A GENERALIZATION OF CESARO POLYNOMIALS IN SEVERAL VARIABLES

NEJLA ÖZMEN

Abstract. Cesaro polynomials were introduced and investigated in 1978, and then have been cited in several articles [1, 2]. In this sequel, by modifying Lin at el. show how to generalize the Cesaro polynomials in one variables to present two generating functions of the generalized Cesaro polynomials $g_n^{(s)}(\lambda, x)$ [6]. Very recently, M. A. Malik introduced and investigated Cesaro polynomials in two and three variables to give their generating functions [3]. Subsequently, N. Özmen investigated the generating functions for the q-analogue of generalized Cesaro polynomials [7]. In this paper, new multivariate generalized Cesaro polynomials will be obtained. Two new generating functions will be given and some special properties of this polynomial will be examined.

1. Introduction

Special functions, with its various sub-branches, provide a very wide field of study that arises not only in various fields of mathematics, but also in the solutions of important problems in many disciplines such as physics, chemistry and biology. This topic is effective enough to make sense of ambiguous questions, especially in physical problems, thus encouraging many people to make notable improvements in this area. As in other sciences, however, outstanding problems are discussed in many disciplines and more general results are attempted. Function generation theory is used in the analysis of discrete problems involving sequences of numbers or sequences of functions and polynomials. This theory has useful applications in many areas of study. In recent years, various interesting applications of various methods of obtaining linear, bilinear, bilateral or mixed multilateral generation functions of special functions (and polynomials) in one, two and more variables have also been explored.

In 2011, Lin et. al. introduced the generalized Cesàro polynomials as follows [6]:

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} {}_2F_1\left[\begin{matrix} -n, & \lambda; \\ & -s-n; \end{matrix} \; x\right]. \tag{1.1}$$

Here, ${}_2F_1$ denotes the Gauss hypergeometric function. It is noted that the special case $\lambda = 1$ of (1.1) reduces immediately to the second one of the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in $g_n^{(s)}(x)$ [10].

A further generalization of the familiar Kampé de Fériet hypergeometric function in two variables, which is called the generalized Lauricella function, was introduced

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by Srivastava and Daoust [15]:

$$F^{A:B^{(1)};...;B^{(n)}}_{C:D^{(1)};...;D^{(n)}}\left(\begin{array}{l}\left[(a):\theta^{(1)},...,\theta^{(n)}\right]:\quad\left[(b^{(1)}):\phi^{(1)}\right];\\ \\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]:\quad\left[(d^{(1)}):\delta^{(1)}\right];\end{array}\right.$$
$$\left.\begin{array}{l}...;\quad\left[(b^{(n)}):\phi^{(n)}\right];\\ \qquad\qquad\qquad\qquad z_1,...,z_n\\ ...;\quad\left[(d^{(n)}):\delta^{(n)}\right];\end{array}\right)$$
$$=\sum_{m_1,...,m_n=0}^{\infty}\Omega(m_1,...,m_n)\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},$$

where for convenience

$$\Omega(m_1,...,m_n):=\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}}\frac{\prod\limits_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod\limits_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}...\frac{\prod\limits_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod\limits_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}.$$

Here, the coefficients

$$\theta_k^{(j)}\ (k=1,...,A;\ j=1,...,n)\text{ and }\phi_k^{(j)}\ (k=1,...,B^{(j)};\ j=1,...,n),$$
$$\psi_k^{(j)}\ (k=1,...,C;\ j=1,...,n)\ \text{ and }\delta_k^{(j)}\ (k=1,...,D^{(j)};\ j=1,...,n)$$

are real constants and $\left(b_{B^{(j)}}^{(j)}\right)$ abridges the array of $B^{(j)}$ parameters $b_k^{(j)}$ $(k=1,...,B^{(j)};\ j=1,...,n)$ with similar commentarys for other sets of parameters [11].

As usual $(\delta)_v$ denotes the Pochhammer symbol by

$$(\delta)_v:=\frac{\Gamma(\delta+v)}{\Gamma(\delta)}=\left\{\begin{array}{cc}1 & (v=0;\delta\in\mathbb{C}/\{0\})\\ \delta(\delta+1)...(\delta+n-1) & (v=n\in\mathbb{N};\delta\in\mathbb{C})\end{array}\right.,$$

it being understood conventionally that $(0)_0:=1$.

In recent years, many researchers have studied multilinear and multilateral generating functions for different type of polynomials. For example, in [12], Liu introduced bilateral generating functions for Lagrange polynomials and Lauricella functions. Similarly, in [13] the authors obtained bilateral generating functions for Chan–Chyan–Srivastava polynomials and generalized Lauricella functions. In 2012, they derived bilateral generating functions for Erkuş–Srivastava polynomials and generalized Lauricella functions (see [11]).We have very recently obtained bilateral generating functions for generalized Cesàro polynomials and generalized Lauricella functions (see [16]). Various generating functions can be found with the [4], [5], [14] and [17] methods.

The main object of this paper is to study several properties of generalized Cesàro polynomials in several variables. Various families of multilinear and multilateral generating functions, their miscellaneous properties and also some special cases are given. In addition, we derive a result giving certain families of bilateral generating functions for generalized Cesàro polynomials in several variables and generalized Lauricella functions.

