Quasi-least square finite element methods for stationary incompressible magnetohydrodynamics problems☆

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Abstract

This article aims to study the Quasi-least square mixed finite element (FE) method for the approximate solution of Magnato-Hydro-Dynamic equations (MHD). The resulting non-linear system of equations are linearized around a characteristic state, resulting in first order linearized least-square models that yield algebraic system of equations with symmetric positive definite coefficient matrices. A central feature of the method is that it does not require (Ladyzhenskaya-Babuška-Brezzi) LBB conditions on the finite-dimensional subspaces and the resulting bilinear form is symmetric and positive definite. Secondly, it only needs to choose the value of a single parameter to find the well-posedenss of the model equations. For the theoretical accuracy and authentication of the method, we investigate existence of the solutions and obtain a priori error estimates. The variables are fluid velocity, fluid pressure and magnetic field. Numerical tests are performed in order to assess the stability and the accuracy of the resulting methods. Result shows good agreement with analytical solutions.

Keywords: Quasi-least-square finite element scheme, stationary magnetohydrodynamic equations, stability, convergence analysis.

Magnetohydrodynamics (MHD) deals with the study of the interaction of electromagnetic fields and conducting fluids. These equations consist of a coupling between Maxwell equations and Navier-Stokes equations. The field of MHD is first introduced by Alfven in [29]. This field has many applications in astrophysics, geophysics and many other engineering fields, such as liquid metal cooling [2], process metallurgy [3], controlled thermonuclear fusion and sea water propulsion [4] etc. Moreover, the hydrodynamical behavior of conducting fluids, e.g., plasmas, liquid metals, and electrolytes, etc., are usually modeled by the MHD system [5]. Since the applied magnetic fields can induce currents which in turn polarizes the fluid and reciprocally changes the magnetic field itself, the governing system is a combination of hydrodynamics (Navier-Stokes equations) and electromagnetism (Maxwell equations). About the extensive study of the theoretical modeling/numerical analysis of the MHD system, we refer to [6, 10, 11] and the references therein.

A mixed finite element method for the MHD equation is discussed in the following articles [8, 14, 25, 15] and uniqueness and existence of stationary equations have been given in [1].

In this contribution, we develop a quasi-least-square finite element scheme (QLSFES) in the L^2 inner product and perform an analysis of existence and convergence for them. The key idea

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of QLSFES is to apply least-square method in L^2-norm to linearized forms of nonlinear problems. We introduce several methods contrast from the previous studied methods. Say we have developed a non-singular solution a branch of nonlinear problems to analyze existence and convergence of approximate solutions of QLSFES. It has many advantages over the least square finite element method. Firstly, only L^2 -norm is used in these methods which is convenient for programming sense. Secondly, by linearizing procedure one can derive simple iterative methods with symmetric positive definite coefficient metrics. So, these simple iterative methods are convergent in a large region of initial functions. Remember that this is strongly contrast to locally convergent properties of any standard least-square finite element schemes for nonlinear problems. Lastly, QLSFES and analysis for their convergence are independent of the algorithm and theory for non-singular solution branches of nonlinear problems with respect to the two viscosities i.e., fluid viscosity ν and magnetic viscosity ν_m. However, in practical engineering applications, many other complicated nonlinear systems are in practice, such as singular solutions of bifurcation-driven multiplicity [22, 23], viscoelastic fluid flows [19, 20], flows and heat transfer [17], fluids with a velocity dependent or temperature-dependent viscosity [], are of great significance, for further study, we refer reader to (see [25]). For the mention fluid flows, algorithms and theories for approximation of the non-singular solutions of nonlinear problems are invalid. QLSFES can be applied to study such type of complicated nonlinear problems.

The remaining work is organized as follows. In the next section, we introduce magnetohydrodynamic equations linear and non-linear forms and QLSFES. While the former has a simpler form than the later. In section 3, we analyze the existence and convergence of the QLSFES. We show the solutions of QLSFES converges to different solutions of the system depending on the initial guess. In section 4, we analyze the convergent rate of the QLSFES in case that the exact solution is nonsingular. In section 5, we give some numerical experiments and some further discussion to end this work.

This work deals with the numerical resolution of the stationary incompressible magnetohydrodynamics system of equations [13]. The unknowns for this problem are velocity field $\mathbf{u}$, the pressure p in the fluid and the magnetic field $\mathbf{B}$ (in fact the magnetic induction).

The unknowns for this problem are velocity field $\mathbf{u}$, the pressure p in the fluid and the magnetic field $\mathbf{B}$ (in fact the magnetic induction) [13, 18]. The non-dimensional magnetohydrodynamics (MHD) equation for the unknowns are as follows

$$\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{R_e}\Delta\mathbf{u} + \nabla p + S(\mathbf{B}\times\mathbf{curl}\ \mathbf{B}) = \mathbf{f} \ \text{ in } \Omega, \tag{0.1}$$

$$\nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega, \tag{0.2}$$

$$\frac{1}{R_m}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{B}) - S(\mathbf{curl}\ (\mathbf{u}\times\mathbf{B})) = 0 \ \text{ in } \Omega, \tag{0.3}$$

$$\nabla\cdot\mathbf{B} = 0 \ \text{ in } \Omega. \tag{0.4}$$

Here R_e, R_m and $S = \frac{M^2}{R_e R_m}$, with $M > 0$ being the hydrodynamic Reynolds number, magnetic Reynolds number, coupling number and Hartman number respectively. In the industrial cases, we have in mind, $R_e \approx 10^5$, $R_m \approx 10^{-1}$ and $S \approx 1$. To find the velocity $\mathbf{u} = (u_1, u_2)$ and pressure field p define in Ω and magnetic field $\mathbf{B} = (b_1, b_2)$. The function $\mathbf{f}$ represents external force term. In the two-dimensional case, the curl operator $\nabla\times$ applied to a vector $\mathbf{B} = (b1, b2)$ is defined as $\nabla\times\mathbf{B} = \frac{\partial b2}{\partial x} - \frac{\partial b1}{\partial y}$ while the cross product of two vectors $\mathbf{u} = (u1, u2)$ and $\mathbf{B} = (b1, b2)$ is given by $\mathbf{u}\times\mathbf{B} = u_1 b_2 - u_2 b_1$.

The system is set on a simply connected and bounded domain $\Omega \in \mathbb{R}^2$ with simplest essential homogeneous boundary conditions on $\partial\Omega$ [7, 21, 9, 18]:

$$
\begin{aligned}
\mathbf{u} &= 0 \text{ on } \partial\Omega, && (0.5)\\
\mathbf{B}\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega, && (0.6)\\
\mathbf{curl}\ \mathbf{B}\times\mathbf{n} &= 0 \text{ on } \partial\Omega, && (0.7)
\end{aligned}
$$

where $\mathbf{n}$ denotes the normal to Ω [25]. Here (0.5) is a no-slip boundary condition implied by the viscous nature of the fluid, (0.6) and (0.7) are known as perfectly conduction wall.

Remark 1. *Instead of the boundary conditions that $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ in (0.6)and $\boldsymbol{curl}\ \boldsymbol{B}\times\boldsymbol{n} = 0$ in (0.7) for the magnetic field $\boldsymbol{B}$, we can equally apply $\boldsymbol{B}\times\boldsymbol{n} = 0$ and $\boldsymbol{n}\times(\nabla\times\boldsymbol{B}) = 0$ which is also frequently used for the MHD system; see, e.g., [11, 12, 24, 21, 26] .*

1. Quasi-least-square finite element schemes

In this section, we discuss the quasi-least-square finite element schemes based on L^2-inner product for the MHD system of equation.

1.1. Notations and definitions

In the next sections, we will use the following Sobolev function spaces: Let $C^\infty(\Omega)$ be the set of functions of infinitely order derivatives and $C_0^\infty(\Omega) = \{\mathbf{a}, \mathbf{b} \in C^\infty(\Omega)\}$; the support of $\mathbf{a}, \mathbf{b}$ are in Ω (In the whole sections, where $\mathbf{a}$,$\mathbf{b}$ are two known functions as Oseen type iterative values using for the linearization of the system of the equations). Moreover, the standard Sobolev function spaces we introduce are as follows:

$$
\begin{aligned}
F: &= [H^1(\Omega)]^{d\times d},\\
\mathbf{X}: &= H_0^1(\Omega)^d,\\
\mathbf{Q}: &= \{q \in H^1(\Omega);\ \int_\Omega q dx = 0\},\\
\mathbf{Y}: &= H_0^1(\Omega),\\
\mathbf{M}: &= [H^1(\Omega)]^d.
\end{aligned}
$$

Let $\mathfrak{U} = (U_{ij})_{d\times d} = (\mathbf{U}_1, \mathbf{U}_2..., \mathbf{U}_d)$ be a matrix function of $d\times d$ degree with column vectors $\mathbf{U}_k$, $1 \leq k \leq d$. For each d-dimensional vector-valued function $\mathbf{u}$ and matrix $\mathfrak{U}$, define $\nabla\mathbf{u} = (u_1, u_2......., u_d)$ and $\nabla\times\mathfrak{U} = (\nabla\times\mathbf{U}_1, \nabla\times\mathbf{U}_2..., \nabla\times\mathbf{U}_d)$, $\nabla\cdot\mathfrak{U}^T = (\nabla\cdot\mathbf{U}_1, \nabla\cdot\mathbf{U}_2..., \nabla\cdot\mathbf{U}_d)^T$ similarly $\mathbf{Z} = curl\mathbf{B}$. By using the symbolical representations, we can easily describe the first order system equivalently to the original system and their algorithms.

1.2. The first order MHD system and QLSFES algorithms

We develop an algorithm to solve the incompressible MHD system (0.1)-(0.4). The first quasi-least-square scheme is based on the following first-order system:

$$
\begin{aligned}
-\nu\nabla\cdot\mathfrak{U}^T + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S(\mathbf{B}\times curl\ \mathbf{B}) &= \mathbf{f} \quad \text{in } \Omega, && (1.1)\\
\mathfrak{U} - \nabla\mathbf{u} &= 0 \quad \text{in } \Omega, && (1.2)\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, && (1.3)\\
\nu_m curl\ (\mathbf{Z}) - S(curl\ (\mathbf{u}\times\mathbf{B})) &= 0 \quad \text{in } \Omega, && (1.4)\\
\mathbf{Z} - curl\mathbf{B} &= 0 \quad \text{in } \Omega, && (1.5)
\end{aligned}
$$

$$\nabla \cdot \mathbf{B} \;=\; 0 \qquad in\ \Omega. \tag{1.6}$$

Here in the upcoming formulation for any $\mathbf{a}, \mathbf{b} \in X$ is supposed to make the nonlinear system in to linear form [17, 19]. Because $\mathbf{a}$ and $\mathbf{b}$ are known functions(usually stand for the approximate solutions for $\mathbf{u}$ and $\mathbf{B}$ in the previous iterative step of the Picard iterations and are also supposed regular). We denote viscosity of the hydrodynamic fluid $\nu = \frac{1}{R_e}$, and magnetic diffusivity $\nu_m = \frac{1}{R_m}$. For further formulation the bilinear form with respect to $(\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})$ and $(\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c})$ in $(\digamma \times \mathbf{X} \times \mathbf{Q} \times \mathbf{Y} \times \mathbf{M})$ can be written as

