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Uniqueness and stability for generalized Caputo fractional differential quasi-variational inequalities

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Abstract: This paper studies a new kind of generalized Caputo fractional differential quasi-variational inequalities in Hilbert spaces. We prove uniqueness and stability of the abstract inequality by using generalized singular Gronwall's lemma, projection operators and contraction principle. Finally, an example is given to illustrate the abstract results.

Keywords: Generalized Caputo fractional differential equation; stability; quasi-variational inequality

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1 Introduction

Differential variational inequality (DVI) was proposed in 2008 by Pang and Stewart [1]. The DVI is a comprehensive modeling paradigm that unifies differential inclusions, dynamic Nash equilibrium problems, evolutionary or time-dependent variational inequalities (see, for example, [2–5] and the cited references therein). Since then, many scholars did some extended research on differential

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variational inequalities. In particular, Some researchers have dedicated to the study of existence and stability for all kinds of differential variational inequalities. For example, Gwinner [6] used the monotonicity method and technique of the Mosco convergence to show the stability of the solution set to a class of DVI. Wang et al [7] studied the existence and convergence results of a class of differential quasivariational inequalities (DQVI) by using Filippov's implicit function lemma and discrete Euler approximation algorithm. We observe that the models of DVI in [6] and DQVI [7] are formulated in finite dimensional spaces. Recently, the study of stability for DVIs in infinite dimensional spaces have attracted considerable attention of several researchers. More precisely, Guo [12] et al. applied theory of semigroups and monotone operators, and variational inequality techniques to obtain a general stability result for the partial differential variational inequality when both the nonlinear mapping and the set of constraints are perturbed by two different parameters. Liu and Sofonea [8] proved the existence and uniqueness of solutions for a class of differential quasi variational inequalities in infinite dimensional spaces by using a generalized fixed point theorem, and applied the results to an elastic contact mechanics model. Loi and Vu [9] studied the Ulam-Hyers stability and uniqueness of a class of differential variational inequalities with nonlocal conditions. It is worth pointing out that Jiang [10] established an existence theorem of the mild solution of a global attractor for the semiflow governed by a fractional differential hemivariational inequality, and Jiang - Wei [11] obtained the global solvability and weakly asymptotic stability of fractional delay mixed variational inequalities by applying a new noncompact measure and a fixed point theorem for a condensing set-valued maps, respectively. Migórski [13] studies the existence theorem of solutions of fractional differential variational inequalities using discrete approximation method in Banach space.

Motivated by the work mentioned above, the main purpose of this paper is to investigate the uniqueness and stability results (Mittag-leffle-Ulam-Hyers stability, Mittag-Leffler-Ulam-Hyers-Rassias stability, generalized Mittag-Leffler-Ulam-Hyers and generalized Mittag-Leffler-Ulam-Hyers-Rassias stability) of solutions of the generalized fractional quasi variational inequalities (GFDQVIs) with Cauchy boundary conditions, which can be considered as further continuous study of the work in [9–11, 13]. The innovation of this paper is twofold. On the one hand, The first novelty arises in the special structure of the problem we consider the generalized fractional derivatives in sense of Caputo-Katugampola including the Caputo and Katugampola fractional derivatives. Moreover, we consider completely nonlinear fractional quasi variational inequality. This is the first novelty of the present work. On the other hand, [9] mainly applied the Filippov implicit function lemma and the Gronwall lemma to obtain Uniqueness and Hyers-Ulam stability results for a class of differential variational inequalities with nonlocal conditions, while we use contraction pinciple and the generalized singular Gronwall lemma to Uniqueness and stability for GFDQVIs. So the methods used in this paper the ones employed in [9]. This is the second novelty of the present work.

This paper is organized as follows. In Section 2, we introduce four definitions of stability and some lemmas. In Section 3, the stability and uniqueness of (2.1) are proved by using projection operator, contraction principle and generalized singular Gronwall lemma. In Section 4, the main results are applied to an example.

2 Preliminaries

Let $I := [0,T]$, H_1, H_2 be a Hilbert space and $D \subset H_2$ be a closed convex subset. We will use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the norm and inner product of a Hilbert space, respectively. In this paper, the space of all continuous and integrable functions $x(\cdot)$ from I to H_1 will respectively represent by the symbol $C(I,H_1)$ $\left[L^P(I,H_1),(P \geq 1)\right]$ with the norm

$$\|x\|_c = \max_{t\in I}\|x\|_c, \|x\|_p = \left(\int_0^T \|x(s)\|^p ds\right)^{\frac{1}{p}}.$$

In this paper, we investigate the uniqueness and stability of solutions for the following fractional differential quasi variational inequalities with Cauchy boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha,\rho}x(t) = f(t,x(t),u(t)), \text{ a.e. t } \in I, \\ \langle v-u(t), g(x(t),u(t))\rangle \geq 0, \forall t \in I, \forall v \in K(u), \\ x(0) = x_0 \end{cases} \tag{2.1}$$

where $f : I \times H_1 \times K \to H_1$, $g : H_1 \times K \to H_2$.
we review some definitions and lemmas which will be used later.

