Fermat collocation method with convergence analysis of Volterra integro- differential equations

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Abstract

this article presents the algorithm for solving a Volterra integro-differential equation using Fermat collocation method. The constructing and the properties of Fermat expansion were displayed. The method depends on modifying the Volterra integro-differential equation with initial conditions to a system of equations in the coefficients, which must be determined, of the expansion. We use properties of the matrices to do this modification. We investigate accurately the convergence and error analysis of the problem. Some examples are solved using this algorithm and the absolute errors were compared with others. The results proved the method is the most accuracy of the others.

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1 Introduction

There are many applications of Volterra integro-differential equations (VIDE) in several sciences mathematics, physics, chemistry, engineering, biology and so on. So we have different methods for solving these equations such as : [1][2] variational iteration method for solving Volterra integro-differential equations of the second kind, and for multi terms and vanishing delays respectively. A finite difference method for a linear first-order Volterra integro-differential equation with delay, and for non-linear Volterra partial integro-differential equations [3][4]. [5] a finite element method of linear Volterra integro-differential equations using boundary conditions. [6] solution using fixed-point method.

Recently, There are the spectral methods for solving Volterra integro-differential equations. These methods small errors and small number of unknowns such as: Chebyshev method [7][8]. Legendre collocation method [9][10]. Solution using Jacobi collocation method [11]. The first order VIDEs with different kinds were solved in many articles: [12][13] equations of the second kind and third kind respectively. We compared the obtained results with collocation method [14], standard collocation method (SCM) and chebyshev-Gauss- Lobatto collocation method (CGLCM) [15], homotopy analysis method (HAM) and finite difference method (FDM) [16].

This article discussed the first-order Volterra integro-differential equation of the third kind: [14]

$$\gamma^{\alpha}V'(\gamma)=\gamma^{\alpha}\eta(\gamma)V(\gamma)+\gamma^{\alpha}\ F(\gamma)+\int_{0}^{\gamma}\rho(\gamma,y)\ V(y)\ dy\ ,\ \gamma\in L=[0,l] \tag{1}$$

Where $V(\gamma)$ is an unknown function. $\eta(\gamma)$, $F(\gamma)\in C\ (L)$. $\rho(\gamma,y)$, $\varkappa(\gamma,y)\in C(A)$, and

$$A=\{(\gamma,y):\ \gamma\in L,\ 0\leq y\leq\gamma\}\,.$$

Supose that $0<\alpha<1$ and $\rho(\gamma,y)=y^{\alpha}\varkappa(\gamma,y)$.

The paper has the following organization: section 2 presents the important properties and the first derivative of Fermat polynomials which are useful in the next sections. In section 3, the algorithm of the method is explained on a first-order Volterra-Fredholm integral equation using operational matrices. The convergence and error analysis of the Fermat polynomials are studied in detail in section 4. We give some numerical examples and compared them with others in section 5, the results showed the efficiency of the method. In the last section, we introduce some conclusions.

2 Properties and the first derivative of Fermat polynomials

In this section, we present the important relations, formulas and the first derivative of Fermat polynomials [17], which are used in the following sections.
The recurrence relations of Fermat polynomials are:

$$\Upsilon_{j+2}(\gamma)=\ 3\gamma\ \Upsilon_{j+1}(\gamma)-2\ \ \Upsilon_{j}(\gamma)\,,\quad j\geq 0 \tag{2}$$

With initial values:

$$\Upsilon_{0}(\gamma)=0,\qquad \Upsilon_{1}(\gamma)=1.$$

$\Upsilon_{j}(\gamma)$ has the Binet's form:

$$\Upsilon_{j}(\gamma)=\begin{cases}\dfrac{\left(3\gamma+\sqrt{9\gamma^{2}-8}\right)^{j}+\left(3\gamma-\sqrt{9\gamma^{2}-8}\right)^{j}}{2^{j}\sqrt{9\gamma^{2}-8}},\gamma\neq\frac{2}{3}\\ 2^{\frac{j}{2}}\sin\left(\frac{\pi}{4}j\right),\gamma=\frac{2}{3}\end{cases},j=1,2,.... \tag{3}$$

and the analytic form:

$$\Upsilon_{k+1}(\gamma)=\sum_{j=0}^{\left\lfloor\frac{k}{2}\right\rfloor}(-2)^{j}\ 3^{k-2j}\binom{k-j}{j}\gamma^{k-2j}\ . \tag{4}$$