$$\begin{aligned}
&L(\mathbf{a}, \mathbf{b}; (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B}), (\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c})) \\
&\quad = \big(-\nu \nabla \cdot \mathfrak{U}^T + (\mathbf{a} \cdot \nabla)\mathbf{u} + \nabla p + S(\mathbf{b} \times curl\ \mathbf{B}), \\
&\quad -\nu \nabla \cdot \mathcal{W}^T + (\mathbf{a} \cdot \nabla)\mathbf{w} + \nabla q + S(\mathbf{b} \times curl\ \mathbf{c})\big) \\
&\quad + (\mathfrak{U} - \nabla \mathbf{u}, \mathcal{W} - \nabla \mathbf{w}) + \gamma^2 (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{w}) \\
&\quad + (\nu_m curl\ \mathbf{Z} - curl\ (\mathbf{u} \times \mathbf{b}), \nu_m curl\ \tau - curl\ (\mathbf{w} \times \mathbf{b})) \\
&\quad + (\mathbf{Z} - curl\ \mathbf{B}, \tau - curl\ \mathbf{c}) + \eta^2 (\nabla \cdot \mathbf{B}, \nabla \cdot \mathbf{c})
\end{aligned} \tag{1.7}$$

where γ and η are the two positive constants to be determined in later sections. Suppose that $(\mathbf{u}, p, \mathbf{B})$ are the solution of system equations $(0.1) - (0.4)$ is in $[H^2(\Omega)]^d \times H^1(\Omega) \times H^2(\Omega)$. Let $\mathfrak{U} = \nabla \mathbf{u}$ and $\mathbf{Z} = curl\ \mathbf{B}$. The solution $(\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})$ satisfies the following quasi-least-squares variational formulation:

$$\begin{aligned}
&L(\mathbf{u}, \mathbf{B}; (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B}), (\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c})) \\
&\quad = (f, -\nu \nabla \cdot \mathcal{W}^T + \mathbf{u} \cdot \nabla \mathbf{w} + \nabla q + S(\mathbf{B} \times curl\ \mathbf{c})) \\
&\quad \forall (\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c}) \ \in \digamma \times \mathbf{X} \times \mathbf{Q} \times \mathbf{Y} \times \mathbf{M}.
\end{aligned} \tag{1.8}$$

1.3. Finite element Variational formulation

For the mathematical setting, some notations used in function spaces are introduced. For $m \in N$ the norm associated with Sobolev space $W^{m,p}(G)$ for $[m \geq 0$ and $1 \leq p \leq \infty]$ by $|| \cdot ||_{W^{m,p}(G)}$, with the special case $W^{m,2}(\Omega)$ being written as $H^m(\Omega)$ with the norm $|| \cdot ||_m$ and seminorm $|\cdot|_m$[16, 28]. If $p = 2$ and $m = 0$ then $W_2^0(\Omega)$. The $L^2(\Omega)$ inner product and norm are denoted by $(\cdot, \cdot)$, $|| \cdot || = || \cdot ||_0$ respectively, the $L^p(\Omega)$ norm by $|| \cdot ||_{L^p}$, with the special cases of $L^2(\Omega)$ and $L^\infty(\Omega)$ norms being written as $|| \cdot ||$ and $|| \cdot ||_\infty$. For the sake of simplicity, we omit G if $G = \Omega$ without any confusion.

In order to find the corresponding variational formulation, we introduce finite element spaces $(\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h) \ \subset (\digamma \times \mathbf{X} \times \mathbf{Q} \times \mathbf{Y} \times \mathbf{M})$ on triangulations T_h. Where T_h is a family of finite element triangulations of the domain Ω and subscript h denotes the largest mesh size of the elements in T_h. Based on the quasi-least-square formulation, we define

QLSFES. Find $(\mathfrak{U}^h, \mathbf{u}^h, p^h, \mathbf{Z}^h, \mathbf{B}^h) \in (\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h)$ such that

$$\begin{aligned}
&L\big(\mathbf{u}^h, \mathbf{B}^h; (\mathfrak{U}^h, \mathbf{u}^h, p^h, \mathbf{Z}^h, \mathbf{B}^h), (\mathcal{W}^h, \mathbf{w}^h, q^h, \tau^h, \mathbf{c}^h)\big) \\
&\quad = \big(f, -\nu \nabla \cdot \mathcal{W}^h + \mathbf{u}^h \cdot \nabla \mathbf{w}^h + \nabla q^h + S(\mathbf{B}^h \times curl\ \mathbf{c}^h)\big) \\
&\quad \forall \ \ (\mathcal{W}^h, \mathbf{w}^h, q^h, \tau^h, \mathbf{c}^h) \in (\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h).
\end{aligned} \tag{1.9}$$

Remark 2. *We have the linear scheme for the MHD equation. It is clear that for a given* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, the bi-linear form* $L(\boldsymbol{a}, \boldsymbol{b}; \cdot, \cdot)$ *are least-square bi-linear form derived by applying the least-squares minimization principle to linearized form of a nonlinear system (1.8). Here we take* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *as a known values for the approximation of* $\boldsymbol{u}$ *and* $\boldsymbol{B}$*. Thus we call this scheme as QLSFES.*

Remark 3. *QLSFES is a simple symmetric form. We introduce the parameters γ and η with the incompressibility conditions, which plays a key role in practical calculation. It is impossible to ensure the coerciveness of $L(\boldsymbol{a}, \boldsymbol{b} : \cdot, \cdot)$ for each $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{X}$. But by choosing suitable γ and η they became positive definite if $\boldsymbol{a}$ and $\boldsymbol{b}$ lies in some bounded function sets which contain solution of the (0.1)-(0.4). In next sections, we would discuss the suitable and judicious choice of the parameters γ and η for the linear form. The well feature of this technique is that some one can use some iterative procedure to find solutions of nonlinear complex problems in a large region of functions without requiring good guess for exact solutions. We confirm this feature in numerical examples in our experimental part.*

2. Existence and convergence of solutions of QLSFES

In this section, we derive existence and convergence of solution of QLSFES. The process of this analysis is divided into four steps:

- We analyze the positive definite property of the bilinear form $L(\mathbf{a}, \mathbf{b} : \cdot, \cdot)$, we prove that for a given γ and η there exist a bounded function set such that $L(\mathbf{a}, \mathbf{b} : \cdot, \cdot)$ is positive definite and continuous as $\mathbf{a}$ and $\mathbf{b}$ lies in this function set.

- In second step, we determine a large region of function set which contains all solutions of the system $(1.1) - (1.6)$. We intend to seek all solutions of the nonlinear system by suitable choice of γ and η, $L(\mathbf{a}, \mathbf{b} : \cdot, \cdot)$ is a positive definite as $\mathbf{a}$ and $\mathbf{b}$ are in this large region of functions.

- To prove existence of solutions of QLSFES, we introduce the nonlinear map such that solutions of QLSFES are fixed points of the map. In step one and step two, we prove that for a suitable parameter γ and η, the nonlinear map is uniquely determined in this bounded function set and maps it to itself (see detail in lemma 2.3 and 2.4). Moreover, by using the fixed point theory, we prove existence of solutions of the QLSFES in theorem 2.1.

- In the last step, we shall prove the convergence of the solutions of QLSFES briefly in theorem 2.2.

  Before going to proceed the main work, we give several known results which are applicable on the next sections. Hence, by Poincaré's inequality and the embedding theory, there exists positive constants C, a_0 and a_1 which are independent on the parameters and any mesh size domain.

$$
\begin{aligned}
a_0 \parallel \mathbf{w} \parallel_{L^4}^2 &\leq \parallel \nabla \mathbf{w} \parallel_0^2, \\
a_1 \parallel \mathbf{w} \parallel_1^2 &\leq \parallel \nabla \mathbf{w} \parallel_0^2, \quad \forall\, \mathbf{w} \in \mathbf{X}. \\
\parallel \mathbf{u} \parallel_{L^4}^2 &\leq C \parallel \mathbf{u} \parallel_1^2 \quad \forall\, \mathbf{u} \in \mathbf{X}.
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
\parallel \mathbf{B} \parallel_0^2 &\leq a_2(\parallel curl\ \mathbf{B} \parallel_0^2 + \parallel \nabla \cdot \mathbf{B} \parallel_0^2), \quad \forall\, \mathbf{B} \in \mathbf{M} && (2.2)\\
\parallel \mathbf{B} \parallel_{L^4}^2 &\leq C \parallel \mathbf{B} \parallel_1^2 \\
\parallel curl\ \mathbf{B} \parallel_0^2 &\leq a_3 \parallel \mathbf{B} \parallel_1^2 && (2.3)\\
\parallel \mathbf{B} \parallel_0^2 &\leq C \parallel curl\ \mathbf{B} \parallel_0^2 && (2.4)\\
&\leq C \parallel \mathbf{B} \parallel_0^2 && (2.5)
\end{aligned}
$$

Two Green's formula on integration by parts [30], are as follows:

$$
\begin{aligned}
(\nabla\ \mathbf{u},\mathbf{w})-(\mathbf{u},\nabla\ \mathbf{w}) &= 0, \quad \forall\ \ \mathbf{u},\mathbf{w}\in\mathbf{X} \\
(\mathbf{a}\cdot\nabla\mathbf{w},\mathbf{w})+\frac{1}{2}(div\ \mathbf{a},\mid\mathbf{w}\mid^2) &= 0, \quad \forall\mathbf{w},\mathbf{a}\in\mathbf{X} \\
(\mathbf{b}\times curl\ \mathbf{B},\mathbf{w})-(\mathbf{u}\times\mathbf{b},curl\ \mathbf{B}) &= 0, \qquad \forall\mathbf{w}\in\mathbf{X}, \quad \forall\mathbf{b},\mathbf{B}\in\mathbf{M} \\
\parallel\mathbf{a}\times\mathbf{b}\parallel_0^2 &\leq C\parallel\mathbf{a}\parallel_{L^\infty}^2\parallel\mathbf{b}\parallel_0^2 \qquad \forall\mathbf{a},\mathbf{b}\in(L^2(\Omega)^{d\times d}).
\end{aligned}
\tag{2.6}
$$

Here $(\cdot,\cdot)$ is an inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$. For each $q\in L^2(\Omega)$ satisfies the boundary condition $(q,1)=0$ there exists $\phi\in[H_0^1(\Omega)]^d$ such that

$$
\begin{aligned}
\nabla\cdot\phi &= q, \\
\parallel\phi\parallel_1 &\leq C_0\parallel q\parallel_0 .
\end{aligned}
\tag{2.7}
$$

Lemma 2.1. *Let constants $K>0,\quad 0<\delta<2a_0min(\nu,\frac{\nu_m}{2})$ and $\gamma>max(K,1)$. There exists two constants α and $\alpha*$ such that for each $\boldsymbol{a}\in\boldsymbol{X},\boldsymbol{b}\ \in\boldsymbol{M}$ satisfying $\parallel\nabla\boldsymbol{a}\parallel_0\leq K$ and $\parallel\nabla\cdot\boldsymbol{a}\parallel_0\leq\delta$, $\parallel\boldsymbol{b}\parallel_{L^\infty}\leq\pi$ and for each $(\mathcal{W}^h,\boldsymbol{w}^h,q^h,\tau^h,\boldsymbol{c}^h)\in\digamma,\boldsymbol{X},\boldsymbol{Q},\boldsymbol{Y},\boldsymbol{M}$,*

$$
\begin{aligned}
(i)\quad \alpha\big[\parallel\nabla\boldsymbol{w}\parallel_0^2+\parallel curl\ \boldsymbol{c}\parallel_0^2\big] &\leq L\big(\boldsymbol{a},\boldsymbol{b};(\mathcal{W},\boldsymbol{w},q,\tau,\boldsymbol{c}),(\mathcal{W},\boldsymbol{w},q,\tau,\boldsymbol{c})\big), \\
(ii)\quad \alpha\alpha^*\big[\parallel\mathcal{W}\parallel_0^2+\parallel\tau\parallel_0^2+\parallel q\parallel_0^2\big] &\leq L\big(\boldsymbol{a},\boldsymbol{b};(\mathcal{W},\boldsymbol{w},q,\tau,\boldsymbol{c}),(\mathcal{W},\boldsymbol{w},q,\tau,\boldsymbol{c})\big),
\end{aligned}
\tag{2.8}
$$

where the constants α and α^* are defined as follows

$$
\alpha = min(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5), \quad \alpha^* = \frac{\alpha}{\alpha+1}, \tag{2.9}
$$