Definition 1 ([14], Definition 1) (Katugampola fractional integral). Given $0 < \alpha \leq 1$, $0 < \rho$. The Katugampola fractional integral of a function $x \in L^1[0,T]$ is defined by

$${}^CI_{0^+}^{\alpha,\rho}x(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}x(s)ds,$$

where $\Gamma(\cdot)$ is a gamma function dened by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

Definition 2 ([14], Definition 2) (Katugampola fractional derivative). Given $0 < \alpha \leq 1$, $0 < \rho$. The Katugampola fractional derivative is defined by

$$D_{0^+}^{\alpha,\rho}x(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}t^{1-\rho}\frac{d}{dt}\int_0^t \frac{s^{\rho-1}x(s)}{\left(t^\rho - s^\rho\right)^\alpha}ds.$$

The Caputo derivative of a constant is equal to zero.

Definition 3 ([14], Definition 3) (Caputo-Katugampola fractional derivative). Given $0<\alpha\leq 1$, $0<\rho$. The Caputo-Katugampola fractional derivative is defined by

$$^{C}D_{0^+}^{\alpha,\rho}x(t)=D_{0^+}^{\alpha,\rho}x(t)=\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}t^{1-\rho}\frac{d}{dt}\int_0^t\frac{s^{\rho-1}x(s)}{(t^{\rho}-s^{\rho})^{\alpha}}ds.$$

Remark 1 If x is an abstract function with values in $L^1[0,T]$, then integrals which appear in Definitions 1-3 are taken in Bochner's sense.

Lemma 1 ([15], Theorem 8) Let $v:I\to R$ be a continuous nonnegative function and $a,b\geq 0$ constants such that

$$v(t)\leq a+b\int_0^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}v(s)ds,\quad \forall t\in I,$$

then

$$v(t)\leq aE_{\alpha}\left(b\Gamma(\alpha)\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right),$$

where E_{α} is the Mittag-Leffler function [16] defined by

$$E_{\alpha}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)},t\in I,\mathrm{Re}(\alpha)>0.$$

Remark 2 ([17], Remark 2.12) There exists a constant $M^*>0$ independent of a such that

$$z(t)\leq M^*a \text{ for all } t\in I.$$

Lemma 2 ([18], Projection methods) Let $K:H_2\rightrightarrows H_2$ be a multifunction with nonempty closed convex values and $F:H_2\to H_2$ be a given mapping. The point $\omega_*\in H_2$ is the solution of variational inequality

$$\langle u-\omega,F(\omega)\rangle\geq 0,\quad \forall u\in K(u),$$

if and only if

$$\omega_*=P_{k(u)}\left(\omega_*-kF\left(\omega_*\right)\right),$$

for every $k>0$,where $P_{K(u)}$ if the projection from H_2 to $K(u)$.

Definition 4 We express the solution of (2.1) by a pair of function which is composed of continuous function $x:I\to H_1$ and continuous function $u\in K(u)$. For a given subset $O\subseteq H_1$, if $(x(\cdot),u(\cdot))$ is the solution of (2.1) and $x(t)\in O$ for all $t\in I$, then it is called the solution of (2.1) on $O\times K(u)$.

For $\varepsilon>0$ and $\varphi\in C(I,\mathbb{R}_+)$, we consider GFDQVIs (2.1) and the following inequalities

$$\left\|{}^{c}D_{0^+}^{\alpha,\rho}y(t)-f(t,y(t),u(t))\right\|\leq\varepsilon,t\in I, \tag{2.2}$$

$$\left\|{}^{c}D_{0^+}^{\alpha,\rho}y(t)-f(t,y(t),u(t))\right\|\leq\varphi(t),t\in I, \tag{2.3}$$

$$\left\|{}^{c}D_{0^+}^{\alpha,\rho}y(t)-f(t,y(t),u(t))\right\|\leq\varepsilon\varphi(t),t\in I. \tag{2.4}$$

Definition 5 ([19], Definition 3.1.) FDQVIs (2.1) is Mittag-Leffler-Ulam-Hyers stable, with respect to Mittag-Leffler function E_α, if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1(I,B)$ of inequality (2.2) there exists a solution $x \in C(I,B)$ of FDQVIs (2.1) with

$$\|y(t) - x(t)\| \leq c\varepsilon E_\alpha(t), \quad \forall t \in I.$$

Definition 6 ([19], Definition 3.2.). FDQVIs (2.1) is genreralized Mittag-Leffler-Ulam-Hyers stable, with respect to E_α, if there exists $\theta \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta(0) = 0$, such that for each solution $y \in C^1(I,B)$ of inequality (2.2) there exists a solution $x \in C(I,B)$ of FDQVIs (2.1) with

$$\|y(t) - x(t)\| \leq \theta(\varepsilon) E_\alpha(t), \quad \forall t \in I.$$

Definition 7 ([19], Definition 3.3.). FDQVIs (2.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable,with respect to φE_α,if there exists $c_\varphi > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1(I,B)$ of inequatity (2.4) there exists a solution $x \in C(I,B)$ of FDQVIs (2.1) with

$$\|y(t) - x(t)\| \leq c_\varphi \varphi(t) \varepsilon E_\alpha(t), \quad \forall t \in I.$$

Definition 8 ([19], Definition 3.4.). FDQVIs (2.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable, with respect to φE_α, if there exists $c_\varphi > 0$ such that for each solution $y \in C^1(I,B)$ of inequality (2.3) there exists a solution $x \in C(I,B)$ of FDQVIs (2.1) with

$$\|y(t) - x(t)\| \leq c_\varphi \varphi(t) E_\alpha(t), \quad \forall t \in I.$$

Remark 3 It is clear that: (i) Definition 5 $\Longrightarrow$ Definiton 6 ; (ii) Definiton 7 $\Longrightarrow$ Definiton 8

Remark 4 A function $y : I \to O$ and a continuous function $u \in K(u)$ is a solution of the inequality (2.1) if and only if there exists a function $h : I \to O$ such that
(i)$\|h(t)\| \leq \varepsilon, t \in I$;
(ii)${}^cD^{\alpha,\rho}_{0^+} y(t) = f(t, y(t), u(t)) + h(t)$.