Let $V(\gamma)$ be written in the expansion of Fermat polynomials:

$$V(\gamma) = \sum_{j=1}^{\infty} e_j \ \Upsilon_j(\gamma). \tag{5}$$

If we approximate $V(\gamma)$ as

$$V(\gamma) \approx V_K(\gamma) = \sum_{j=1}^{K+1} e_j \quad \Upsilon_j(\gamma) = E^T \ \Phi(\gamma), \tag{6}$$

Where

$$\Phi(\gamma) = [\Upsilon_1(\gamma), \Upsilon_2(\gamma), ..., \Upsilon_{K+1}(\gamma)]^T, \tag{7}$$

and the coefficients

$$E^T = [e_1, e_2, ..., e_{K+1}]. \tag{8}$$

must be determined.

The first derivative of Fermat polynomials: from eq. (7)

$$\frac{d \ \Phi(\gamma)}{d \ \gamma} = H^{(1)} \ \Phi(\gamma),$$

where $H^{(1)} = \left(h_{mn}^{(1)}\right)$ is $(K+1) \times (K+1)$ matrix and has the form

$$h_{mn}^{(1)} = \begin{cases} 3(n+1)\,2^{\frac{m-n-1}{2}} & m > n, \ (m+n) \ odd \\ 0 & otherwise \end{cases} \tag{9}$$

3 Fermat collocation method

In this section, we use Fermat polynomials to approximate the solution of the first-order Volterra integro-differential eq. (1). Substitute the expansion (6) in eq. (1), we have

$$\begin{aligned} \sum_{j=1}^{K+1} e_j \ \ h_{mn}^{(1)} \ \Upsilon_j(\gamma) &= \eta(\gamma) \sum_{j=1}^{K+1} e_j \ \ \Upsilon_j(\gamma) + F(\gamma) + \\ & \int_0^{\gamma} \gamma^{-\alpha} \rho(\gamma, y) \ \sum_{j=1}^{K+1} e_j \ \ \Upsilon_j(y) \ dy \,. \end{aligned} \tag{10}$$

Let

$$g_j(\gamma) = h_{mn}^{(1)} \ \Upsilon_j(\gamma) - \eta(\gamma) \ \Upsilon_j(\gamma) - \int_0^{\gamma} \gamma^{-\alpha} \rho(\gamma, y) \ \Upsilon_j(y) \ dy.$$

So eq. (10) has the form

$$\sum_{j=1}^{K+1} e_j \; g_j\left(\gamma\right) = F\left(\gamma\right).$$

There exist $K+1$ roots. So we have a system of equations

$$\sum_{j=1}^{K+1} e_j \; g_j\left(\gamma_i\right) = F\left(\gamma_i\right).$$

The matrix form of this equation is

$$G^T \; E = F,$$

Where

$$G = (g_{ji}), \qquad i, j = 1, 2, ...K+1,$$

and

$$F = [F(\gamma_1), F(\gamma_2), ..., F(\gamma_{K+1})]^T.$$

We can determine the unknown constants by the following equation:

$$E = (G^T)^{-1} F.$$

4 Convergence and error analysis

In this section, the convergence and error analysis of Fermat expansion of VIDE are discussed. The following theorems are satisfied:

Theorem 1 *If $V(\gamma)$ is defined on $[0,1]$ and $\left|V^{(j)}(0)\right| \leq \ell^j$, $j \geq 0$ where ℓ is a positive constant and if $V(\gamma)$ has the expansion:*

$$V(\gamma) \; = \sum_{j=1}^{\infty} e_j \; \Upsilon_j\left(\gamma\right).$$

Then:

1) $|e_j| \leq \dfrac{\ell^{j-1}}{3^j \; (j-1) \; !} \; \cosh\left(\sqrt{2}\upsilon\right)$

2) The series converges absolutely.