$\alpha_i, 1\leq i\leq 5$, are given by

$\alpha^*=min\big(\frac{1}{2(1+\alpha)},\frac{1}{2(C_0)^2\alpha},\frac{1}{4\nu^2C_0^2(1+\alpha)},\frac{a_0^2a_1}{2C_0^2(\delta+K)^2a_1}\big)$

Proof. From the given conditions and the Young's inequality, we can get

$$
\begin{aligned}
\parallel\nabla\mathbf{w}\parallel_{L^2}^2 &= (\nabla\mathbf{w}-\mathcal{W},\nabla\mathbf{w})+\nu^{-1}\big[(-\nu\nabla\cdot\mathcal{W}^T+(\mathbf{a}\cdot\nabla)\mathbf{w}+\nabla q+S\mathbf{b}\times curl\ \mathbf{c},\mathbf{w})\big] \qquad (2.10)\\
&+\big[\frac{1}{2\nu}(\nabla\cdot\mathbf{a}\mathbf{w},\mathbf{w})\big]+\nu^{-1}(q,\nabla\cdot\mathbf{w})+\nu^{-1}[(S\mathbf{b}\times curl\ \mathbf{c},\mathbf{w})] \\
&\leq \frac{1}{2a_1^2\epsilon_0\nu^2}\parallel-\nu\nabla\cdot\mathcal{W}^T+(\mathbf{a}\cdot\nabla)\mathbf{w}+\nabla q+S\mathbf{b}\times curl\ \mathbf{c}\parallel_{L^2}^2 \\
&+\frac{1}{2\epsilon_0}\parallel\nabla\mathbf{w}-\mathcal{W}\parallel_{L^2}^2+\frac{\gamma^2}{4\epsilon_1}\parallel\nabla\cdot\mathbf{w}\parallel_{L^2}^2+\frac{\epsilon_1}{\gamma^2\nu^2}\parallel q\parallel_{L^2}^2+(2\epsilon_0+\frac{\delta}{2\nu^2a_0})\parallel\nabla\mathbf{w}\parallel_{L^2}^2 \\
&+\frac{S^2\pi^2}{4\nu^2a_0\epsilon_0}\parallel curl\mathbf{c}\parallel_{L^2}^2 \qquad (2.11)
\end{aligned}
$$

and

$$
\parallel\mathcal{W}\parallel_{L^2}^2 \leq 2\big[\parallel\nabla\mathbf{w}\parallel_{L^2}^2+\parallel\nabla\mathbf{w}-\mathcal{W}\parallel_{L^2}^2\big]. \tag{2.12}
$$

Let us suppose ϕ satisfies (2.7), we get the following estimation

$$
\begin{aligned}
\parallel q\parallel_{L^2}^2 &= -(-\nu\nabla\cdot\mathcal{W}^T+(\mathbf{a}\cdot\nabla)\mathbf{w}+\nabla q+S\ \mathbf{b}\times curl\ \mathbf{c},\phi) \\
&+\nu(\mathcal{W},\nabla\phi)+\big((\mathbf{a}\cdot\nabla)\mathbf{w}+S\ \mathbf{b}\times curl\ \mathbf{c},\phi\big)
\end{aligned}
\tag{2.13}
$$

$$\leq \Bigg[\| (-\nu\nabla\cdot\mathcal{W}^T + (\mathbf{a}\cdot\nabla)\mathbf{w} + \nabla q + S\ \mathbf{b}\times curl\ \mathbf{c}) \|_{L^2}$$
$$+ \nu \| \mathcal{W} \|_{L^2} + \frac{1}{a_0}(\| \nabla\cdot\mathbf{a} \|_{L^2} + \| \nabla\mathbf{a} \|_{L^2}) \| \nabla\mathbf{w} \|_{L^2} + S\pi \| curl\ \mathbf{c} \|_{L^2} \Bigg] \| \phi \|_{H^1},$$

by using the (2.7), implies

$$\| q \|_{L^2}^2 \leq 2C_0^2 \Bigg[\| -\nu\nabla\cdot\mathcal{W}^T + (\mathbf{a}\cdot\nabla)\mathbf{w} + \nabla q + S\ \mathbf{b}\times curl\ \mathbf{c} \|_{L^2}^2 \tag{2.14}$$
$$+ \nu^2 \| \mathcal{W} \|_{L^2}^2 + \frac{(\delta+K)^2}{a_0^2} \| \nabla\mathbf{w} \|_{L^2}^2 + S^2\pi^2 \| curl\ \mathbf{c} \|_{L^2}^2 \Bigg],$$

At the same time,

$$\| curl\ \mathbf{c} \|_{L^2}^2 = (curl\ \mathbf{c}, curl\ \mathbf{c} - \tau) + \nu_m^{-1}(\mathbf{c}, \nu_m curl\ \tau - curl\ (\mathbf{w}\times\mathbf{b})) \tag{2.15}$$
$$+ \nu_m^{-1}(\mathbf{c}, curl\ (\mathbf{w}\times\mathbf{b}))$$

$$\| curl\ \mathbf{c} \|_{L^2}^2 \leq 4 \| curl\ \mathbf{c} - \tau \|_{L^2}^2 + \frac{4\nu_m a_2}{{\nu_m}^2 a_2 - \delta} \| \nu_m\ curl\ \tau - curl(\mathbf{w}\times\mathbf{b}) \|_{L^2}^2 \tag{2.16}$$
$$+ \frac{2\pi^2}{\nu_m^2} \| \nabla\mathbf{w} \|_{L^2}^2 + \frac{1}{4} \| \nabla\cdot\mathbf{c} \|_{L^2}^2 .$$

By putting (2.14) and (2.16) in equation (2.10), we get

$$\| \nabla\mathbf{w} \|_{L^2}^2 \leq \frac{1}{2a_1^2\epsilon_0\nu^2} \| (-\nu\nabla\cdot\mathcal{W}^T + (\mathbf{a}\cdot\nabla)\mathbf{w} + \nabla q + S\mathbf{b}\times curl\ \mathbf{c}) \|_{L^2}^2 \tag{2.17}$$
$$+ \frac{1}{2\epsilon_0} \| \nabla\mathbf{w} - \mathcal{W} \|_{L^2}^2 + \frac{\gamma^2}{4\epsilon_1} \| \nabla\cdot\mathbf{w} \|_{L^2}^2 + (\epsilon_0 + \frac{\delta}{2\nu a_0}) \| \nabla\mathbf{w} \|_{L^2}^2$$
$$+ \frac{a_2 S^2}{4\nu^2\epsilon_1} \| \nu_m curl\tau - curl(\mathbf{w}\times\mathbf{b}) \|_{L^2}^2 + 2\epsilon_1 \| curl\mathbf{c} \|_{L^2}^2$$
$$+ \epsilon_1 \| \nabla\cdot\mathbf{c} \|_{L^2}^2 + \frac{S^2\nu_m^2}{4\nu^2\epsilon_1} \| curl\mathbf{c} - \tau \|_{L^2}^2$$
$$+ \frac{2\epsilon_1 C_0^2}{\gamma^2\nu^2} \Bigg[\| -\nu\nabla\cdot\mathcal{W}^T + (\mathbf{a}\cdot\nabla)\mathbf{w} + \nabla q + S\ \mathbf{b}\times curl\ \mathbf{c} \|^2$$
$$+ \nu^2 \| \mathcal{W} \|_{L^2}^2 + \frac{(\delta+K)^2}{a_0^2} \| \nabla\mathbf{w} \|_{L^2}^2 + S^2\pi^2 \| curl\ \mathbf{c} \|_{L^2}^2 \Bigg]$$
$$\leq \left(\frac{1}{2a_1^2\epsilon_0\nu^2} + \frac{2\epsilon_1 C_0^2}{\gamma^2\nu^2} \right) \| (-\nu\nabla\cdot\mathcal{W}^T + (\mathbf{a}\cdot\nabla)\mathbf{w} + \nabla q + S\mathbf{b}\times curl\ \mathbf{c}) \|_{L^2}^2$$
$$+ \left(\frac{4\epsilon_1 C_0^2}{\gamma^2} + \frac{1}{2\epsilon_0} \right) \| \nabla\mathbf{w} - \mathcal{W} \|_{L^2}^2 + \frac{\gamma^2}{4\epsilon_1} \| \nabla\cdot\mathbf{w} \|^2$$
$$+ \left(\epsilon_0 + \frac{\delta}{2\nu a_0} + \frac{2\epsilon_1 C_0^2(\delta+K)^2}{\gamma^2\nu^2 a_0^2} + \frac{4\epsilon_1 C_0^2}{\gamma^2} \right) \| \nabla\mathbf{w} \|_{L^2}^2$$
$$+ \left(\frac{2\epsilon_1 C_0^2 S^2\pi^2}{\gamma^2\nu^2} + 2\epsilon_1 \right) \| curl\ \mathbf{c} \|_{L^2}^2$$

$$+\frac{a_2S^2}{4\nu^2\epsilon_1}\parallel \nu_m curl\tau - curl(\mathbf{w}\times\mathbf{b})\parallel_{L^2}^2 +\epsilon_1\parallel \nabla\cdot\mathbf{c}\parallel_{L^2}^2 +\frac{S^2\nu_m^2}{4\nu^2\epsilon_1}\parallel curl\mathbf{c}-\tau\parallel_{L^2}^2 .$$

Letting the parameters
$\epsilon_0=\frac{1}{4}(1-\frac{\delta}{2\nu a_0})$, $\epsilon_1=\frac{a_0\nu\nu_m^2(2a_0\nu-\delta)}{16(C_0^2(\delta+K)^2\nu_m^2+2C_0^2\nu^2\nu_m^2a_0^2+2a_0^2C_0^2S^2\pi^4+2a_0^2\nu^2\pi^2)}$,
we have

$$\begin{aligned}
\parallel\nabla\mathbf{w}\parallel_{L^2}^2 \leq & \left(\frac{1}{4a_1^2\epsilon_0\nu^2}+\frac{C_0^2\epsilon_1}{\epsilon_0\nu^2}\right)\parallel(-\nu\nabla\cdot\mathcal{W}^T+(\mathbf{a}\cdot\nabla)\mathbf{w}+\nabla q+S\mathbf{b}\times curl\ \mathbf{c})\parallel_{L^2}^2 \\
&+\left(\frac{2C_0^2\epsilon_1}{\epsilon_0}+\frac{1}{4\epsilon_0^2}\right)\parallel\nabla\mathbf{w}-\mathcal{W}\parallel_{L^2}^2+\frac{\gamma^2}{8\epsilon_0\epsilon_1}\parallel\nabla\cdot\mathbf{w}\parallel_{L^2}^2 \\
&+\left(\frac{\epsilon_1C_0^2S^2\pi^2}{4\epsilon_0\nu^2}+\frac{3\epsilon_1}{4\epsilon_0}\right)\parallel\nabla\cdot\mathbf{c}\parallel_{L^2}^2 \\
&+\left(\frac{a_2S^2}{8\epsilon_0\epsilon_1\nu^2}+\frac{4a_2\epsilon_1C_0^2S^2\pi^2}{\epsilon_0\nu^2\nu_m^2}+\frac{4a_2\epsilon_1}{\epsilon_0\nu_m^2}\right)\parallel\nu_m curl\tau-curl(\mathbf{w}\times\mathbf{b})\parallel_{L^2}^2 \\
&+\left(\frac{S^2\nu_m^2}{8\epsilon_0\epsilon_1\nu^2}+\frac{4\epsilon_1C_0^2S^2\pi^2}{\epsilon_0\nu^2}+\frac{4\epsilon_1}{\epsilon_0}\right)\parallel curl\mathbf{c}-\tau\parallel_{L^2}^2 .
\end{aligned}\tag{2.18}$$

we estimate $\parallel\tau\parallel_{L^2}^2$ again:

$$\begin{aligned}
\parallel\tau\parallel_{L^2}^2 &\leq 2\big[\parallel curl\ \mathbf{c}-\tau\parallel_{L^2}^2+\parallel curl\ \mathbf{c}\parallel_{L^2}^2\big] \\
&\leq\ 2\parallel curl\ \mathbf{c}-\tau\parallel_{L^2}^2+\frac{4\nu_m\alpha_0}{\nu_m\alpha_0-\delta}\big[\parallel curl\ \mathbf{c}-\tau\parallel_{L^2}^2+\frac{1}{\alpha_1{\nu_m}^2}\parallel\nu_m\ curl\ (\mathbf{b}\times\mathbf{w})\parallel_{L^2}^2\big].
\end{aligned}\tag{2.19}$$

