \frac{P(y)}{P(1)}, \frac{H(y)}{H(1)}, \frac{U(y)}{U(1)}, \frac{V(y)}{V(1)}, \frac{W(y)}{W(1)}, \chi_\theta$ and χ_z with respect to the similarity variable y for different values of $\bar{b}, M_A^{-2}, \lambda_1$ and γ. From Figs. 1-4, it is observed that the reduced density, pressure, radial fluid velocity and the axial component of vorticity vector increase, whereas the reduced magnetic field, azimuthal fluid velocity, axial fluid velocity and the azimuthal component of the vorticity vector decrease monotonically on moving from the shock front $y = 1$ to the axis of symmetry. The variations in non-ideal parameter, Alfven-Mach number, ambient azimuthal velocity exponent and adiabatic exponent have remarkable effects on the flow variables which are discussed as follows:

5.1. Effect of non-ideal parameter $\bar{b}$ on the flow variables.

From Fig. 1, we can see the effect of the non-ideal parameter $\bar{b}$ on the profiles of the flow variables for $\gamma = 5/3, M_A^{-2} = 0.02$ and $\lambda_1 = 0.3$. We observe that the increment in the parameter $\bar{b}$ causes the reduced density, pressure, radial fluid velocity and the axial component of the vorticity vector to decrease (see Figs. 1(a, b, d, h)), whereas the reduced magnetic field, azimuthal fluid velocity, axial fluid velocity and the azimuthal component of the vorticity vector to increase (see Figs. 1(c, e, f, g)) behind the shock front.