Proof. See [17] ■

Theorem 2 *Let $V(\gamma)$ satisfy the assumptions stated in theorem (1). Moreover $\varepsilon_K(\gamma) = |V(\gamma) - V_K(\gamma)| = \sum_{j=K+2}^{\infty} e_j \; \Upsilon_j\left(\gamma\right)$ be the truncation error so:*

$$|\varepsilon_K(\gamma)| < \frac{2\upsilon^{K+1}}{3\,(K+1)!}\cosh\left(\sqrt{2}\upsilon\right), \qquad \upsilon = \frac{2\ell}{3}$$

Proof. See [17] ■

Lemma 3 *The derivative of* Υ_j *is denoted by the following relation:*

$$|\Upsilon_j'| \leq 2^j j^2$$

Proof. Apply the derivative to the right-hand side of expansion (4) and observe that $\gamma < 1$, then by induction on j, the Lemma is satisfied ■

Theorem 4 *Let* $V(\gamma) = \sum\limits_{j=1}^{\infty} e_j \quad \Upsilon_j(\gamma)$ *be the exact solution of eq.* (1)*, which satisfies eq.* (4)*, and* $V_K(\gamma) = \sum\limits_{j=1}^{K+1} e_j \quad \Upsilon_j(\gamma)$ *be its approximation. If*

$$R_K(y) = \left| V_K'(\gamma) - \eta(\gamma)V_K(\gamma) - \int\limits_0^{\gamma} \gamma^{-\alpha}\rho(\gamma, y)\ V_K(y)\ dy - F(\gamma) \right|,$$

$$\Re_K = \max_{0\leq\gamma\leq l} R_K(\gamma),$$

and if $|\eta(\gamma)| \leq \eta_1, |\varkappa(\gamma,\gamma| \leq \varkappa_1$*. Where* η_1 *and* $\varkappa_1$ *are positive constants. Then we have the following global error estimate:*

$$\Re_K \leq \frac{2}{3}\frac{\upsilon^{K+1}e^{\upsilon}\sigma}{(K+1)!}\cosh\left(\sqrt{2}\upsilon\right)$$

Where

$$\sigma = \max\left\{(K+1)^2, \left(\upsilon + \frac{1}{2}(4+K)\right)^2, \eta_1, \varkappa_1\frac{L}{\alpha+1}\right\}.$$

Proof. *From eq.* (1)*, we have*

$$F(\gamma) = V'(\gamma) - \eta(\gamma)V(\gamma) - \int\limits_0^{\gamma} \gamma^{-\alpha}\rho(\gamma, y)\ V(y)\ dy\ .$$

So

$$R_K(\gamma) \leq \sum_{j=K+2}^{\infty} |e_j|\ \ |\Upsilon_j'(\gamma)| + |\eta(\gamma)|\ \ |\varepsilon_K(\gamma)| + \left|\gamma^{-\alpha}\varkappa(\gamma,\gamma)\int\limits_0^{\gamma} y^{\alpha}dy\right|\ \ |\varepsilon_K(\gamma)|\,.$$

From theorems 1, 2 and Lemma 3:

$$\Re_K \leq \frac{2}{3}\cosh\left(\sqrt{2}\upsilon\right) \left\{ \sum_{j=K+2}^{\infty} \frac{\upsilon^{j-1}}{(j-1)!} j^2 + \left(\eta_1 + \varkappa_1 \frac{L}{\alpha+1}\right) \frac{\upsilon^{K+1}}{(K+1)!} \right\},$$

but

$$\sum_{j=K+2}^{\infty} \frac{\upsilon^{j-1}}{(j-1)!} j^2 = \frac{1}{\Gamma(1+K)} \{(3+\upsilon+K)\ \upsilon^{1+K} +$$

$$\left(1+3\upsilon+\upsilon^2\right)\left(\Gamma(1+K) - \Gamma(1+K,\upsilon)\right)\ e^{\upsilon}\}.$$