From (2.16) and (2.18), we know that the end of condition given in equation (2.8 (i)) holds for the α defined by (2.8)and (2.9), while by (2.8 (ii)), (2.12)(2.14) and (2.19) also holds □

Remark 4. *In lemma 2.1, there are three parameters K, δ and γ. Parameters K and δ are two constrain conditions for the gradient and divergence of $\boldsymbol{a}$ in the lemmas below, we would see that K is related to bound of the right hand side terms. By using K, we can know the bounded function set which contains all of solutions of the model equations. On the other hand, it is not easy to treat divergence-free condition in practical calculation. Here we deal divergence-free condition to $\parallel\nabla\cdot\boldsymbol{a}\parallel_{L^2}\leq\delta$ and $\parallel\nabla\cdot\boldsymbol{b}\parallel_{L^2}\leq\pi$. The coefficient α is independent of K and γ but it depends upon δ and π, α becomes smaller if δ is larger. Hence lemma 2.1 gives formula to calculate the parameter α. This will be useful to determine the parameter γ in practical applications. The parameter γ is a penalty factor to control the constrain condition for divergence of the velocity field. The parameter α^* has actually no practical applications.*

Remark 5. *It is necessary to understand that the magnetic field is a solenoidal field so it is not consider as compressible or incompressible more, we refer reader to the work of stabilization magnetohydrodynamic equation [10]. To relax or penalize, we do not have any coefficient for the so called curl or divergence of magnetic field in this segment. We may consider this problem in our future work, where we will consider two inf-sup conditions for the magnetohydrodynamics model equations.*

In practical applications, the entirely positive definite property of $L(\mathbf{a},\mathbf{b};\cdot,\cdot)$ for all $\mathbf{a}\in\mathbf{X}$ is not necessary and similarly assumption for term $\mathbf{b}$ is applied. However, one seek approximate solutions in the bounded function set which contains all exact soluitions.We need to find this set. To this end, introduce function sets in $\mathcal{F} \times \mathbf{X} \times \mathbf{Q} \times \mathbf{Y} \times \mathbf{M}$

$$\widetilde{\mathscr{L}}(\delta, K, \rho) = \Big[(\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c}); \| \nabla \cdot \mathbf{w} \|_{L^2} \le \delta, \quad [\| \nabla \mathbf{w} \|_{L^2}^2 + \| \, curl \ \mathbf{c} \|_{L^2}^2]^{1/2} \le K, \quad [\| \mathcal{W} \|_{L^2}^2 + \| \tau \|_{L^2}^2 + \| q \|_{L^2}^2]^{1/2} \le \frac{K}{\sqrt{\rho}}\Big], \tag{2.20}$$

$$\widetilde{\mathscr{L}}^h(\delta, K, \rho) = \widetilde{\mathscr{L}}(\delta, K, \rho) \cap [\mathcal{F}^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h]. \tag{2.21}$$

The following lemma shows that all solutions of (1.1)-(1.6) are in the bounded set $\widetilde{\mathscr{L}}$ for some given parameters δ, K, ρ.

Lemma 2.2. *Let $0 < \delta < 2a_0 min(\nu, \frac{\nu_m}{2})$. Assume that α and α^* satisfy (2.8 i and ii) and K satisfies*

$$\| \boldsymbol{f} \|_{L^2} \le min(\sqrt{\alpha}, 1)K \tag{2.22}$$

All solutions of system equations (1.1)-(1.6) are in the function $\widetilde{\mathscr{L}}(\delta, K, \alpha^*)$.

Proof.

Let $(\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})$ be one solution of the system . For all $\gamma \ge 0$, we see that

$$\begin{aligned} &L(\mathbf{u}, \mathbf{B}; (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B}), (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})) \\ &\quad = (f, -\nu \nabla \cdot \mathfrak{U}^T + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p + S\mathbf{B} \times curl \mathbf{B}). \end{aligned} \tag{2.23}$$

This implies that

$$L(\mathbf{u}, \mathbf{B}; (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B}), (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})) \le \| \mathbf{f} \|_{L^2}^2 . \tag{2.24}$$

We have the condition $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \mathbf{B} = 0$ *in* Ω,

$$\begin{aligned} \| \nabla \mathbf{u} \|_{L^2}^2 &= \nu^{-1}(-\nu \nabla \cdot \mathfrak{U}^T + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + S\mathbf{B} \times curl \ \mathbf{B}, \mathbf{u}) \\ &+ (\nabla \mathbf{u} - \mathfrak{U}, \nabla \mathbf{u}) - \nu^{-1}(S\mathbf{B} \times curl \ \mathbf{B}, \mathbf{u}) \\ &\le \nu^{-1}(-\nu \nabla \cdot \mathfrak{U}^T + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + S\mathbf{B} \times curl \ \mathbf{B}, \mathbf{u}) \\ &+ (\nabla \mathbf{u} - \mathfrak{U}, \nabla \mathbf{u}) + \frac{S}{\nu}(\nu_m curl \mathbf{Z} - curl(\mathbf{u} \times \mathbf{B}), \mathbf{B}) + \frac{S\nu_m}{\nu}(curl \mathbf{B} - \mathbf{Z}, curl \mathbf{B}) \\ &\le \ \frac{2}{a_1^2 \nu^2} \| -\nu \nabla \cdot \mathfrak{U}^T + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + S\mathbf{B} \times curl \ \mathbf{B} \|_{L^2}^2 \\ &+ 2 \| \nabla \mathbf{u} - \mathfrak{U} \|_{L^2}^2 + \frac{1}{4} \| \nabla \mathbf{u} \|_{L^2}^2 \\ &+ \frac{2a_2 S^2}{\nu^2} \| \nu_m curl \mathbf{Z} - curl(\mathbf{u} \times \mathbf{B}) \|_{L^2}^2 + \frac{1}{4} \| curl \mathbf{B} \|_{L^2}^2 + \frac{2S^2 \nu_m^2}{\nu^2} \| curl \mathbf{B} - \mathbf{Z} \|_{L^2}^2 \end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\| curl\ \mathbf{B} \|_{L^2}^2 \leq 2 \| curl\ \mathbf{B} - \mathbf{Z} \|_{L^2}^2 + \frac{2a_2}{{\nu_m}^2} \| \nu_m\ curl\ \mathbf{Z} - curl(\mathbf{u} \times \mathbf{B}) \|_{L^2}^2 \\ + \frac{2}{a_1^2 S^2 \nu_m^2} \| -\nu \nabla \cdot \mathfrak{U}^T + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p + S\mathbf{B} \times curl\ \mathbf{B} \|_{L^2}^2 \\ + \frac{2\nu^2}{S^2 \nu_m^2} \| \nabla \mathbf{u} - \mathfrak{U} \|_{L^2}^2 + \frac{1}{4} \| \nabla \mathbf{u} \|_{L^2}^2 + \frac{1}{4} \| curl\ \mathbf{B} \|_{L^2}^2 .\end{aligned} \tag{2.26}$$

Hence

$$\widetilde{\alpha}[\| \nabla \mathbf{u} \|_{L^2}^2 + \| curl\mathbf{B} \|_{L^2}^2] \leq L(\mathbf{u}, \mathbf{B}; (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B}), (\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})) \leq \| \mathbf{f} \|_{L^2}^2, \tag{2.27}$$

where $\widetilde{\alpha} = min(\frac{a_1^2 S^2 \nu^2 \nu_m^2}{4(S^2 \nu_m^2 + \nu^2)}, \frac{S^2 \nu_m^2}{4(S^2 \nu_m^2 + \nu^2)}, \frac{\nu^2}{4a_2 S^2}, \frac{\nu^2}{4S^2 \nu_m^2})$. Hence it can be easily checked that $\alpha \leq \widetilde{\alpha}$. Because $\| \nabla \mathbf{u} \|_{L^2}^2 \leq K$. By taking $\gamma \geq max(1, K)$ and using lemma 3.1, we know that $(\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})$ is in $\widetilde{\mathscr{L}}(\delta, K, \alpha^*)$.

□

We seek approximate solution of the system (1.1) -(1.6). To this end, we define a nonlinear map from $\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h$ into $\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h$ given as

$$\mathcal{L}(\mathcal{W}^h, \mathbf{w}^h, q^h, \tau^h, \mathbf{c}^h) = (\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h)$$

such that for each $(\overline{\mathcal{W}}^h, \overline{\mathbf{w}}^h, \overline{q}^h, \overline{\tau}^h, \overline{\mathbf{c}}^h) \in \digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h$,

$$\begin{aligned}&\mathcal{L}(\mathbf{w}^h, \mathbf{c}^h; (\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h), (\overline{\mathcal{W}}^h, \overline{\mathbf{w}}^h, \overline{q}^h, \overline{\tau}^h, \overline{\mathbf{c}}^h)) \\ &= (f, -\nu \nabla \cdot \overline{\mathcal{W}}^h + \mathbf{w}^h \cdot \nabla \overline{\mathbf{w}}^h + \nabla \overline{q}^h + S\overline{\mathbf{B}}^h \times \overline{\tau}^h)\end{aligned} \tag{2.28}$$

So it is clear that the system (2.28) is linear with respect to $(\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h)$. For this non-linear map, we have the following estimated results.

Lemma 2.3. *Assume that conditions of the lemma 2.1 and lemma 2.2 holds, the non-linear map* $\mathcal{L}$ *from* $\widetilde{\mathscr{L}}^h(\delta, K, \alpha^*)$ *to* $\digamma^h \times \boldsymbol{X}^h \times \boldsymbol{Q}^h \times \boldsymbol{Y}^h \times \boldsymbol{M}^h$ *is uniquely defined.*

Note that this lemma 2.3 is the direct corollary of the lemma 2.1.