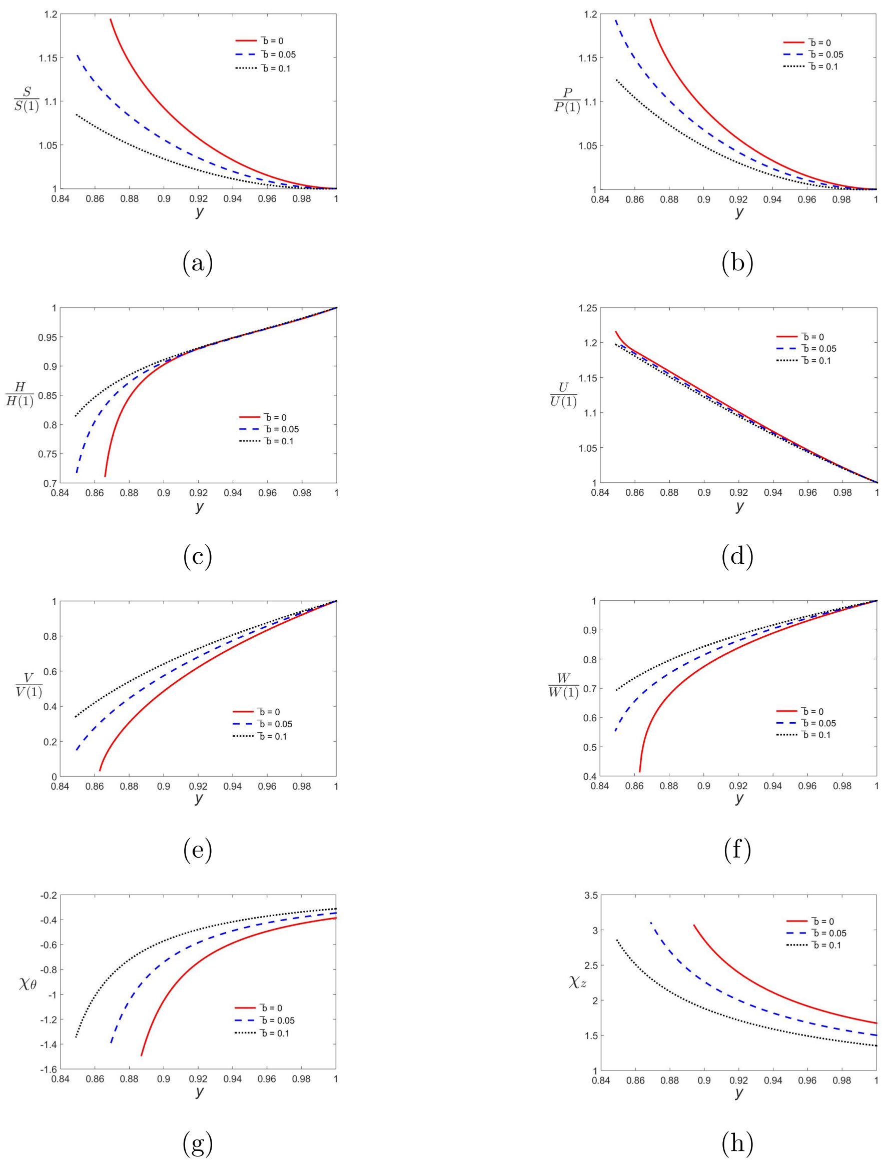

Fig. 1. Flow patterns with $\gamma = 5/3, M_A^{-2} = 0.02$ and $\lambda_1 = 0.3$.

5.2. Effect of the parameter M_A^{-2} on the flow variables.

Figure 2 depicts the effect of M_A^{-2} (the strength of ambient magnetic field) on the profiles of the flow variables for $\gamma = 5/3, \bar{b} = 0.05$ and $\lambda_1 = 0.3$. As we increase the value of M_A^{-2}, all the flow variables except the axial component of vorticity vector increase (see Figs. 2(a, b, c, d, e, f, g)). The axial component of vorticity vector decreases with an increase in the value of M_A^{-2} behind the shock front (see Fig. 2(h)).