Then

$$\Re_K \leq \frac{2}{3}\frac{\upsilon^{K+1}e^{\upsilon}}{(K+1)!}\cosh\left(\sqrt{2}\upsilon\right)\ \{(3+\upsilon+K)(K+1) +$$

$$\left(1+3\upsilon+\upsilon^2\right) + \eta_1 + \varkappa_1 \frac{L}{\alpha+1}\},\ e^{-\upsilon} < 1$$

after some calculations:

$$\Re_K \leq \frac{2}{3}\frac{\upsilon^{K+1}e^{\upsilon}}{(K+1)!}\cosh\left(\sqrt{2}\upsilon\right) \left\{ (K+1)^2 + \left(\upsilon + \frac{1}{2}(4+K)\right)^2 + \eta_1 + \varkappa_1 \frac{L}{\alpha+1} \right\}.$$

From the assumptions of the theorem, the proof is completed. ■

5 Numerical examples

In this section, we solve the Volterra integro-differential equations using Fermat collocation (FC) method and compare them with other methods:

Example 5 *Suppose that the following VIDE [14]*

$$\begin{cases} \gamma^{\frac{1}{2}}\ V'(\gamma) = \frac{1}{20}\gamma V(\gamma) + F(\gamma) + \int\limits_0^{\gamma} y^{\frac{1}{2}}\ V(y)\ dy\ , \quad \gamma \in [0,1]\,. \\ V(0) = 0. \end{cases} \tag{11}$$

The exact solution of eq. (11) is $V(\gamma) = \gamma^{\frac{9}{2}}$ and

$$F(\gamma) = \frac{9}{2}\gamma^4 - \frac{1}{20}\gamma^{\frac{11}{2}} - \frac{1}{6}\gamma^6.$$

Table 1 compares the absolute error obtained by FCM with obtained by the collocation method (TC) [14]. We notice that the absolute error in the proposed method is better than the other. In Figure 1, the results are displayed at $K = 8, 16,\ 32$ and the convergence is exponential.

Table 1: Comparison between absolute errors with different values of K

K	FCM	CM [14]
8	3.7×10^{-7}	2.65×10^{-3}
16	2.1×10^{-7}	1.39×10^{-3}
32	1.3×10^{-6}	7.09×10^{-4}

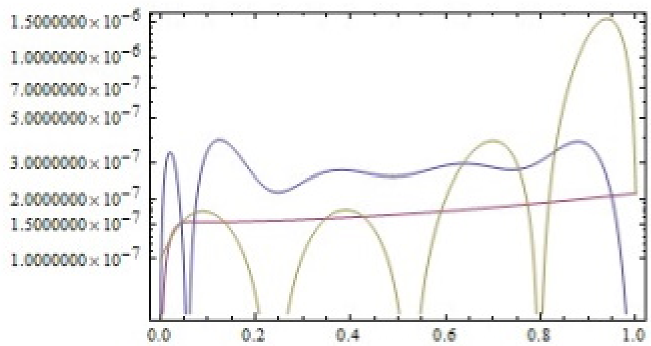

Figure 1: Graph of the error at K=8, 16 and 32

Example 6 *Suppose that the following VIDE [15][16]*

$$\begin{cases} V'(\gamma) = -V(\gamma) + F(\gamma) - \displaystyle\int_0^\gamma y\, V(y)\, dy\ , \quad \gamma \in [0,1]\,. \\ V(0) = 10. \end{cases} \tag{12}$$