Lemma 2.4. *Assume that conditions of the lemma 3.1 and 3.2 hold and that* γ *satisfies*

$$\gamma \geq \| \boldsymbol{f} \|_{L^2} \max(1, 1/\delta, 1/\sqrt{\alpha}). \tag{2.29}$$

Here now the operator $\mathcal{L}$ *maps* $\widetilde{\mathscr{L}}^h(\delta, K, \alpha^*)$ *to itself.*

Proof. Hence from equation (2.28)

$$\mathcal{L}(\mathbf{w}^h, \mathbf{c}^h; (\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h), (\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h)) \leq \| \mathbf{f} \|_{L^2}^2 .$$

since we have $\| \nabla \mathbf{w}^h \|_{L^2} \leq K$ and $\| \nabla \cdot \mathbf{w}^h \|_{L^2} \leq \delta$ so equation (2.8) tends to

- $\| \nabla \widehat{\mathbf{w}}^h \|_{L^2}^2 + \| curl\ \widehat{\mathbf{c}}^h \|_{L^2}^2 \leq \frac{1}{\alpha} \| \mathbf{f} \|_{L^2}^2 \leq K^2;$

- $\| \widehat{\mathcal{W}}^h \|_{L^2}^2 + \| \widehat{\tau}^h \|_{L^2}^2 + \| \widehat{q}^h \|_{L^2}^2 \leq \frac{1}{\alpha\alpha^*} \| \mathbf{f} \|_{L^2}^2 \leq \frac{K^2}{\alpha^*}$

Furthermore,

$$\| \nabla \cdot \widehat{\mathbf{w}}^h \|_{L^2} \leq \gamma^{-1} \| \mathbf{f} \|_{L^2} \leq \delta.$$

Therefore, $(\widehat{\mathcal{W}}^h, \widehat{\mathbf{w}}^h, \widehat{q}^h, \widehat{\tau}^h, \widehat{\mathbf{c}}^h) \in \widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$. This proves lemma 3.4 completed □

Theorem 2.1. *Assume that $0 < \delta < 2a_0 min(\nu, \frac{1}{\nu_m})$, K satisfies (2.22), γ satisfies (2.29), α and α^* satisfy (2.8 i and ii). So QLSFES has at least one solution in $\widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$. However, all the solutions of the nonlinear system (1.9) are in $\widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$.*

Proof. By lemma 2.4, the operator $\mathcal{L}$ maps the boundary function set $\widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$ into itself under the conditions of theorem2.1. On the other hand, it follows Browners theory of fixed point that the nonlinear system (1.9) has atleast one solution in $\widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$. It is obvious that all of solutions of the system (1.9) are in $\widetilde{\mathscr{L}^h}(\delta, K, \alpha^*)$. Hence the proof of theorem 2.1 is completed. □

Next, we study convergence of solutions of QLSFES. To this end we have the following convergence result.

Theorem 2.2. *Assume that conditions of theorem 2.1 hold and $(\mathfrak{U}^h, \boldsymbol{u}^h, p^h, \boldsymbol{Z}^h, \boldsymbol{B}^h)$ is one sequence of solutions of QLSFES as $h \to 0$. Then solution sequence $(\mathfrak{U}^h, \boldsymbol{u}^h, p^h, \boldsymbol{Z}^h, \boldsymbol{B}^h)$ can be divided into several sub sequences which weakly convergence to different solutions of the first-order system (1.1)-(1.6). In particular, components $\boldsymbol{u}^h, \boldsymbol{B}^h$ of these weakly convergent sub sequences strongly convergence to the corresponding limit components in $[H^s(\Omega)^d] \times H^s(\Omega)$ for $0 \leq s < 1$.*

Before going to prove theorem 2.1 we need the following lemmas to assist.

Lemma 2.5. *Let us consider $0 < \delta < 2a_0\nu$. For a given function $\overline{\boldsymbol{a}} \in [H^1(\Omega)]^d$ satisfying $\| \nabla \cdot \overline{\boldsymbol{a}} \|_{L^2} \leq \delta$ and a given vector-valued function $\overline{\boldsymbol{f}} \in [L^2(\Omega)]^d$. The following boundary value problem*

$$\begin{aligned} -\nu\Delta \boldsymbol{u}^* + \overline{\boldsymbol{a}} \cdot \nabla \boldsymbol{u}^* + \nabla p &= \overline{\boldsymbol{f}}, \quad \text{in } \Omega; & (2.30)\\ \nabla \cdot \boldsymbol{u}^* &= 0, & (2.31)\\ (p^*, 1) &= 0, \\ \boldsymbol{u}^* &= 0, \quad \text{on } \partial\Omega; \end{aligned}$$

has one unique solution $(\boldsymbol{u}^*, p^*)$ in $H^2(\Omega) \times H^1(\Omega)$.

For the proof of lemma 2.5 see [17] appendix 1. Hence we will use the embedding theory between Sobolev spaces and some results reported in [16, 17] as the following lemmas given bellow.

Lemma 2.6. *Let $\mathcal{G}$ be a Hilbert space and F be a bounded function set in $\mathcal{G}$, i.e., there exist a constant $K > 0$ such that $\| f \|_{\mathcal{G}} \leq K$ for each $f \in F$. The function set F is weakly compact in $\mathcal{G}$, i.e., one weakly convergent sub-sequence $\{f_n\}_{n=1}^{\infty}$ with weak limitation f can be extracted from F in such a way that*

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle, \ \forall g \in \mathcal{G}, \qquad (2.32)$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathcal{G}$.

Lemma 2.7. *Let F be a bounded function set in $H^1(\Omega)$, such that there exists a positive constant K such that $\| v \|_{H^1(\Omega)} \leq K$ for each $v \in F$. Then the function set F is strongly compact in $H^s(\Omega)$ for each $0 \leq s < 1$, i.e., one sub sequence $\{f_n\}_{n=1}^{\infty}$, which is strongly convergent in $H^s(\Omega)$ as $n \to \infty$, which can be extracted from F.*

Lemma 2.8. *There exists a positive constant C such that for $0 \le s \le 1$ and each $v \in H^s(\Omega)$*

$$\| v \|_{L^q(\Omega)} \le C \| v \|_{H^s(\Omega)}, \qquad \forall\ 1 \le q \le \frac{2d}{d-2s}.$$

Now we resume theorem 2.2.

Proof. It follows from theorem 2.1 that sequence $(\mathfrak{U}^h, \mathbf{u}^h, p^h, \mathbf{Z}^h, \mathbf{B}^h)$ of solutions of QLSFE are bounded in the Hilbert space [17]. As consequence of lemma 3.6 and 3.7, they can be divided into several sub sequences which are weakly convergent in $L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$. For a weakly convergent sub sequence of solutions of QLSFEM, without loss of generality and for the sake of simplicity, we still represent it by $(\mathfrak{U}^h, \mathbf{u}^h, p^h, \mathbf{Z}^h, \mathbf{B}^h)$ and its weak limitation by $(\overline{\mathfrak{U}}, \overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{Z}}, \overline{\mathbf{B}})$ in $[L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)\ ]$. Furthermore, by lemma 2.7 we know that components $(\mathbf{u}^h, \mathbf{B}^h)$ also strongly converges to $(\overline{\mathbf{u}}, \overline{\mathbf{B}})$ in $[H^s(\Omega) \times H^s(\Omega)]$ for $0 \le s < 1$.

We shall prove that $(\overline{\mathfrak{U}}, \overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{Z}}, \overline{\mathbf{B}})$ is one solution of the first order system (1.1)-(1.6). As result, $(\overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{B}})$ is one solution of the model equation. To end this, we introduce auxiliary functions $(\mathbf{u}^*, p^*)$ such that

$$\begin{aligned} \overline{\mathbf{u}} \cdot \nabla \mathbf{u}^* - \Delta \mathbf{u}^* + \nabla p^* &= \mathbf{f} - S(\ \overline{\mathbf{B}} \times curl\ \overline{\mathbf{B}}),\ in\ \Omega; &(2.33a)\\ \nabla \cdot \mathbf{u}^* &= 0,\ in\ \Omega; &(2.33b)\\ \mathbf{u}^* &= 0,\ on\ \partial\Omega; &(2.33c)\end{aligned}$$

similarly for $\mathbf{B}^*$

$$\begin{aligned} \frac{1}{\mu\sigma} curl\ (curl\ \mathbf{B}^*) - curl\ (\overline{\mathbf{u}} \times \mathbf{B}^*) &= \mathbf{0},\ in\ \ \Omega; &(2.34a)\\ \nabla \cdot \mathbf{B}^* &= 0,\ in\ \ \Omega; &(2.34b)\\ \mathbf{B}^* &= 0,\ in\ \ \partial\Omega; &(2.34c)\end{aligned}$$

systems (2.33a) and (2.34a) are two independent linear systems. Lemma 3.5 shows that these two system of equations are uniquely solvable and $(\mathbf{u}^*, p^*) \in [H^2(\Omega)]^d \times [H^1(\Omega)]^d$. Let $\mathfrak{U}^* = \nabla \mathbf{u}^*$ and $\mathbf{Z}^* = curl \mathbf{B}^*$. It is straight-forward that $(\overline{\mathfrak{U}}, \overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{Z}}, \overline{\mathbf{B}})$ is one solution of the first-order system (1.1)-(1.6) if $(\overline{\mathfrak{U}}, \overline{\mathbf{u}}, \overline{p}, \overline{\mathbf{Z}}, \overline{\mathbf{B}}) = (\mathfrak{U}^*, \mathbf{u}^*, p^*, \mathbf{Z}^*, \mathbf{B}^*)$. We shall prove this fact in three steps.
Firstly, we prove that

$$\| -\nu \nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^*)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^*) + \nabla(\overline{p} - p^*) \|_{L^2(\Omega)} \ = \ 0. \qquad (2.35a)$$

Secondly, we prove that

$$\begin{aligned} (\overline{\mathfrak{U}} - \mathfrak{U}^*) - \nabla(\overline{\mathbf{u}} - \mathbf{u}^*) &= 0,\ in\ \Omega; &(2.36a)\\ \nabla \cdot (\overline{\mathbf{u}} - \mathbf{u}^*) &= 0,\ in\ \Omega. &(2.36b)\end{aligned}$$

Thirdly, it follows from (2.35a) and (2.36a-2.36b) that

$$\begin{aligned} -\nu\Delta(\overline{\mathbf{u}} - \mathbf{u}^*) + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^*) + \nabla(\overline{p} - p^*) &= 0,\ in\ \Omega; &(2.37a)\\ \nabla \cdot (\overline{\mathbf{u}} - \mathbf{u}^*) &= 0,\ in\ \Omega. &(2.37b)\\ \overline{\mathbf{u}} - \mathbf{u}^* &= 0,\ on\ \partial\Omega. &(2.37c)\end{aligned}$$

The system (2.37a) implies that $(\overline{\mathfrak{U}}, \overline{\mathbf{u}}, \overline{p}) = (\mathfrak{U}^*, \mathbf{u}^*, p^*)$ because the system (2.37a) is of unique trivial solution. Similarly, we can prove $(\overline{\mathbf{Z}}, \overline{\mathbf{B}}) = (\mathbf{Z}^*, \mathbf{B}^*)$ □

The proof of (2.35a) is as follows. It is clear that (2.35a) is equivalent to

$$(-\nu\nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^*)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^*) + \nabla(\overline{p} - p^*), \varphi) \quad = \quad 0, \qquad \forall \varphi \in [C_0^\infty]^d. \tag{2.38}$$

For each $\varphi \in [C_0^\infty(\Omega)]^d$, we may introduce new functions known as test functions ($\mathbf{v}$,q) as

$$\begin{aligned} -\nu\Delta\mathbf{v} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{v} + \nabla q &= \varphi, \ in \ \Omega, && (2.39a)\\ \nabla \cdot \mathbf{v} &= 0, \ in \ \Omega, && (2.39b)\\ \mathbf{v} &= 0, \ on \ \partial\Omega. && (2.39c) \end{aligned}$$