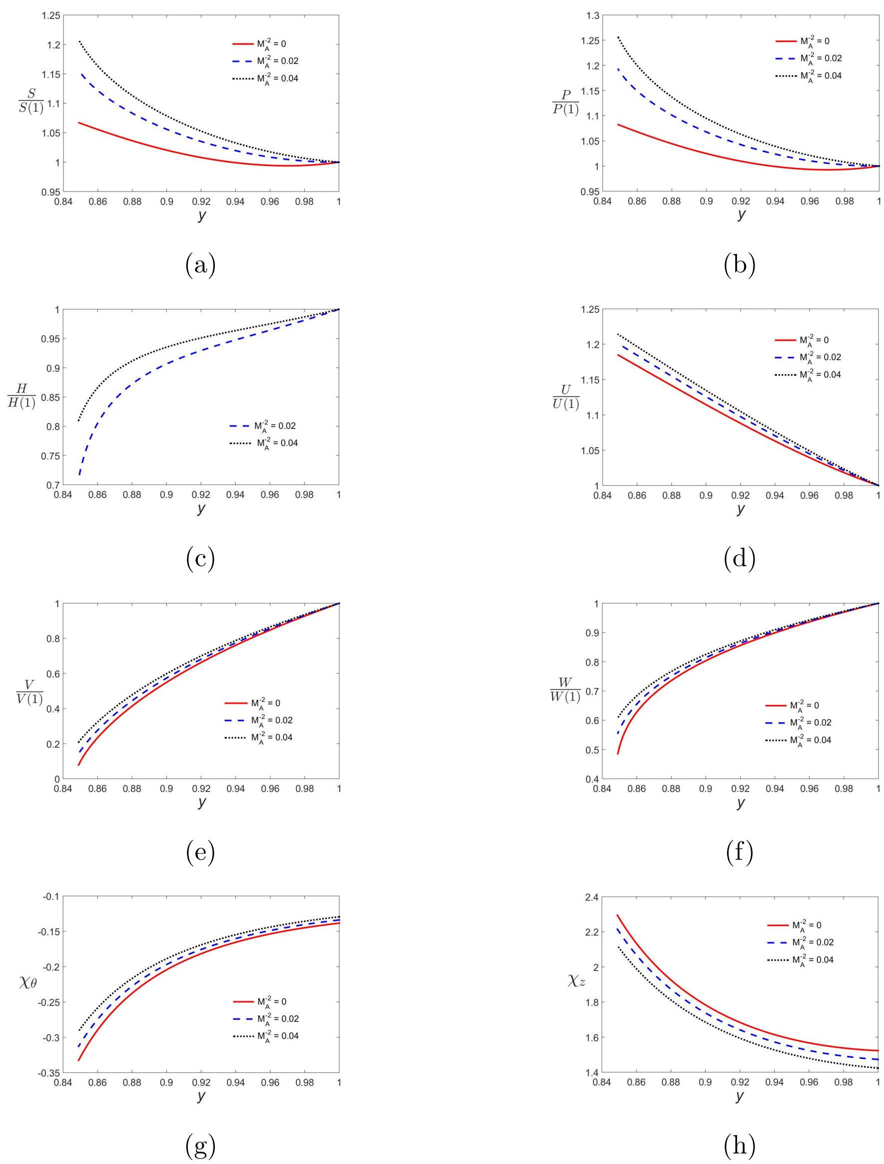

Fig. 2. Flow patterns with $\gamma = 5/3, \bar{b} = 0.05$ and $\lambda_1 = 0.3$.

5.3. **Effects of ambient azimuthal or axial fluid velocity exponent $\lambda_1 (= \lambda_2)$ on the flow variables.**

Figure 3 shows the effects of $\lambda_1 (= \lambda_2)$ on the profiles of the flow variables for $\gamma = 5/3, \bar{b} = 0.05$ and $M_A^{-2} = 0.02$. On increasing the value of λ_1, the reduced density, pressure, radial fluid velocity increase (see Figs. 3(a, b, d)), whereas the reduced azimuthal fluid velocity, axial fluid velocity and the azimuthal component of vorticity vector decrease (see Figs. 3(e, f, g)) as we move towards the axis of symmetry from the shock front. The reduced magnetic field increases near the shock front and decreases near the axis of symmetry (see Fig. 3(c)), whereas the axial component of vorticity vector has reverse effect, i.e., it decreases near the shock front and increases near the axis of symmetry (see Fig. 3(h)) with an increase in the value of $\lambda_1 (= \lambda_2)$.