The exact solution of equation (12) is $V(\gamma) = 10 - \gamma e^{-\gamma}$ and

$$F(\gamma) = \left(\gamma^2 + 2\gamma + 1\right) e^{-\gamma} + 5\gamma^2 + 8.$$

In Table 2, there is a comparison between the absolute errors of the present method with standard collocation method (SCM) and Chebyshev-Gauss- Lobatto collocation method (CGLCM) [15], finite difference method (FDM) and homotopy analysis method (HAM) [16] at $K = 12$ and different values of γ. In Figure 2, we illustrate the results of the present method at $K = 8, 10, 12$ and 16. The Figure shows that the convergence is exponential and the errors in this method are the best especially when the values of K are large.

Table 2: Maximum absolute errors with various values of K

γ	FCM	SCM [15]	$CGLCM$ [15]	FDM [16]	HAM [16]
0.0000	1.8×10^{-15}	0	0	0	0
0.0714	1×10^{-13}	1.7×10^{-6}	5.9×10^{-6}	2.9×10^{-4}	5.2×10^{-7}
0.1429	10×10^{-14}	1.9×10^{-6}	2.3×10^{-5}	3×10^{-4}	3×10^{-7}
0.2143	8.5×10^{-14}	1.8×10^{-6}	4.9×10^{-5}	5.4×10^{-4}	2.8×10^{-6}
0.2857	8.2×10^{-14}	1.6×10^{-6}	7.9×10^{-5}	5.1×10^{-4}	1.5×10^{-5}
0.3571	7.5×10^{-14}	1.5×10^{-6}	1.1×10^{-4}	7.2×10^{-4}	4.6×10^{-5}
0.4286	6.8×10^{-14}	1.3×10^{-6}	1.2×10^{-4}	6.5×10^{-4}	1.1×10^{-4}
0.5000	6.6×10^{-14}	1.1×10^{-6}	1.3×10^{-4}	8.2×10^{-4}	2.4×10^{-4}
0.5714	5.9×10^{-14}	8.4×10^{-7}	1×10^{-4}	7.4×10^{-4}	4.7×10^{-4}
0.6429	5.3×10^{-14}	6.2×10^{-7}	5.9×10^{-5}	8.6×10^{-4}	8.5×10^{-4}
0.7143	4.4×10^{-14}	5×10^{-7}	4.5×10^{-6}	7.7×10^{-4}	1.5×10^{-3}
0.7857	4.3×10^{-14}	4.3×10^{-7}	6.9×10^{-5}	8.6×10^{-4}	2.4×10^{-3}
0.8571	3.7×10^{-14}	3×10^{-7}	1×10^{-4}	7.7×10^{-4}	3.7×10^{-3}
0.9286	3.2×10^{-14}	1.4×10^{-8}	5.6×10^{-5}	8.2×10^{-4}	5.6×10^{-4}
1.0000	3.4×10^{-14}	4.1×10^{-7}	1.5×10^{-4}	7.3×10^{-4}	8.3×10^{-4}

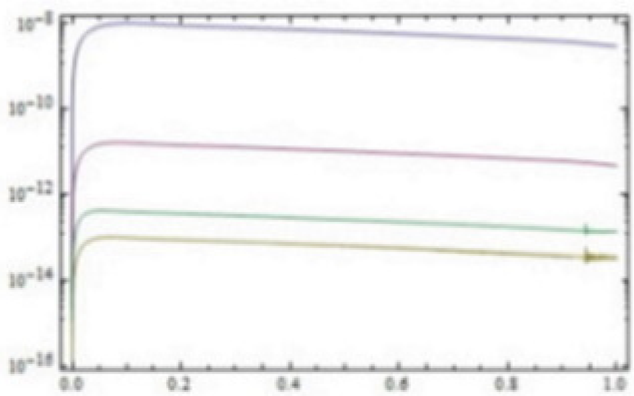

Figure 2: Graph of the error at K=8, 10, 12 and 16

Example 7 *Suppose that the following VIDE [15][16]*

$$\begin{cases} V'(\gamma) = -V(\gamma) + \int\limits_0^\gamma e^{1-\gamma}\, V(y)\ dy\ , \quad \gamma \in [0,1]\,. \\ V'(0) = 1. \end{cases} \tag{13}$$

The exact solution of equation (13) *is* $V(\gamma) = e^{-\gamma}\cosh\gamma$.