Lemma 3 Let $0 < \alpha \leq 1$, if $y : I \to O$ is a solution of the inequality (2.2). Then y is a solution of the following integral inequality

$$\left\| y(t) - x_0 - \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} f(s, y(s), u(s)) ds \right\| \leq \frac{T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} \varepsilon.$$

proof By Remark 4, we have

$${}^cD^{\alpha,\rho}_{0^+} y(t) = f(t, y(t), u(t)) + h(t), \forall t \in I.$$

Then

$$y(t) - x_0 = \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} f(x, y(s), u(s)) ds + \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} h(s) ds.$$

This implies that

$$y(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds + \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} h(s)ds.$$

Thus, we can get

$$\begin{aligned} \|y(t) - x_0 - &\frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds\| \\ &= \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} \|h(s)\|ds \\ &\le \frac{\rho^{1-\alpha}}{\Gamma(a)} \varepsilon \int_0^t s^{\rho-1} \left(t^\rho - s^\rho\right)^{\alpha-1} ds \\ &\le \frac{t^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} \varepsilon \\ &\le \frac{T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} \varepsilon. \end{aligned}$$

□

Lemma 4 ([16], Theorem 1.9) Define the mapping $N : B_{H_1}(0,R) \to B_{H_1}(0,R)$. For all $x, y \in B_{H_1}(0,R)$

$$\|Nx - Ny\|_C \le a\|x - y\|_C.$$

If $0 < a < 1$, then T has a unique fixed point.

3 Main Results

Suppose there are the following conditions:

(A1) There exist $R, a, L_{1g}, L_{2g} > 0$ and positive functions $\alpha_f(\cdot), \beta_f(\cdot) \in L^1[0,T]$ such that

$$\|f(t,z_1,w_1) - f(t,z_2,w_2)\| \le \alpha_f(t)\|z_1 - z_2\| + \beta_f(t)\|w_1 - w_2\|,$$

$$\|g(z_1,w_1) - g(z_2,w_2)\| \le L_{1g}\|z_1 - z_2\| + L_{2g}\|w_1 - w_2\|,$$

$$a\|w_1 - w_2\|^2 \le \langle w_1 - w_2, g(z,w_1) - g(z,w_2)\rangle,$$

for $a.e.t \in I$, all $z, z_1, z_2 \in B_{H_1}(0,R)$ and $w_1, w_2 \in H_2$,where

$$B_{H_1}(0,R) = \{z \in H_1 : \|z\| \le R\};$$

(A2) For every $(z,w) \in B_{H_1}(0,R) \times H_1$ the mapping $f(\cdot,z,w) : I \to H_1$ is measurable and $\|f(\cdot,0,0)\| \in L^1[0,T]$;

(A3) There exists a constant $\tau \ge 0$ satisfying

$$\left\|P_{K(u)}(z) - P_{K(v)}(z)\right\| \le \tau\|u - v\|, \forall u,v,z \in H_2;$$

$$(\tau + \sqrt{1 - \frac{\alpha^2}{L_{2g}^2}}) \le 1.$$

(A4) If $\varphi \in B_{H_1}(0,R)$ is be an increasing function, then there exists $\lambda_\varphi > 0$ such that

$$\frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \varphi(s) ds \leq \lambda_\varphi \varphi(t). \text{ for each } t \in I.$$

Note that in case (A2), the constants R is the same one as in (A1).

Lemma 5 Let (A1) and (A3) hold.Then for every $z \in B_{H_1}(0,R)$, the set

$$\mathrm{SOL}(K(u), g(z,\cdot)) = \{\omega \in K(u) : \langle v-\omega, g(z,\omega)\rangle \geq 0, \forall v \in K(u)\}$$

consists of only one element.

proof For every $z \in B_{H_1}(0,R)$, from Lemma 2, it follows that

$$\langle v-\omega, g(z,\omega)\rangle \geq 0, \forall v \in K(u),$$

if and only if

$$\omega = P_{K(u)}(\omega - kg(z,\omega))\big), \text{ for } k > 0. \tag{3.1}$$

Consider the mapping $Q : K(u) \to K(u)$,

$$Q(\omega) = P_{K(u)}(\omega - kg(z,\omega)).$$

According to (A1), let $\frac{\alpha}{L_{2g}} \leq 1$, we obtain

$$\begin{aligned}
&\|w_1 - w_2 - k(g(z,w_1) - g(z,w_2))\|^2 \\
&= \langle w_1 - w_2 - k(g(z,w_1) - g(z,w_2)), w_1 - w_2 - k(g(z,w_1) - g(z,w_2))\rangle \\
&= \|w_1 - w_2\|^2 + k^2 \|g(z,w_1) - g(z,w_2)\|^2 - 2k\langle w_1 - w_2, g(z,w_1) - g(z,w_2)\rangle \\
&\leq \left(1 + k^2 L_{2g}^2 - 2ka\right) \|w_1 - w_2\|^2 .
\end{aligned} \tag{3.2}$$