From Lemma 2.5, we may have $(\mathbf{v}, q) \in [H^2(\Omega)]^d \times H^1(\Omega)$. Letting $V = \nabla \mathbf{v}$, we get

$$\begin{aligned} &(-\nu\nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^*)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^*) + \nabla(\overline{p} - p^*), \varphi) \\ &\quad = \left[\Big(- \nu\nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^h)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^h) + \nabla(\overline{p} - p^h) + S\ \overline{\mathbf{B}} \times curl\ \overline{\mathbf{B}} - \mathbf{B}^h, \varphi\Big)\right] + ((\overline{\mathbf{u}} - \mathbf{u}^h \cdot \nabla)\mathbf{u}^h, \varphi) \\ &+ \left(- \nu\nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) + S\ \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), ((\overline{\mathbf{u}} - \mathbf{u}^h) \cdot \nabla)\mathbf{v}\right) \\ &+ \Big[(-\nu\nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) + S\ \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), \\ &\nu\nabla \cdot \mathbf{V}^T + (\mathbf{u}^h \cdot \nabla)\mathbf{v} + \nabla q + (\mathfrak{U}^h - \nabla\mathbf{u}^h, \mathbf{V} - \nabla\mathbf{v}) + \gamma^2(\nabla \cdot (\mathbf{u}^h - \mathbf{u}^*), \nabla \cdot \mathbf{v})\Big]. \end{aligned} \tag{2.40}$$

The property of weak convergence of $(\mathfrak{U}^h, \mathbf{u}^h, p^h, \tau^h, \mathbf{B}^h)$ be the one solution of the systems (1.1)-(1.6). For any positive γ, we may further see that

$$\begin{aligned} &\mid (-\nu\nabla\cdot(\overline{\mathfrak{U}} - \mathfrak{U}^h)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^h) + \nabla(\overline{p} - p^h) + S\ \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), \varphi) \mid \\ &\quad \leq \mid (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^h) + S\ \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), \varphi \mid \\ &\quad + \nu \mid \nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^h, \nabla\varphi) \mid + \mid (p - p^h, \nabla \cdot \varphi) \mid \to 0, \qquad as\ h \to 0 \end{aligned} \tag{2.41}$$

Thus from consequence of the lemma 2.8 and the strong convergence property of $(\mathbf{u}^h, \mathbf{B}^h)$, we have the following result

$$\| \overline{\mathbf{u}} - \mathbf{u}^h \|_{L^4} \quad \leq \quad C \| \overline{\mathbf{u}} - \mathbf{u}^h \|_{H^{d/4}} \to 0, \quad as \quad h \to 0,$$

and

$$\begin{aligned} &\mid (((\overline{\mathbf{u}} - \mathbf{u}^h) \cdot \nabla)\mathbf{u}^h, \varphi) \mid + \mid -\nu\nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* \\ &\quad + \nabla(p^h - p^*) + \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), ((\overline{\mathbf{u}} - \mathbf{u}^h) \cdot \nabla\mathbf{v}) \mid \\ &\quad \leq C \| \overline{\mathbf{u}} - \mathbf{u}^h \|_{L^4} [\| \nabla\mathbf{u}^h \|_0 \| \varphi \|_{L^4} + \| f \|_0 \| \nabla\mathbf{v} \|_{L^4}]. \end{aligned} \tag{2.42}$$

From the system of equations (1.1)-(1.6), we can drive estimate as

$$\begin{aligned} &\big(- \nu\nabla\cdot(\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) \\ &\quad + \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), -\nu\nabla \cdot \mathbf{V}^T + (\mathbf{u}^h \cdot \nabla)\mathbf{v} + \nabla q\big) \end{aligned}$$

$$
\begin{aligned}
&+ (\mathfrak{U}^h - \nabla \mathbf{u}^h, \mathbf{V} - \nabla \mathbf{v}) + (\nabla \cdot (\mathbf{u}^h - \mathbf{u}^*), \nabla \cdot \mathbf{v}) \\
&= \inf_{(\mathbf{V}^h, \mathbf{v}^h, q^h) \in (F^h \times X^h \times Q^h)} \Big[\big(-\nu \nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) \\
&+ \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), -\nu \nabla \cdot (\mathbf{V} - \mathbf{V}^h)^T + (\mathbf{u}^h \cdot \nabla)(\mathbf{v} - \mathbf{v}^h) + \nabla(q - q^h) \big) \\
&+ (\mathfrak{U}^h - \nabla \mathbf{u}^h, \mathbf{V} - \mathbf{V}^h - \nabla(\mathbf{v} - \mathbf{v}^h)) + \gamma^2 (\nabla(\mathbf{u}^h - \mathbf{u}^*), \nabla \cdot (\mathbf{v} - \mathbf{v}^h)) \Big] \qquad (2.43) \\
&\to 0, h \to 0.
\end{aligned}
$$

We substitute equations (2.41),(2.42),(2.43) into (2.40) yields (2.38). Now we give the proof of the second step (2.36a)-(2.36b). BY weak convergence of solution sequence and approximation properties of finite element spaces we see that for each $(\mathcal{W}, \mathbf{w}) \in [C_0^\infty(\Omega)]^{d \times d} \times [C_0^\infty(\Omega)]^d$

$$
\begin{aligned}
&(\overline{\mathfrak{U}} - \mathfrak{U}^* - \nabla(\overline{\mathbf{u}} - \mathbf{u}^*), \mathcal{W} - \nabla \mathbf{w}) + \gamma^2 (\nabla \cdot (\overline{\mathbf{u}} - \mathbf{u}^*), \nabla \cdot \mathbf{w}) \qquad (2.44) \\
&\quad = \big(-\nu \nabla \cdot (\overline{\mathfrak{U}} - \mathfrak{U}^h)^T + (\overline{\mathbf{u}} \cdot \nabla)(\overline{\mathbf{u}} - \mathbf{u}^h) + \nabla(\overline{p} - p^h) + \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), -\nu \nabla \cdot \mathcal{W}^T + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{w} \big) \\
&\quad + (\overline{\mathfrak{U}} - \mathfrak{U}^h - \nabla(\overline{\mathbf{u}} - \mathbf{u}^h), \mathcal{W} - \nabla \mathbf{w}) + \gamma^2 (\nabla \cdot (\overline{\mathbf{u}} - \mathbf{u}^h), \nabla \cdot \mathbf{w}) + (((\overline{\mathbf{u}} - \mathbf{u}^h) \cdot \nabla)\mathbf{u}^h, -\nu \nabla \cdot \mathcal{W}^T + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{w}) \\
&\quad + (-\nu \nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) + \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), ((\overline{\mathbf{u}} - \mathbf{u}^h) \cdot \nabla)\mathbf{w}) \\
&\quad + \inf_{(\mathcal{W}^h, \mathbf{w}^h) \in (F^h \times X^h)} \Big[(-\nu \nabla \cdot (\mathfrak{U}^h - \mathfrak{U}^*)^T + (\mathbf{u}^h \cdot \nabla)\mathbf{u}^h - (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u}^* + \nabla(p^h - p^*) + \overline{\mathbf{B}} \times curl(\overline{\mathbf{B}} - \mathbf{B}^h), \\
&\quad - \nu \nabla \cdot (\mathcal{W} - \mathcal{W}^h)^T + (\mathbf{u}^h \cdot \nabla)(\mathbf{w} - \mathbf{w}^h)) + (\mathfrak{U}^h - \mathfrak{U}^* - \nabla(\mathbf{u}^h - \mathbf{w}^*), \mathcal{W} - \mathcal{W}^h - \nabla(\mathbf{w} - \mathbf{w}^h)) \\
&\quad + \gamma^2 (\nabla \cdot (\mathbf{u}^h - \mathbf{u}^*), \nabla \cdot (\mathbf{w} - \mathbf{w}^h)) \Big] \to 0, \text{as} \quad h \to 0.
\end{aligned}
$$

Now the prove of theorem 2.2 is completed. This theorem 2.2 shows the solutions of QLSFES in general cases. Near to the singular solution, we only can obtain weak convergence of sub-sequences of approximate solutions. However, in the next section, we acquire the prove of the strong convergence of uniformly convergent rate in cases of non-singular solutions of the magnetohydrodynamics system of equations.

3. Convergent rate of non-singular solution of QLSFES

In this section, we analyze convergent rate of solutions of the non-singular solutions of the system equations (**??**)-(**??**). A solution of the system (**??**)-(**??**) is termed as the non-singular solution if this solution is an isolated solution and the first-order differential approximation of the system is non-singular at this solution, for more detail we refer reader to (see [17]). We consider for each $(F_0^*, F_1^*, f^*) \in [H^{-1}(\Omega)]^d \times [L^2(\Omega)]^{d \times d} \times [L^2(\Omega)]$, the linear system

$$
\begin{aligned}
-\nu \nabla \cdot \mathcal{W}^T + (\mathbf{u} \cdot \nabla)\mathbf{w} + \nabla q + S(\mathbf{B} \times curl\ \mathbf{c}) &= F_0^* \quad \text{in } \Omega, && (3.1) \\
\mathcal{W} - \nabla \mathbf{w} &= F_1^* \quad \text{in } \Omega, && (3.2) \\
\nabla \cdot \mathbf{w} &= f^* \quad \text{in } \Omega, && (3.3) \\
\mathbf{w} &= 0 \quad \text{on } \partial\Omega, (q, 1) = 0 && (3.4) \\
\nu_m\ curl\ \tau - S(curl\ (\mathbf{u} \times \mathbf{c})) &= 0 \quad \text{in } \Omega, && (3.5) \\
\tau - curl\ \mathbf{c} &= 0 \quad \text{in } \Omega, && (3.6) \\
\mathbf{c} &= 0 \quad \text{on } \partial\Omega, && (3.7) \\
\nabla \cdot \mathbf{c} &= 0 \quad \text{in } \Omega, && (3.8)
\end{aligned}
$$

has unique solution $(\mathcal{W}, \mathbf{w}, q, \tau, \mathbf{c}) \in [L^2(\Omega)]^{d\times d} \times [H_0^1(\Omega)]^d \times [L^2(\Omega)] \times [H_0^1(\Omega)]$. There exists the positive constant C such that

$$\| \mathcal{W}^T \|_0 + \| \mathbf{w} \|_1 + \| q \|_0 + \| \mathbf{c} \|_0 + \| \tau \|_0 \leq C \| F_0^* \|_{H^{-1}} \| F_1^* \|_0 \| f^* \|_0 \qquad \text{in } \Omega. \tag{3.9}$$

For further analysis, we assume that the following Stokes equations as

$$\begin{aligned} -\nu\Delta \mathbf{w} + \nabla q &= f \qquad \text{in } \Omega, && (3.10)\\ \mathbf{w} &= 0 \qquad \text{on } \partial\Omega, \quad (q,1)=0, && (3.11)\\ \nabla\cdot\mathbf{w} &= f^* \qquad \text{in } \Omega, . && (3.12)\\ & && (3.13)\end{aligned}$$

and

$$\begin{aligned} -\nu\Delta w &= g \qquad \text{in } \Omega, && (3.14)\\ w &= 0 \qquad \text{on } \partial\Omega. && (3.15)\\ & && (3.16)\end{aligned}$$

have H^{2+s} regularity for any $s \in (0,1]$ so the solution $(\mathbf{w}, q, w) \in [H^{2+s}(\Omega)]^d \times [H^{1+s}(\Omega)] \times [H^{2+s}(\Omega)]$ and

$$\| \mathbf{w} \|_{H^{2+s}} + \| q \|_{H^{1+s}} \leq \| f \|_{H^s}, \tag{3.17}$$

and

$$\| w \|_{H^{2+s}} \leq C \| g \|_{H^s} . \tag{3.18}$$

Here, we assume that the finite element spaces $\digamma^h \times \mathbf{X}^h \times \mathbf{Q}^h \times \mathbf{Y}^h \times \mathbf{M}^h$ possess the approximation properties such that there exists the approximation order r of the finite element space which is defined as $(r \geq 1)$ and C such that

$$\begin{aligned} \inf_{(\mathcal{W}^h \in \digamma^h)} [\| \mathcal{W} - \mathcal{W}^h \|_0 + h \| \mathcal{W} - \mathcal{W}^h \|_1] &\leq Ch^{r+1} \| \mathcal{W} \|_{r+1} \, \forall \mathcal{W} \in [H^{r+1}]^{d\times d}; && (3.19)\\ \inf_{(\mathbf{v}^h \in \mathbf{X}^h)} [\| \mathbf{v} - \mathbf{v}^h \|_0 + h \| \mathbf{v} - \mathbf{v}^h \|_1] &\leq Ch^{r+1} \| \mathbf{v} \|_{r+1} \, \forall \mathbf{v} \in [H^{r+1}]^{d\times d}, \forall \mathbf{v} \in \mathbf{X} \bigcap [H^{r+1}]^d; &&\\ \inf_{(q^h \in \mathbf{Q}^h)} [\| q - q^h \|_0 + h \| q - q^h \|_1] &\leq Ch^{r+1} \| q \|_{r+1} \, \forall q \in [H^{r+1}]^{d\times d}, \forall q \in \mathbf{Q} \bigcap [H^{r+1}]^d; &&\\ \inf_{(\tau^h \in \mathbf{M}^h)} [\| \tau - \tau^h \|_0 + h \| \tau - \tau \|_1] &\leq Ch^{r+1} \| \tau \|_{r+1} \, \forall \tau \in [H^{r+1}]^{d\times d}, \forall \tau \in \mathbf{M} \bigcap [H^{r+1}]^d. &&\\ \inf_{(\mathbf{c}^h \in \mathbf{Y}^h)} [\| \mathbf{c} - \mathbf{c}^h \|_0 + h \| \mathbf{c} - \mathbf{c}^h \|_1] &\leq Ch^{r+1} \| \mathbf{c} \|_{r+1} \, \forall \mathbf{c} \in [H^{r+1}]^{d\times d}, \forall \mathbf{c} \in \mathbf{Y} \bigcap [H^{r+1}]^d; && \end{aligned}$$

where r is the approximation order of the finite element spaces.