Table 3 lists The numerical results obtained by the proposed method for $K = 16$ and different values of γ. We observe the errors in this method are the least for large values of K . The absolute errors of this method at $K = 8$, 10, 12, 16 are plotted in Figure 3. We observe from the Figure that the convergence is exponential.

Table 3: Results of absolute errors for various values of K

γ	FCM	SCM [15]	$CGLCM$ [15]	FDM [16]	HAM [16]
0.0000	0	0	0	0	0
0.0833	4.1×10^{-13}	9.8×10^{-6}	4.6×10^{-5}	1.8×10^{-2}	1.9×10^{-9}
0.1667	3.7×10^{-13}	9.8×10^{-6}	7.6×10^{-5}	2.2×10^{-3}	3.1×10^{-10}
0.2500	3.4×10^{-13}	8.4×10^{-6}	7.5×10^{-5}	1.9×10^{-3}	1.1×10^{-9}
0.3333	3.4×10^{-13}	7.7×10^{-6}	4.1×10^{-5}	4.5×10^{-3}	8.4×10^{-11}
0.4167	3.3×10^{-13}	7.6×10^{-6}	2.1×10^{-5}	2.1×10^{-2}	2.7×10^{-9}
0.5000	3.1×10^{-13}	7.3×10^{-6}	8.8×10^{-5}	7.1×10^{-3}	3.1×10^{-10}
0.5833	3.1×10^{-13}	6.8×10^{-6}	1.3×10^{-5}	1.1×10^{-2}	1.2×10^{-9}
0.6667	3×10^{-13}	6.2×10^{-7}	1.2×10^{-5}	8.2×10^{-3}	5.6×10^{-10}
0.7500	2.9×10^{-13}	5.6×10^{-7}	3.5×10^{-5}	3.4×10^{-3}	1.3×10^{-9}
0.8333	2.9×10^{-13}	5.4×10^{-7}	8.8×10^{-5}	8.2×10^{-3}	6.8×10^{-10}
0.9167	2.8×10^{-13}	5.2×10^{-7}	1.3×10^{-5}	2.9×10^{-3}	5.2×10^{-9}
1.0000	2.6×10^{-13}	4.8×10^{-7}	1.6×10^{-5}	3.3×10^{-3}	9.5×10^{-9}

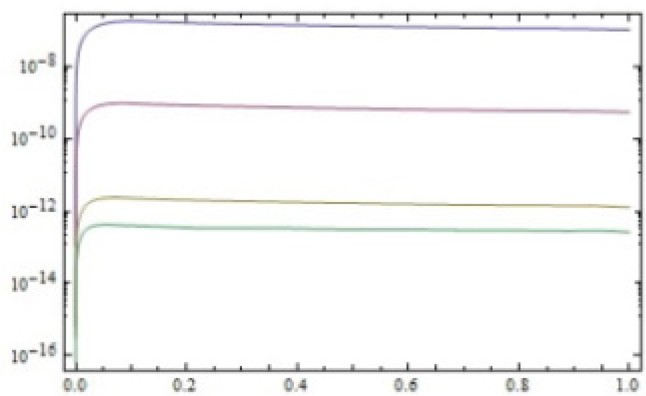

Figure 3: Graph of the absolute error at K=8, 10, 12 and 16

Example 8 *Suppose that the following VIDE*

$$\begin{cases} V'(\gamma) = -\gamma V(\gamma) + F(\gamma) - \int\limits_0^{\gamma} V(y)\ dy\ , \quad \gamma \in [0,1]\,. \\ V'(0) = 0. \end{cases} \tag{14}$$

The exact solution of equation (14) is $V(\gamma) = \sin\gamma$ and

$$F(\gamma) = 1 + \gamma \sin\gamma.$$

In Table 4, the errors of this method are displayed at $K = 8, 10, 12$ and 16. The absolute errors in the proposed method are very small for large values K. The errors are plotted in Figure 4. It is clear from the Figure that the absolute errors decrease drastically with increasing the number of steps.