We now consider the mapping $Q : K(u) \to K(u)$,

$$Q(\omega) = P_{K(u)}(\omega - kg(z,\omega)).$$

In the sequel, according to (A3), nonexpansive property of projection mapping and (3.2), for every $\omega_1, \omega_2 \in K$, we have

$$\begin{aligned}
\|Q(\omega_1) - Q(\omega_2)\| &= \left\|P_{K(\omega_1)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_2 - kg(z,\omega_2))\right\| \\
&= \|P_{K(\omega_1)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_1 - kg(z,\omega_1)) \\
&+ P_{K(\omega_2)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_2 - kg(z,\omega_2))\| \\
&\leq \tau\|\omega_1 - \omega_2\| + \|\omega_1 - \omega_2 - k(g(z,\omega_1) - g(z,\omega_2))\| \\
&\leq (\tau + \sqrt{1 + k^2 L_{2g}^2 - 2ka})\|\omega_1 - \omega_2\| .
\end{aligned} \tag{3.3}$$

In particular, we choose $k = \frac{\alpha}{L_{2g}^2}$, then we get

$$\|Q(\omega_1) - Q(\omega_2)\| \leq (\tau + \sqrt{1 - \frac{\alpha^2}{L_{2g}^2}})\|\omega_1 - \omega_2\| .$$

Thus, the above formula and condition (A3) indicate that the mapping Q is contractive. As a consequence, the set $\mathrm{SOL}(K(u), g(z,\cdot))$ contains only one element for each $z \in B_{H_1}(0,R)$.

□

Define the mappings $\pi : H_1 \to B_{H_1}(0,R)$, $U : H_1 \to K(u)$ and $\Phi : I \times H_1 \to H_1$,

$$\pi(z) = \begin{cases} z, & \text{if } z \in B_{H_1}(0,R), \\ \frac{R}{\|z\|} z, & \text{if } z \notin B_{H_1}(0,R), \end{cases}$$

$$U(z) = \mathrm{SOL}(K(u), g(\pi(z), \cdot)),$$

and

$$\Phi(t,z) = f(t, \pi(z), U(z)).$$

Lemma 6 Let condition (A1) hold. Then there exists $\lambda > 0$ such that

$$\|U(z_1) - U(z_2)\| \le \lambda \|z_1 - z_2\|, \tag{3.4}$$

for all $z_1, z_2 \in H_1$.

proof In fact, for every $z_1, z_2 \in H_1$, from Lemma 5 it follows that the sets $U(z_1)$ and $U(z_2)$ consist of only one element, respectively. Put $w_1 = U(z_1)$ and $w_2 = U(z_2)$. Similar the proof of (3.2) and (3.3), applying (A1) and (A3), we have

$$\begin{aligned} \|\omega_1 - \omega_1\| &= \left\|P_{K(\omega_1)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_2 - kg(z,\omega_2))\right\| \\ &= \|P_{K(\omega_1)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_1 - kg(z,\omega_1)) \\ &+ P_{K(\omega_2)}(\omega_1 - kg(z,\omega_1)) - P_{K(\omega_2)}(\omega_2 - kg(z,\omega_2))\| \\ &\le \tau\|\omega_1 - \omega_2\| + \|\omega_1 - \omega_2 - k(g(z,\omega_1) - g(z,\omega_2))\|. \end{aligned} \tag{3.5}$$

Choosing arbitrarily $k \in \left(0, \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$ we obtain

$$\begin{aligned} \left(2ak - k^2\left(L_{1g}^2 + L_{1g}^2\right) - \tau\right)\|\omega_1 - \omega_2\|^2 - 2kL_{1g}\|\omega_1 - \omega_2\|\,\|z_1 - z_2\| \\ -k^2\left(L_{1g}^2 + L_{1g}^2\right)\|z_1 - z_2\| \le 0. \end{aligned}$$

So, put

$$\lambda := \frac{kL_{1g} + \sqrt{k^2L_{1g}^2 + 4k\left(L_{1g}^2 + L_{2g}^2\right)\left(2ak - k^2\left(L_{1g}^2 + L_{2g}^2\right) - \tau\right)}}{2\left[2ak - k^2\left(L_{1g}^2 + L_{2g}^2\right) - \tau\right]}, \tag{3.6}$$

we have

$$\|w_1 - w_2\| \le \lambda\|z_1 - z_2\|.$$

□

Lemma 7 Let conditions (A1)-(A3) hold. Then $\Phi(\cdot,\cdot)$ is a Carathéodory mapping and there exist positive functions $L_\Phi(\cdot), \gamma_\Phi(\cdot) \in L^1_+[0,T]$ such that

$$\|\Phi(t,z_1) - \Phi(t,z_2)\| \le L_\Phi(t)\|z_1 - z_2\|,$$

for all $z_1, z_2 \in H_1$ a.e.t $\in I$.