Theorem 3.1. *Assume that the conditions of theorem 2.2 holds. Let $(\mathfrak{U}^h, \boldsymbol{u}^h, p^h, \boldsymbol{Z}^h, \boldsymbol{B}^h)$ be the one approximate solutions of QLSFES which weakly converges to one solution $(\mathfrak{U}, \boldsymbol{u}, p, \boldsymbol{Z}, \boldsymbol{B})$ of the first order system 1.1-1.6. Suppose the solution $(\mathfrak{U}, \boldsymbol{u}, p, \boldsymbol{Z}, \boldsymbol{B})$ is non-singular, then the solution sequence $(\mathfrak{U}^h, \boldsymbol{u}^h, p^h, \boldsymbol{Z}^h, \boldsymbol{B}^h)$ is strongly convergent in $[L^2(\Omega)]^d \times [L^2(\Omega)] \times [L^2(\Omega)]^{d\times d} \times [H^1(\Omega)]$ as $h \to 0$.*

Moreover, the priori error estimate holds

$$\parallel \mathfrak{U}-\mathfrak{U}^h \parallel_0 + \parallel \boldsymbol{u}-\boldsymbol{u}^h \parallel_1 + \parallel p-p^h \parallel_0 + \parallel \boldsymbol{Z}-\boldsymbol{Z}^h \parallel_0 + \parallel \boldsymbol{B}-\boldsymbol{B}^h \parallel_1 \quad \leq \quad Ch^r \tag{3.20}$$

Proof. Suppose $(\mathfrak{U}^h, \mathbf{u}^h, p^h, \mathbf{Z}^h, \mathbf{B}^h)$ be a sequence of solutions of discreite QLSFES, which is weakly converge to one non-singular continuous solution $(\mathfrak{U}, \mathbf{u}, p, \mathbf{Z}, \mathbf{B})$ of the lowest order system which is actually first order linear system (1.1)-(1.6) in $[L^2(\Omega)]^d \times [L^2(\Omega)] \times [L^2(\Omega)]^{d\times d} \times [H^1(\Omega)]$ as $h \to 0$. By using the curl operator identity and incompressibility conditions we can write

$$\begin{aligned} curl(\mathbf{u}\times\mathbf{B}) &= \big((\nabla\cdot\mathbf{B})\mathbf{u} + (\mathbf{B}\cdot\nabla)\mathbf{u}\big) - \big((\nabla\cdot\mathbf{u})\mathbf{B} + (\mathbf{u}\cdot\nabla)\mathbf{B}\big) \\ &= (\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} \end{aligned}$$

Then, we note that

$$\begin{aligned} \text{(a)} \qquad & -\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) + ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u} + \nabla(p-p^h) \\ & \quad + S(\mathbf{B}\times curl(\mathbf{B}-\mathbf{B}^h)) \\ & = -\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) \\ & \quad + S(\mathbf{B}\times curl(\mathbf{B}-\mathbf{B}^h) + ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h), \qquad in \quad \Omega; \\ \text{(b)} \qquad & \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - curl\ ((\mathbf{u}-\mathbf{u}^h)\times(\mathbf{B}-\mathbf{B}^h)) \\ = \ & \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) + \{(\mathbf{B}\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) + (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)\mathbf{u}\} \\ & + \{(\mathbf{u}\cdot\nabla)(\mathbf{B}-\mathbf{B}^h) + (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{B}\} \\ = \ & \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) + \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h + (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} \\ & + \{(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h + (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)\} + ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \\ & ((\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h) \end{aligned}$$

By using (3.9) we have the following inequality

$$\begin{aligned} & \parallel \mathfrak{U}-\mathfrak{U}^h \parallel_0 + \parallel \mathbf{u}-\mathbf{u}^h \parallel_1 + \parallel p-p^h \parallel_0 + \parallel \mathbf{Z}-\mathbf{Z}^h \parallel_0 + \parallel \mathbf{B}-\mathbf{B}^h \parallel_1 \\ & \leq C\{\parallel -\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) \\ & + S\mathbf{B}\times curl\ (\mathbf{B}-\mathbf{B}^h) \parallel_0^2\} + \parallel \mathfrak{U}-\mathfrak{U}^h - \nabla(\mathbf{u}-\mathbf{u}^h) \parallel_0^2 + \gamma^2 \parallel \nabla\cdot(\mathbf{u}-\mathbf{u}^h) \parallel_0^2 \\ & + \parallel \quad \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} \\ & \qquad - \{(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h - (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)\} \parallel_1^2 \\ & + \parallel \mathbf{Z}-\mathbf{Z}^h - curl\ (\mathbf{B}-\mathbf{B}^h) \parallel_0^2 + \eta^2 \parallel \nabla\cdot(\mathbf{B}-\mathbf{B}^h) \parallel_0^2 + \parallel (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \parallel_{H^{-1}}^2 \\ & + \parallel (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \parallel_{H^{-1}} + \parallel (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h) \parallel_{H^{-1}} \end{aligned} \tag{3.21}$$

To find the error estimates, we would like to bound terms on the right-hand side of (3.21) as following

$$\begin{aligned} & \parallel (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \parallel_{H^{-1}}^2 + \parallel (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \parallel_{H^{-1}} + \parallel (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h) \parallel_{H^{-1}} \\ & \leq \quad C \parallel (\mathbf{u}-\mathbf{u}^h) \parallel_{L^4}^2 [\parallel \nabla(\mathbf{u}-\mathbf{u}^h) + \nabla(\mathbf{B}-\mathbf{B}^h) \parallel_0^2] + \parallel (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) \parallel_{H^{-1}} \end{aligned} \tag{3.22}$$

It follows from (1.1)-(1.6) and (1.9) we get

$$\parallel -\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h)$$

$$
\begin{aligned}
&+ S\mathbf{B} \times curl(\mathbf{B}-\mathbf{B}^h) \|_0^2\} + \| \mathfrak{U} - \nabla\mathbf{u} - (\mathfrak{U}^h - \nabla\mathbf{u}^h) \|_0^2 + \| \nabla\cdot(\mathbf{u}-\mathbf{u}^h) \|_0^2 \\
&+ \| \quad \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} \\
&\qquad - \{(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h - (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)\} \|_1^2 \\
&= \inf_{(\mathcal{W}^h,\mathbf{w}^h,q^h,\tau^h,\mathbf{c}^h)\in(F^h\times\ \mathbf{X}^h\times\mathbf{Q}^h\times\mathbf{Y}^h\times\mathbf{M}^h)} \Bigg[\Bigg(-\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) \\
&+ S\mathbf{B}\times curl(\mathbf{B}-\mathbf{B}^h), -\nu\nabla\cdot(\mathfrak{U}-\mathcal{W}^h)^T + (\mathbf{u}^h\cdot\nabla(\mathbf{u}-\mathbf{w}^h)) + \nabla(p-q^h) + S\mathbf{B}\times curl(\mathbf{B}-\mathbf{c}^h))\Bigg) \\
&\quad + (\mathfrak{U}-\nabla\mathbf{u}-(\mathfrak{U}^h-\mathbf{u}^h), \mathfrak{U}-\mathcal{W}^h-\nabla(\mathbf{u}-\mathbf{w}^h)) + (\nabla\cdot(\mathbf{u}-\mathbf{u}^h), \nabla\cdot(\mathbf{u}-\mathbf{w}^h)) \\
&\quad + \Bigg(\nu_m curl(\mathbf{Z}-\mathbf{Z}^h) - \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} \\
&\qquad - \{(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h - (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)\}), \\
&\nu_m curl(\mathbf{Z}-\tau^h) - \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{B}-\mathbf{c}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} \\
&- \{(\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h - (\mathbf{u}-\mathbf{w}^h)\cdot\nabla)(\mathbf{B}-\mathbf{c}^h)\}) + (\nabla\cdot(\mathbf{B}-\mathbf{B}^h), \nabla\cdot(\mathbf{B}-\mathbf{c}^h))\Bigg)\Bigg] \\
&- \nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) + S\mathbf{B}\times curl\ (\mathbf{B}-\mathbf{B}^h), ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u}) \\
&+ (\nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - \{(\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\} - \{(\mathbf{u}\cdot\nabla)\mathbf{u} \\
&\qquad - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h - (\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)), ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h) + ((\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{B}-\mathbf{B}^h)
\end{aligned}
\tag{3.23}
$$

To control the last terms on the right-hand side of (3.23), we intend to introduce auxiliary functions $(\mathbf{v}, q, \mathbf{c})$ such that

$$
\begin{aligned}
-\nu\Delta\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + \nabla q + S\mathbf{B}\times curl(\mathbf{c}) &= ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u} && (3.24)\\
\nabla\cdot\mathbf{v} &= 0, \text{ in } \Omega; \quad (q,1)=0; && (3.25)\\
\nu_m curl\ curl(\mathbf{c}) - (\mathbf{B}\cdot\nabla)\mathbf{c} - (\mathbf{u}\cdot\nabla)\mathbf{v} &= ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u} + ((\mathbf{B}-\mathbf{B}^h)\cdot\nabla)\mathbf{B} && (3.26)
\end{aligned}
$$