Table 4: Comparison between the absolute errors with various values of K

K	FCM
8	1.9×10^{-9}
10	2.4×10^{-12}
12	3.4×10^{-15}
16	4.6×10^{-14}

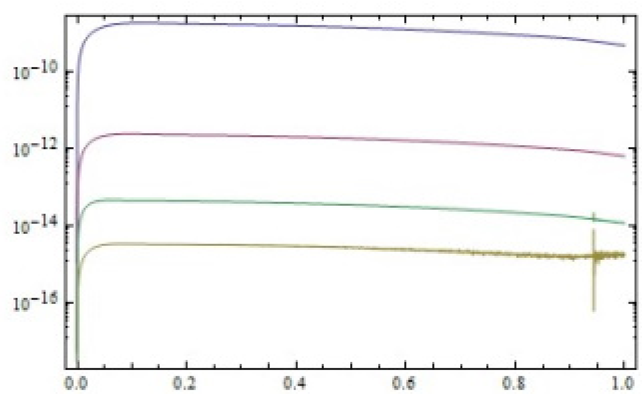

Figure 4: Graph of the error at K=8, 10, 12 and 16

Example 9 *Suppose that the following VIDE*

$$\begin{cases} \gamma^{\frac{1}{2}} V'(\gamma) = -\gamma V(\gamma) + F(\gamma) - \int\limits_0^{\gamma} V(y)\ dy\ , \quad \gamma \in [0,1]\,. \\ V'(0) = 0. \end{cases} \tag{15}$$

The exact solution of equation (15) is $V(\gamma) = \sin\gamma$ and

$$F(\gamma) = 1 - \cos\gamma + \sqrt{\gamma}\cos\gamma + \gamma\sin\gamma.$$

Table 5 presents the absolute error for the studied method with various values of K then displayed them in Figure 5. It is clear from the Figure that the absolute errors decrease drastically with increasing the number of steps.

Table 5: The absolute errors at different values of K

K	FCM
8	1.8×10^{-9}
10	2.3×10^{-12}
12	2×10^{-15}
16	8.1×10^{-14}

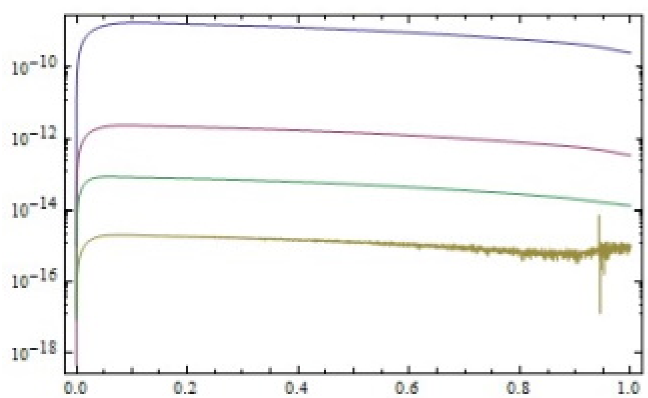

Figure 5: Graph of the error at K=8, 10, 12 and 16

6 Conclusion

The numerical solution of Volterra integro-differential equations is the aim of this article. We use the collocation method based on the operational matrix of Fermat polynomials. Then transform the VIDEs for five examples to a system of linear algebraic equations that are solved by Mathematica software and the errors are evaluated. The spectral results, that are obtained, point that this algorithm is high adequacy, viable, easy in applications. The convergence and error analysis are discussed minutely. This method can solve different types of differential, integral and integro-differential equations.

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