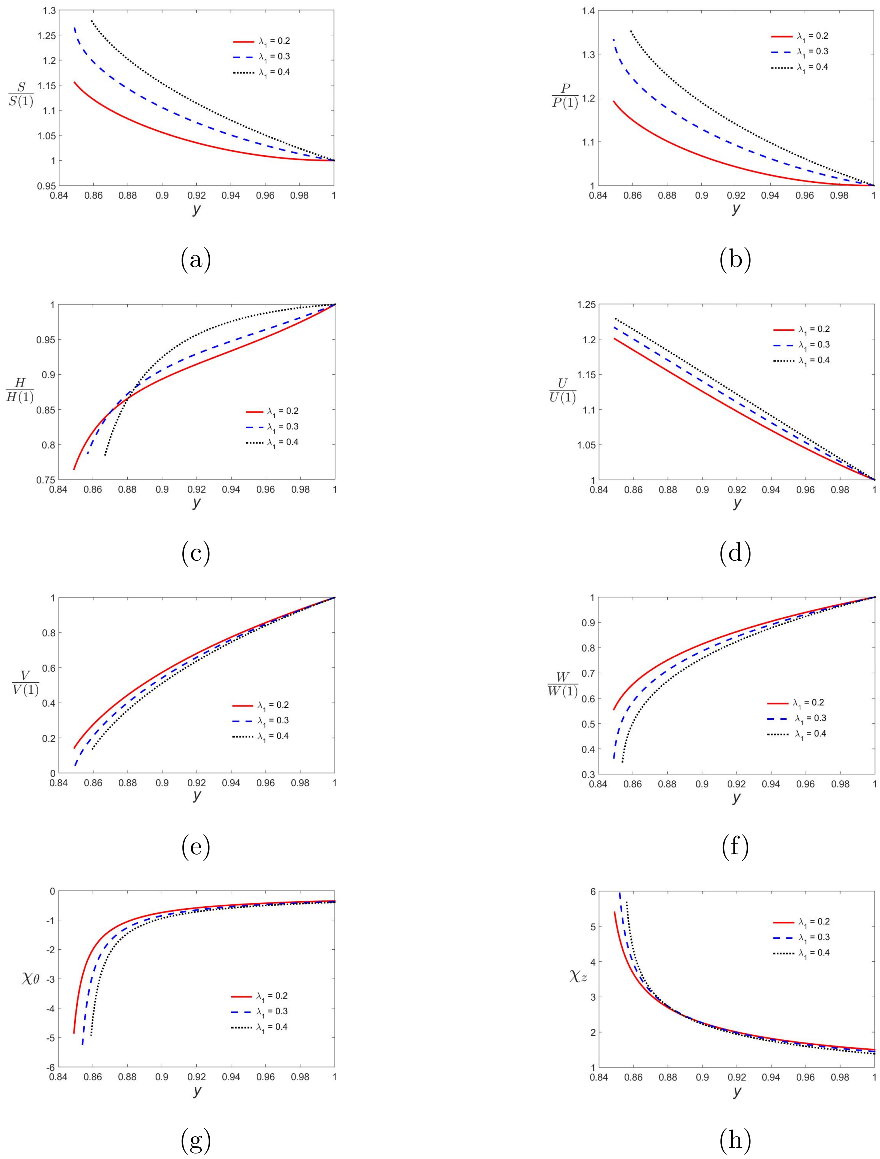

Fig. 3. Flow patterns $\gamma = 5/3, \bar{b} = 0.05$ and $M_A^{-2} = 0.02$.

5.4. **Effect of adiabatic exponent γ on the flow variables.**
From Fig. 4, we can see the effect of γ on the profiles of the flow variables for $\bar{b} = 0.05, M_A^{-2} = 0.02$ and $\lambda_1 = 0.3$. With an increase in the value of γ, the reduced density, pressure, radial fluid velocity and the axial component of vorticity vector decrease (see Figs. 4(a, b, d, h)), whereas the reduced magnetic field, azimuthal fluid velocity, axial fluid velocity and the azimuthal component of vorticity vector increase (see Figs. 4(c, e,

f, g)) behind the shock front. The variation in the values of γ and $\bar{b}$ has the same effects on the flow variables (see Figs. (1, 4)).