Hence this system (3.24)-(3.26) possesses one unique solution $(\mathbf{v}, q, \mathbf{c})$ in $[H^2(\Omega)]^d \times H^1(\Omega) \times H^2(\Omega)$. Let us consider $\mathbf{V} = \nabla\mathbf{v}$ and $\tau = curl\ \mathbf{c}$, then by auxiliary functions $(\mathbf{V}, \mathbf{v}, q, \tau, \mathbf{c})$ we have the following inequality

$$
\begin{aligned}
&\Big(-\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) + S(\mathbf{B}-\mathbf{B}^h)\times curl\mathbf{B}, ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u}\Big) \\
&\quad + (\quad \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - (\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h - (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h, \\
&\qquad ((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{u} + ((\mathbf{B}-\mathbf{B}^h)\cdot\nabla)\mathbf{B} \quad) \\
&\quad = \inf_{(\mathbf{V}^h,\mathbf{v}^h,q^h,\tau^h,\mathbf{c}^h)\in(F^h\times\ \mathbf{X}^h\times\mathbf{Q}^h\times\mathbf{Y}^h\times\mathbf{M}^h)} \Bigg[\Bigg(-\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{u}^h + \nabla(p-p^h) \\
&\quad + S(\mathbf{B}-\mathbf{B}^h)\times curl\mathbf{B}, -\nu\nabla\cdot(\mathbf{V}-\mathbf{V}^h)^T + (\mathbf{u}^h\cdot\nabla(\mathbf{v}-\mathbf{v}^h)) + \nabla(p-q^h) + S(\mathbf{c}-\mathbf{c}^h)\times curl\mathbf{B})\Bigg) \\
&\qquad + (\mathfrak{U}-\mathfrak{U}^h-\nabla(\mathbf{u}-\mathbf{u}^h), \mathbf{V}-\mathbf{V}^h-\nabla(\mathbf{v}-\mathbf{v}^h)) + \gamma^2(\nabla\cdot(\mathbf{u}-\mathbf{u}^h), \nabla\cdot(\mathbf{v}-\mathbf{v}^h)) \\
&\quad (\mathbf{Z}-\mathbf{Z}^h + curl(\mathbf{B}-\mathbf{B}^h), \tau-\tau^h + curl(\mathbf{c}-\mathbf{c}^h)) + \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h) - (\mathbf{B}\cdot\nabla)\mathbf{B} - (\mathbf{B}^h\cdot\nabla)\mathbf{u}^h \\
&\quad - (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}^h\cdot\nabla)\mathbf{B}^h, \nu_m curl\ (\tau-\tau^h) - (\mathbf{B}^h\cdot\nabla)(\mathbf{c}-\mathbf{c}^h) - (\mathbf{u}-\mathbf{u}^h\cdot\nabla)(\mathbf{v}-\mathbf{v}^h)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T+(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{u}^h+\nabla(p-p^h)\\
&+S(\mathbf{B}-\mathbf{B}^h)\times curl\mathbf{B},((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{v}\Big)+\Big(\ \ \nu_m curl\ (\tau-\tau^h)-(\mathbf{B}\cdot\nabla)\mathbf{B}-(\mathbf{B}^h\cdot\nabla)\mathbf{u}^h\\
&\quad-(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{B}^h,((\mathbf{u}-\mathbf{u}^h)\cdot\nabla)\mathbf{v}+((\mathbf{B}-\mathbf{B}^h)\cdot\nabla)\mathbf{c}\ \ \Big)\\
&\quad\le\Big[\ \|-\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T+(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{u}^h+\nabla(p-p^h)+S(\mathbf{B}-\mathbf{B}^h)\times curl\mathbf{B}\ \|_0^2\\
&\quad+\|\ \mathfrak{U}-\mathfrak{U}^h-\nabla(\mathbf{u}-\mathbf{u}^h)\ \|_0^2+\gamma^2\ \|\ \nabla\cdot(\mathbf{u}-\mathbf{u}^h)\ \|_0^2+\|\ \mathbf{Z}-\mathbf{Z}^h+curl(\mathbf{B}-\mathbf{B}^h)\ \|_0^2\\
&\quad+\ \ \nu_m\ \|\ curl\ (\tau-\tau^h)-(\mathbf{B}\cdot\nabla)\mathbf{B}-(\mathbf{B}^h\cdot\nabla)\mathbf{u}^h-(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{B}^h\ \|_0^2\ \Big]\\
&\quad+C\ \Big\{h^{2r}\big[\ \|\ \mathbf{u}\ \|_{H^{r+1}}^2+\|\ \mathbf{u}\ \|_{H^{r+1}}^2+\|\ p\ \|_{H^{r+1}}^2+\|\ \mathbf{B}\ \|_{H^{r+1}}^2\ \big]\Big\}\\
&+\Big\{h^{2s}\big[\ \|\ \mathbf{v}\ \|_{H^{s+2}}^2+\|\ q\ \|_{H^{s+1}}^2\ \big]+\|\ \mathbf{u}-\mathbf{u}^h\ \|_{L^4}^2\ (\|\ \mathbf{v}\ \|_{H^2}^2+\|\ \tau\ \|_{H^2}^2)\Big\}
\end{aligned}
\tag{3.27}
$$

By substituting (3.17)-(3.18),(3.27) into (3.23), we get

$$
\begin{aligned}
&\|\ -\nu\nabla\cdot(\mathfrak{U}-\mathfrak{U}^h)^T+(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{u}^h+\nabla(p-p^h)+S(\mathbf{B}-\mathbf{B}^h)\times curl(\mathbf{B})\ \|_0^2\\
&\quad+\|\ \mathfrak{U}-\nabla\mathbf{u}-(\mathfrak{U}^h-\nabla\mathbf{u}^h)\ \|_0^2+\|\ \nabla\cdot(\mathbf{u}-\mathbf{u}^h)\ \|_0^2\\
&\quad+\|\ \ \ \nu_m curl\ (\mathbf{Z}-\mathbf{Z}^h)-\{(\mathbf{B}\cdot\nabla)\mathbf{B}-(\mathbf{B}^h\cdot\nabla)\mathbf{u}^h-(\mathbf{B}-\mathbf{B}^h)\cdot\nabla)(\mathbf{u}-\mathbf{u}^h)\}\\
&\quad-(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{u}^h\cdot\nabla)\mathbf{B}^h-(\mathbf{u}-\mathbf{u}^h)\cdot\nabla(\mathbf{B}-\mathbf{B}^h)\}\ \|_1^2\\
&\quad\le C\ \ \Big\{h^{2r}\big[\ \|\ \mathbf{u}\ \|_{H^{r+2}}^2+\|\ p\ \|_{H^{r+1}}^2+\|\ \mathbf{B}\ \|_{H^{r+2}}^2\ \big]\}\\
&\quad+(h^{2s}+\|\ \mathbf{u}-\mathbf{u}^h\ \|_{L^4}^2)\big[\ \|\ (\mathbf{u}-\mathbf{u}^h)\cdot\nabla\mathbf{u}\ \|_{H^s}^2+\|\ (\mathbf{u}-\mathbf{u}^h)\cdot\nabla\mathbf{B}\ \|_{H^s}^2\ \big]\Big\}.
\end{aligned}
\tag{3.28}
$$

Having substitution of equation (3.22) and (3.28) into (3.21), also the value of $s=0$ we get

$$
\begin{aligned}
\|\ \mathfrak{U}-\mathfrak{U}^h\ \|_0&+\|\ \mathbf{u}-\mathbf{u}^h\ \|_1+\|\ p-p^h\ \|_0+\|\ \mathbf{Z}-\mathbf{Z}^h\ \|_0+\|\ \mathbf{B}-\mathbf{B}^h\ \|_1\\
&\le C\ \ \Big\{h^{r}\big[\ \|\ \mathbf{u}\ \|_{H^{r+2}}^2+\|\ p\ \|_{H^{r+1}}^2+\|\ \mathbf{B}\ \|_{H^{r+2}}^2\ \big]\\
&+(1+\|\ \mathbf{u}-\mathbf{u}^h\ \|_{L^4}^2)\ \|\ \mathbf{u}-\mathbf{u}^h\ \|_{L^4}^2)\big[\ \|\ \mathbf{u}\ \|_{H^2}+\|\ \mathbf{B}\ \|_{H^2}\ \big]\Big\}.
\end{aligned}
\tag{3.29}
$$

and that as $0<s<1$

$$
\begin{aligned}
\|\ \mathfrak{U}-\mathfrak{U}^h\ \|_0&+\|\ \mathbf{u}-\mathbf{u}^h\ \|_1+\|\ p-p^h\ \|_0+\|\ \mathbf{Z}-\mathbf{Z}^h\ \|_0+\|\ \mathbf{B}-\mathbf{B}^h\ \|_1\\
&\le C\ \ \Big\{h^{r}\big[\ \|\ \mathbf{u}\ \|_{H^{r+2}}^2+\|\ p\ \|_{H^{r+1}}^2+\|\ \mathbf{B}\ \|_{H^{r+2}}^2\ \big]\\
&+(h^{s}+\|\ \mathbf{u}-\mathbf{u}^h\ \|_{L^4}^2)\ \|\ \mathbf{u}-\mathbf{u}^h\ \|_{H^1}^2)\big[\ \|\ \mathbf{u}\ \|_{H^3}+\|\ \mathbf{B}\ \|_{H^3}\ \big]\Big\}.
\end{aligned}
\tag{3.30}
$$

So that if $\| \mathbf{u} - \mathbf{u}^h \|_{L^4}$ is convergence to o as $h \to 0$, then (3.28) yields

$$\| \mathfrak{U} - \mathfrak{U}^h \|_0 + \| \mathbf{u} - \mathbf{u}^h \|_1 + \| p - p^h \|_0 + \| \mathbf{Z} - \mathbf{Z}^h \|_0 + \| \mathbf{B} - \mathbf{B}^h \|_1 \to 0, \text{as} \quad h \to 0, \tag{3.31}$$

if $s = 0$, then (3.30) leads to (3.20), if $0 < s < 1$. This gives a completion of the theorem proof. □

4. Numerical examples

In this section we illustrate several numerical experiments to show effect of the schemes and to verify theoretical results. We introduce a very simple iterative algorithm to solve nonlinear systems result from the scheme in section 2. The basic purpose of our experiments is to find optimal convergence rates of our method. We take initiate by considering with smooth solution and one problem with a singular solution. Then, we consider the numerical approximations as common in magnetohydrodynamic research examples known as Hartman channel flow.

4.1. Two-dimensional problem with a smooth solution

First, we verify the teoretical result for a problem with a smooth solution.

We consider the following two dimensional problem. We set $\Omega = [0,1] \times [0,1]$ with a Dirichlet boundary conditions on all the boundaries. We consider the components of velocity $\mathbf{u}$ and magnetic field $\mathbf{B}$ as (u_1, u_2) and (B_1, B_2) for convenience. We choose the source term f that the analytical solution is of the form

$$\begin{aligned} u_1 &= x^2(x-1)^2 y(y-1)(2y-1); \\ u_2 &= -x(x-1)(2x-1)y^2(y-1)^2; \\ P &= (2x-1)(2y-1); \\ B_1 &= sin(\pi x)cos(\pi y); \\ B_2 &= -sin(\pi y)cos(\pi x). \end{aligned}$$

We construct this example to show the convergence rate concerning the L^2-error norm. In Table 1, we illustrate the convergence of the errors in the approximations of the hydrodynamic and magnetic variables. We observe that $|| \mathbf{u} - \mathbf{u}^h ||_0$, $|| p - p^h ||_0$,$|| \mathbf{B} - \mathbf{B}^h ||_0$ converge to zero as the mesh is refined.

Table 1: The error estimate for MHD with standard FE $P_{1b} - P_1 - P_1$ pair.

h	$\|\|\mathbf{u}-\mathbf{u}^h\|\|_0$	Rate	$\|\|p-p^h\|\|_0$	Rate	$\|\|\mathbf{B}-\mathbf{B}^h\|\|_0$	Rate
1/4	0.22122	-	0.20562	-	0.12221	-
1/8	0.10267	1.10	0.06430	1.6	0.04332	1.4
1/16	0.04883	1.07	0.02187	1.5	0.01579	1.4
1/32	0.02390	1.03	0.00724	1.5	0.00621	1.4
1/64	0.01185	1.01	0.00247	1.5	0.00253	1.4

5. Conclusion

In this article, a quasileast square method for the Magnetohydrodynamic (MHD) model e-quations is presented. The mixed finite element method needs some stability conditions for the uniqueness and existence known as inf-sup conditions or LBB conditions. These conditions are not always satisfied with the finite elements, by choosing quasileast square method these conditions are circumvented and do not need for the stability condition.

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