proof It is easy to see that $\Phi(\cdot,\cdot)$ is a Carathéodory mapping sine $U(\cdot), \pi(\cdot)$ are continuous and $f(\cdot,\cdot,\cdot)$ is a Carathéodory mapping on $I \times B_{H_1}(0,R) \times K(u)$. From the definiion of $\Phi(\cdot,\cdot)$, Lemma 7 and (A1) it follows that for every $z_1, z_2 \in H_1$ and a.e.t $\in I$ the following estimations hold.

$$\begin{aligned}\|\Phi(t,z_1) - \Phi(t,z_2)\| &= \|f(t,\pi(z_1),U(z_1)) - f(t,\pi(z_2),U(z_2))\| \\ &\le \alpha_f(t)\|\pi(z_1) - \pi(z_2)\| + \beta_f(t)\|U(z_1) - U(z_2)\| \\ &\le L_\Phi(t)\|z_1 - z_2\|,\end{aligned}$$

where $L_\Phi(t) = \alpha_f(t) + \lambda\beta_f(t), t \in I$.

□

Consider the following problem

$$\begin{cases} {}^C D^{\alpha,\rho}_{0^+} x(t) = \Phi(t,x(t)), \text{for a.e. } t \in I, \\ \qquad\qquad x(0) = x_0. \end{cases} \tag{3.7}$$

It is obvious that:

(i) If $(x(\cdot),u(\cdot))$ is a solution of (2.1) on $B_{H_1}(0,R) \times K(u)$, then $x(\cdot)$ is a solution of (3.7) on $B_{H_1}(0,R)$;
(ii) If $x : I \to B_{H_1}(0,R)$ is a solution of (3.7), then there exists a continuous function $u : I \to K(u)$ such that $(x(\cdot),u(\cdot))$ is a solution of (2.1) on $B_{H_1}(0,R) \times K(u)$.

Lemma 8 Let conditions (A1)-(A3) hold. In addition, assume that there exists $k \in \left(0, \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$, such that

$$\frac{\rho^{-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|\alpha_f\|_1 + \lambda\|\beta_f\|_1\right) < 1 \tag{3.8}$$

and

$$\overline{L}E_\alpha\left(\left(\|\alpha_f\|_1 + \lambda\|\beta_f\|_1\right)\Gamma(\alpha)\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) \le R. \tag{3.9}$$

where $\overline{L} = \left(\|x_0\| + \frac{\rho^{1-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\|\beta_f\|_1\right)\right)$, λ is the constant from (3.6).

proof Let $x_*(\cdot)$ be the solution of (3.9), i.e.,

$$x_*(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}\Phi(s,x_*(s))\,ds, t \in I.$$

Thus, we get

$$\begin{aligned}\|x_*(t)\| &\leq \|x_0\| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \left\| s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}\Phi\left(s, x_*(s)\right)\right\| ds \\ &\leq \|x_0\| + \frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}\|\Phi\left(s, x_*(s)\right)\| ds,\end{aligned}$$

for all $t \in I$.

Now, for all $(z,w) \in B_{H_1}(0,R) \times K(u)$ and a.c. $s \in I$, since

$$\|f(s,z,w)\| \leq \|f(s,0,0)\| + \alpha_f(s)\|z\| + \beta_f(s)\|w\|.$$

we have

$$\begin{aligned}\|\Phi(s,z)\| &\leq \|\Phi(s,0)\| + L_\Phi(s)\|z\| = \|f(s,0,U(0))\| + L_\Phi(s)\|z\| \\ &\leq \|f(s,0,0)\| + \beta_f(s)\|U(0)\| + L_\Phi(s)\|z\|,\end{aligned}$$

for all $z \in B_{H_1}(0,R)$ and a.c. $s \in I$.

Thus we have

$$\begin{aligned}\|x_*(t)\| &\leq \|x_0\| + \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t \|\Phi\left(s,x_*(s)\right)\| ds \\ &\leq \|x_0\| + \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\textstyle\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}\left(\|f(s,0,0)\| + \beta_f(t)\|U(0)\| + L_\Phi(s)\|x_*(s)\|\right) ds \\ &\leq \|x_0\| + \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\left\|\beta_f\right\|_1\right)\textstyle\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1} ds \\ &+ \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1} L_\Phi(s)\|x_*(s)\| ds \\ &\leq \|x_0\| + \tfrac{\rho^{1-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\left\|\beta_f\right\|_1\right) \\ &+ \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}\|\|L_\Phi(s)\|_1\|\,\|x_*(s)\| ds,\end{aligned}$$

Hence, by applying Lemma 1 and (3.9), for all $t \in I$ the following estimations hold true.

$$\begin{aligned}\|x_*(t)\| &\leq \|x_0\| + \tfrac{\rho^{1-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\left\|\beta_f\right\|_1\right) \\ &+ \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t s^{\rho-1}\left(t^\rho - s^\rho\right)^{\alpha-1}\|\|L_\Phi(s)\|_1\|\,\|x_*(s)\| ds \\ &\leq \left(\|x_0\| + \tfrac{\rho^{1-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\left\|\beta_f\right\|_1\right)\right) E_\alpha\left(\|L_\Phi(s)\|_1 \Gamma(\alpha)\left(\tfrac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) \\ &\leq \left(\|x_0\| + \tfrac{\rho^{1-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\|f(\cdot,0,0)\|_1 + \|U(0)\|\left\|\beta_f\right\|_1\right)\right) E_\alpha\left(\left(\left\|\alpha_f\right\|_1 + \lambda\left\|\beta_f\right\|_1\right)\Gamma(\alpha)\left(\tfrac{T^\rho - s^\rho}{\rho}\right)^\alpha\right) \\ &\leq R.\end{aligned}$$

The last inequality means that $x_*(\cdot) \in B_C(0,R)$.