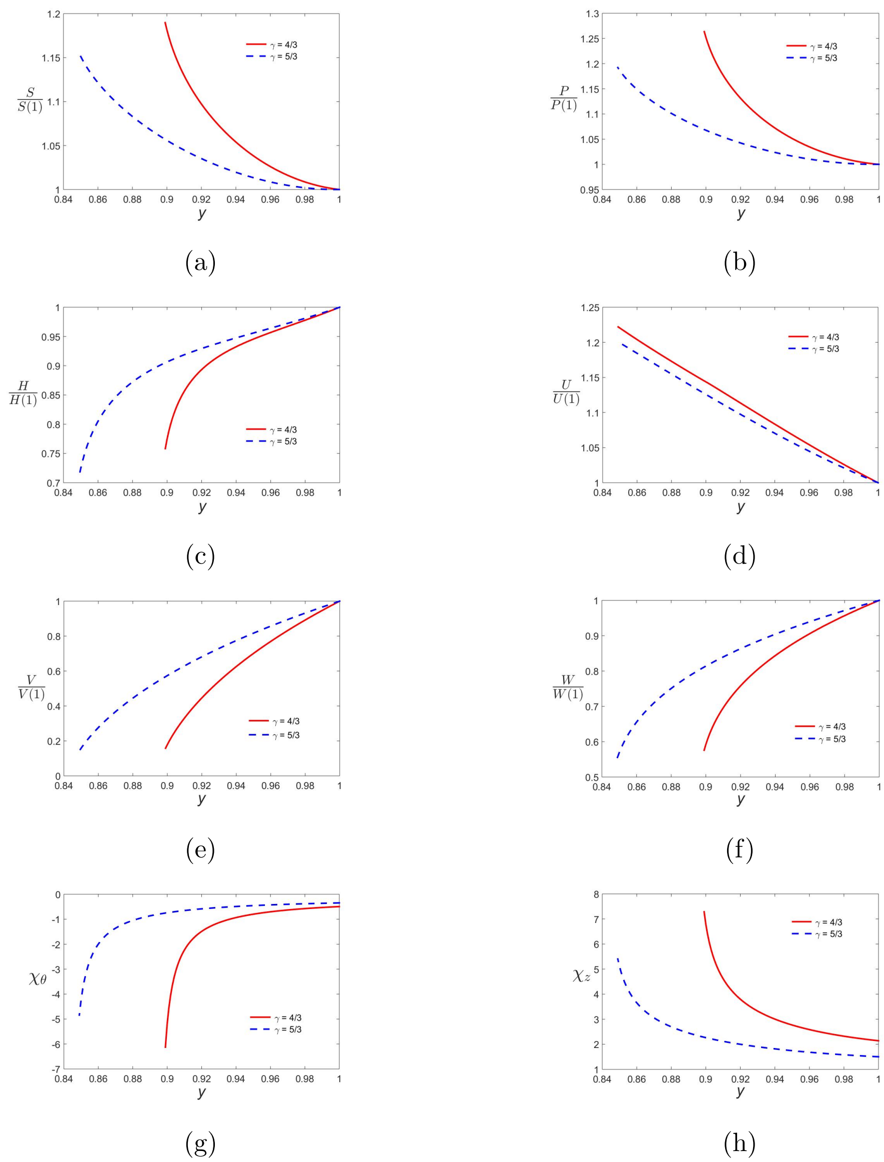

Fig. 4. Flow patterns with $M_A^{-2} = 0.02, \bar{b} = 0.05$ and $\lambda_1 = 0.3$.

6. **Conclusion**

The present work is concerned with the study of one-dimensional cylindrical shock waves in a rotating isothermal flow of a non-ideal gas with the effect of the axial magnetic field. We have used a one-parameter Lie group of transformations to obtain the similarity solutions for the problem under consideration. We determined the infinitesimal generators of the Lie group of transformations for the system of non-linear PDEs (2.1) under which the system of PDEs remains invariant and produces similarity solutions. Based on

TABLE 1. Values of density ratio (β) for different values of $\gamma, \bar{b}$ and M_A^{-2}.

γ	$\bar{b}$	M_A^{-2}	β
5/3	0	0	0.250000
		0.02	0.271701
		0.04	0.292116
5/3	0.05	0	0.287500
		0.02	0.302928
		0.04	0.318450
5/3	0.1	0	0.325000
		0.02	0.336067
		0.04	0.347657
4/3	0	0	0.142857
		0.02	0.185149
		0.04	0.218112
4/3	0.05	0	0.185714
		0.02	0.212525
		0.04	0.237865
4/3	0.1	0	0.228571
		0.02	0.245565
		0.04	0.263610

the arbitrary constants occurring in the expressions for the generators of the Lie group of transformations, we obtained four cases of possible solutions. Among all the possibilities, only three cases admit similarity solutions. In the first and third cases, the shock path follows the power law, whereas in the second case, the shock path follows the exponential law. We discussed the first case in detail in Section 5. The results have been shown through tables and graphs. The present study concludes the following:

(i) The density ratio β increases as we increase the values of $\gamma, \bar{b}$ or M_A^{-2}.

(ii) The reduced density, pressure, radial fluid velocity and the axial component of the vorticity vector increase, whereas the reduced magnetic field, azimuthal fluid velocity, axial fluid velocity and the azimuthal component of the vorticity vector decrease monotonically on moving inwards from the shock front $y = 1$ to the axis of symmetry.

(iii) Increment in the parameter $\bar{b}$ causes the reduced density, pressure, radial fluid velocity and the axial component of the vorticity vector to decrease, whereas the reduced magnetic field, azimuthal fluid velocity, axial fluid velocity and the azimuthal component of vorticity vector to increase behind the shock front.

(iv) Increase the value of M_A^{-2} results in a decrease in the axial component of the vorticity vector; however, other flow variables increase in the region behind the shock front when we increase the value of M_A^{-2}.

(v) On increasing the value of $\lambda_1(=\lambda_2)$ the reduced density, pressure, radial fluid velocity increase, whereas the reduced azimuthal fluid velocity, axial fluid velocity and the azimuthal component of vorticity vector decrease as we move towards the axis of symmetry from the shock front. The reduced magnetic field increases near the shock front and decreases near the axis of symmetry, whereas the axial component of the vorticity vector has a reverse effect, i.e., it decreases near the shock front and increases near the axis of symmetry with an increase in the value of $\lambda_1(=\lambda_2)$.

(vi) The parameters γ and $\bar{b}$ have the same effects on the flow variables.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

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