□

Lemma 9 We suppose that conditions (A1)-(A3) hold. Then the solution set of (3.7) on $B_{H_1}(0,R)$ contains only one element.

proof Suppose that (3.7) has two different solutions on $B_{H_1}(0,R)$:

$$x_1(\cdot) : I \to B_{H_1}(0,R),$$

$$x_2(\cdot) : I \to B_{H_1}(0,R).$$

For a.e. $t \in I$. define the mapping $N : B_{H_1}(0,R) \to B_{H_1}(0,R)$. We have

$$\begin{aligned}
\|Nx_1(t) - Nx_2(t)\|_c &= \max_{t\in I} \left\| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \Phi(s, x_1(s))\, ds - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \Phi(s, x_2(s))\, ds \right\| \\
&\leq \max_{t\in I} \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \|\Phi(s, x_1(s)) - \Phi(s, x_2(s))\|\, ds \\
&\leq \max \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \|L_\Phi(s)\|_1 \|x_1(s) - x_2(s)\|\, ds \\
&\leq \frac{\rho^{-\alpha} T^{\alpha\rho}}{\Gamma(\alpha+1)} \left(\|\alpha_f\|_1 + \lambda \|\beta_f\|_1\right) \max \|x_1(t) - x_2(t)\|,
\end{aligned}$$

by applying (3.8), we have

$$\|Nx_1(t) - Nx_2(t)\|_c \leq \frac{\rho^{-\alpha} T^{\alpha\rho}}{\Gamma(\alpha+1)} \left(\|\alpha_f\|_1 + \lambda \|\beta_f\|_1\right) \max \|x_1(t) - x_2(t)\| < \|x_1(t) - x_2(t)\|_c .$$

Hence, By Lemma 4, there are $Nx(t) = x(t)$ for all $t \in I$. As a consequence, the problem (2.1) has a unique solution on $B_{H_1}(0,R) \times K(u)$.

□

Theorem 1 Under the hypothesesof Lemma 9, let's prove that problem (2.1) is Mittag-Leffler-Hyers–Ulam stable on $B_{H_1}(0,R) \times K(u)$.

proof Let $u_* \in K(u)$ be such that the pair $(x_*(\cdot), u_*(\cdot))$ is the unique solution of (2.1) on $B_{H_1}(0,R) \times K(u)$. For every $\varepsilon > 0$ and for any pair $(y(\cdot), u(\cdot))$ consisting of a continuous function $y : I \to B_{H_1}(0,R)$ and a continuous function $u \in K(u)$. That satisfifies (2.2), the following estimations hold true for a.e. $t \in I$:

$$\begin{aligned}
\|y(t) - x_*(t)\| &= \left\| y(t) - x_0 - \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, x_*(s), u(s))\, ds \right\| \\
&\leq \|y(t) - x_0 - \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, y(s), u(s)) ds \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, y(s), u(s)) ds \\
&- \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, x_*(s), u(s))\, ds\| \\
&\leq \left\| y(t) - x_0 - \frac{\rho^{1-\alpha}}{\Gamma(a)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s, y(s), u(s)) ds \right\| \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \|\Phi(s, y(s)) - \Phi(s, x_*(s))\|\, ds,
\end{aligned}$$

Remark 4 conclude that

$$\begin{aligned}
\|y(t) - x_*(t)\| &\leq \frac{T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} \varepsilon + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} \|L_\Phi(s)\|_1 \|y(s) - x_*(s)\|\, ds \\
&\leq \frac{T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} E_\alpha \left(\|L_\Phi(s)\|_1 \Gamma(\alpha) \left(\frac{T^\rho - s^\rho}{\rho} \right)^\alpha \right) \varepsilon.
\end{aligned}$$

Consequently, problem (2.1) is Mittag-Leffler-Hyers-Ulam stable on $B_{H_1}(0,R) \times R$.

□

Theorem 2 If (A1)-(A4) holds, then (2.1) is generalized Mittag-Leffler-Hyers-Ulam-Rassias stable.
proof By inequality (2.3) and condition (A4), we have

$$\begin{aligned} &\left\| y(t)-x_0-\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds\right\| \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1}\varphi(s)ds \leq \lambda_\varphi\varphi(t), t\in I. \end{aligned}$$

From these relation it follows

$$\begin{aligned} \|y(t)-x_*(t)\| &= \left\| y(t)-x_0-\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f\left(s,x_*(s),u(s)\right)ds\right\| \\ &\leq \| y(t)-x_0-\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds \\ &+\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds \\ &-\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f\left(s,x_*(s),u(s)\right)ds\| \\ &\leq \left\| y(t)-x_0-\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1} f(s,y(s),u(s))ds\right\| \\ &+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1}\|\Phi(s,y(s))-\Phi\left(s,x_*(s)\right)\|ds \\ &\leq \lambda_\varphi\varphi(t)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1}\|L_\Phi(s)\|_1\|y(s)-x_*(s)\|ds. \end{aligned}$$

By Lemma 1 and Remark 2, there exists a constant $M_f^*>0$ independent of $\lambda_\varphi\varphi(t)$ such that

$$\|y(t)-x_*(t)\| \leq M_f^*\lambda_\varphi\varphi(t) := c_{f,\varphi}\varphi(t), t\in I.$$

Thus, the equation (2.1) is generalized Mittag-Leffler-Hyers-Ulam-Rassias stable.

□

Theorem 3 Relation (3.9) satisfies if and only if:

$$\frac{\rho^{-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\left\|\alpha_f\right\|_1+\lambda\left\|\beta_f\right\|_1\right)<1 \tag{3.10}$$

and

$$\bar{L}E_\alpha\left(\left\|\alpha_f\right\|_1\Gamma(\alpha)\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right)<R. \tag{3.11}$$

proof It is obvious condition (3.9) is equivalent to the next condition: there exists $k\in\left(0,\frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$ such that

$$0>\frac{\rho^{-\alpha}T^{\alpha\rho}}{\Gamma(\alpha+1)}\left(\left\|\alpha_f\right\|_1+\lambda\left\|\beta_f\right\|_1\right)-1$$

and

$$0 \geq \overline{L} E_\alpha \left(\left(\|\alpha_f\|_1 + \lambda \|\beta_f\|_1 \right) \Gamma(\alpha) \left(\frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) - R.$$

For (3.7) we can express $\lambda = \lambda(k)$ as: $\lambda(k) = \omega_1(k) + \sqrt{\omega_2(k) + \omega_3(k)}$, where

$$\omega_1(k) = \frac{kL_{1g}}{2\left[2ak - k^2\left(L_{1g}^2 + L_{2g}^2\right) - \tau\right]},$$

$$\omega_2(k) = \frac{kL_{1g}}{2\left[2ak - k^2\left(L_{1g}^2 + L_{2g}^2\right) - \tau\right]},$$

$$\omega_3(k) = \frac{k\left(L_{1g}^2 + L_{2g}^2\right)}{2\left[2ak - k^2\left(L_{1g}^2 + L_{2g}^2\right) - \tau\right]}.$$

It is easy to see that these functions are continuous and increasing on $\left(0, \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$. Thus, $\lambda(k)$ is continuous and increasing on $\left(0, \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$ and $\lim_{k\to 0} \lambda(k) = 0$ and $\lim_{k\to \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}} \lambda(k) = \infty$. Hence, the functions

$$\varphi_1(k) = \frac{\rho^{-\alpha} T^{\alpha\rho}}{\Gamma(\alpha+1)} \left(\|\alpha_f\|_1 + \lambda(k) \|\beta_f\|_1 \right)$$

and

$$\varphi_2(k) = \overline{L} E_\alpha \left(\left(\|\alpha_f\|_1 + \lambda(k) \|\beta_f\|_1 \right) \Gamma(\alpha) \left(\frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) - R,$$

are continuous and increasing on $\left(0, \frac{a+\sqrt{a^2-4\tau\left(L_{1g}^2+L_{2g}^2\right)}}{2\left(L_{1g}^2+L_{2g}^2\right)}\right)$. Moreover,

$$\lim_{k\to 0} \varphi_1(k) = \frac{\rho^{-\alpha} T^{\alpha\rho}}{\Gamma(\alpha+1)} \left(\|\alpha_f\|_1 \right) - 1,$$

$$\lim_{k\to 0} \varphi_2(k) = \overline{L} E_\alpha \left(\|\alpha_f\|_1 \Gamma(\alpha) \left(\frac{t^\rho - s^\rho}{\rho} \right)^\alpha \right) - R.$$

So, condition (3.11) satisfifies if and only if

$$\lim_{k\to 0} \varphi_1(k) < 0,$$

$$\lim_{k\to 0} \varphi_2(k) < 0,$$

or equivalently, condition (3.11) satisfifies if and only if condition (3.9) satisfifies.

□

Now, let's consider problem (2.2) for the case when:

$$f(t,z,\omega)=\tilde{f}(t,z)+B(t,\omega),$$

$$g(z,\omega)=G(z)+F(\omega),$$

where $\tilde{f}:I\times H_1\to H_1$, $B:I\times K(u)\to H_1$, $G:H_1\to H_2$, and $F:K(u)\to H_2$ are continuous mapping that satisfies the following conditions:

(H1) There exist positive functions $\alpha_{\tilde{f}}(\cdot),\beta_B(\cdot)\in L^1[0,T]$ and constants $L_G,L_F,a>0$ such that

$$\left\|\tilde{f}(t,z_1)-\tilde{f}(t,z_2)\right\|\leq\alpha_{\tilde{f}}(t)\left\|z_1-z_2\right\|,$$

$$\left\|B(t,\omega_1)-B(t,\omega_2)\right\|\leq\beta_B(t)\left\|\omega_1-\omega_2\right\|,$$

$$\left\|G(z_1)-G(z_2)\right\|\leq L_G\left\|z_1-z_2\right\|,$$

$$\left\|F(\omega_1)-F(\omega_2)\right\|\leq L_F\left\|\omega_1-\omega_2\right\|,$$

$$a\left\|\omega_1-\omega_2\right\|^2\leq\left\langle\omega_1-\omega_2,F(\omega_1)-F(\omega_2)\right\rangle.$$

(H2) For every $(z,\omega)\in H_1\times K(u)$ the functions $\tilde{f}(\cdot,z),B(\cdot,z):I\to H_1$ are measurable and $\|\tilde{f}(\cdot,0)+B(\cdot,0)\|\in L^1[0,T]$.

(H3) There exists a constant $\tau\geq 0$ satisfying

$$\left\|P_{K(u)}(z)-P_{K(v)}(z)\right\|\leq\tau\|u-v\|,\forall u,v,z\in H_2,$$

$$(\tau+\sqrt{1-\frac{\alpha^2}{L_{2g}^2}})\leq 1,$$

for all $u(\cdot),v(\cdot)\in C(I,H_1)$.

(H4) If $\varphi\in B_{H_1}(0,R)$ is be an increasing function, then there exists $\lambda_\varphi>0$ such that

$$\frac{\rho^{1-\alpha}}{\Gamma(a)}\int_0^t s^{\rho-1}\left(t^\rho-s^\rho\right)^{\alpha-1}\varphi(s)ds\leq\lambda_\varphi\varphi(t).\text{ for each }t\in I.$$

The following statement can be easily followed from Lemma 9 and Theorem 3.

Theorem 4 If the conditions (H1)-(H3) are satisfied, the problem (2.1) is Mittag-Leffler-Hyers-Ulam stable on $H_1\times K(u)$ and has a unique solution. In addition, if the condition (H4) is also satisfied, the problem (2.1) is generalized Mittag-Leffler-Hyers-Ulam-Rassias stable on $H_1\times K(u)$ and has a unique solution.

4 An Example

Consider fractional differential quasi variational inequalities with Cauchy boundary (see [9] Applications)

$$\begin{cases} {}^{c}D_{0^+}^{\alpha,\rho}x(t) = Ax(t) + Bu(t), \text{ for a.e. } t \in I \\ y(t) = Cx(t) + Du(t) \\ 0 \leq u(t) \perp y(t) \geq 0, \forall t \in I \\ x(0) = x_0 \end{cases} \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$; $x_0 \in \mathbb{R}^n$ is a given vector; This problem can be regarded as a control system, where $x(\cdot)$ is the state function; $u(\cdot)$, $y(\cdot)$ are input and output functions, respectively. Here,

$$K(u) = \{w = (w_1, \ldots, w_m) \in \mathbb{R}^m : w_i \geq 0; i = 1, \ldots, m\}.$$

Next, there is a concrete example.

Example 1 consider the following problem

$$\begin{cases} {}^{C}D_{0^+}^{0.8,0.97}x(t) = \frac{2x(t)}{1+x(t)^{9.65}} - u(t), \text{ for a.e.t } \in [0,1], \\ y(t) = -x(t) + u(t), \\ 0 \leq u(t) \perp y(t) \geq 0, \forall t \in [0,1], \\ x(0) = 0.5, \end{cases} \tag{4.2}$$

where $x(\cdot) : [0,1] \to R$, $u(t) : [0,1] \to K(u)$, here

$$K(u) = \{z : z \geq 0\}.$$

It is clear that, it is Mittag-Leffler-Ulam-Hyers stable and has a unique solution. In order to find this solution, we use the trial and error method. First, let $u(t) = 0, \forall \in [0,1]$, Then $u(t) \in K(u)$ and $u(t) \perp y(t), \forall \in [0,1]$. Consider Cauchy problem:

$$\begin{cases} {}^{c}D_{0^+}^{0.8,0.97}x(t) = \frac{2x(t)}{1+x(t)^{9.65}} \text{ for } a.e.t \in [0,1], \\ x(0) = 0.5. \end{cases}$$

All next figures were done with Matlab 2020a. The numerical solution is shown in Fig.1

Since, $x(t) > 0, \forall \in [0,1]$, Hence $y(t) = -x(t) + u(t) < 0, \forall \in [0,1]$. That contradicts with $y(t) \geq 0, \forall \in [0,1]$. Now, let us take $u(t) = x(t), \forall t \in [0,1]$. Then the function $x(t)$ is the solution of Cauchy problem, and its numerical solution is shown in Fig.2

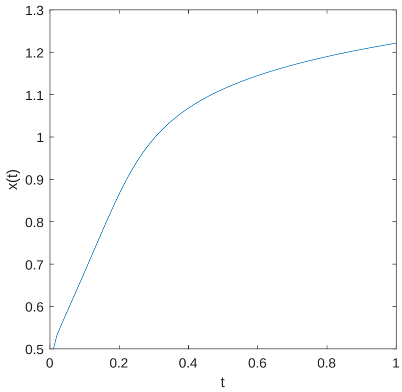

Fig. 1 $x(t), u(t) = 0$

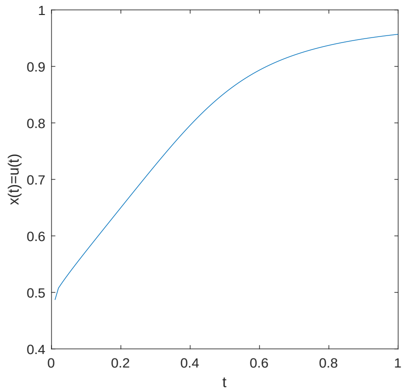

Fig. 2 $x(t) = u(t)$

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