2. Generalized Cesàro polynomials in several variables and their generating functions

Definition 2.1. *We define an extension of the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in m variables by*

$$g_n^{(s)}(\lambda_1,...,\lambda_m,x_1,...,x_m) = \sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\sum_{r_3=0}^{n-r_1-r_2}\cdots\sum_{r_m=0}^{n-r_1-r_2-...-r_{m-1}}\binom{s+n}{n} \times \frac{(-n)_{\delta_m}\prod_{j=1}^{m}(\lambda_j)_{r_j}}{(-s-n)_{\delta_m}}\frac{\prod_{j=1}^{m}(x_j)^{r_j}}{\prod_{j=1}^{m}(r_j)!} \tag{2.1}$$

where $m, n \in \mathbb{N}$ and

$$\delta_m = \sum_{j=1}^{m} r_j. \tag{2.2}$$

It is noted that the special case $m = 1$ of (2.1) reduces immediately to the second one of the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in (1.1).

Now, we will present two generating functions for the multivariable Cesàro polynomials $g_n^{(s)}(\lambda_1,...,\lambda_m,x_1,...,x_m)$.

Theorem 2.1. *The following generating function holds true:*

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_m,x_1,...,x_m)t^n = (1-t)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j},\ (m\in\mathbb{N}). \tag{2.3}$$

Proof. Let's start by remembering a formal manipulation of the double series:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k). \tag{2.4}$$

Let S show the first member of the (2.3) claim. If we substitute for the multivariable Cesàro polynomials from the definition (2.1) into the left hand-side of (2.3) and we apply (2.4) to the resulting expression with $k = r_1, ..., r_m$ consecutively, then we have

$$S = \sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\sum_{n=0}^{\infty}\binom{s+(n+\delta_m)}{(n+\delta_m)}\frac{(-(n+\delta_m))_{\delta_m}\prod_{j=1}^{m}(\lambda_j)_{r_j}}{(-s-(n+\delta_m))_{\delta_m}}\frac{\prod_{j=1}^{m}(x_j)^{r_j}}{\prod_{j=1}^{m}(r_j)!}t^{n+\delta_m}.$$

If we use the identity

$$(-m-p)_p = (-1)^p\frac{(m+p)!}{m!}\ (m,p\in\mathbb{N}_0)$$

to the first factor of the denominator of the last fraction, we get

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\binom{s+n}{n}t^n\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\frac{\prod_{j=1}^{m}(\lambda_j)_{r_j}\prod_{j=1}^{m}(x_j)^{r_j}}{\prod_{j=1}^{m}(r_j)!}t^{\delta_m} \\ &= \sum_{n=0}^{\infty}(s+1)_n\frac{t^n}{n!}\sum_{r_1=0}^{\infty}\frac{(\lambda_1)_{r_1}(x_1t)^{r_1}}{r_{1!}}\cdots\sum_{r_m=0}^{\infty}\frac{(\lambda_m)_{r_m}(x_mt)^{r_m}}{r_{m!}}, \end{aligned}$$

where δ_m is given by (2.2).

By applying the generalized binomial theorem

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!} \quad (|z| < 1)$$

to the last resulting equation, we attain our aim. □

Remark 2.1. *In the case of $m = 1$, Theorem 2.1 for the multivariable Cesàro polynomials reduces to the generating function of the generalized Cesàro polynomials given in* [16].

Theorem 2.2. *The following generating function for the multivariable Cesàro polynomials holds true:*

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{n+m}{n} g_{n+m}^{(s)}(\lambda_1, ..., \lambda_k, x_1, ..., x_k) t^n \\ = \quad &(1-t)^{-s-m-1} \prod_{j=1}^{k} (1-x_j t)^{-\lambda_j} g_m^{(s)} \left(\lambda_1, ..., \lambda_k, \frac{x_1(1-t)}{1-x_1 t}, ..., \frac{x_k(1-t)}{1-x_k t} \right), \end{aligned}$$

where $|t| < \min \{|x_1|, ..., |x_k|\}$.

Proof. Replacing t by $t+u$ in (2.3), we get

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, ..., \lambda_k, x_1, ..., x_k)(t+u)^n = (1-t-u)^{-s-1} \prod_{j=1}^{k} (1-x_j t - x_j u)^{-\lambda_j}.$$

If we use the binomial expansion the left hand-side of the last relation, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, ..., \lambda_k, x_1, ..., x_k) \sum_{m=0}^{n} \binom{n}{m} t^{n-m} u^m \\ = \quad &(1-t)^{-s-1} \left(1 - \frac{u}{1-t}\right)^{-s-1} \\ &\times \prod_{j=1}^{k} (1-x_j t)^{-\lambda_j} \prod_{j=1}^{k} \left(1 - \frac{x_j u}{1-x_j t}\right)^{-\lambda_j}. \end{aligned}$$

After some calculations, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{m} g_{n+m}^{(s)}(\lambda_1, ..., \lambda_k, x_1, ..., x_k) t^n u^m \\ = \quad &(1-t)^{-s-1} \prod_{j=1}^{k} (1-x_j t)^{-\lambda_j} \sum_{m=0}^{\infty} (1-t)^{-m} \\ &\times g_m^{(s)} \left(\lambda_1, ..., \lambda_k, \frac{x_1(1-t)}{1-x_1 t}, ..., \frac{x_k(1-t)}{1-x_k t} \right) u^m. \end{aligned}$$

From the coefficients of u^m on the both sides of the last equality, one can get the desired result. □

3. Bilinear and Bilateral Generating Functions

In this section, we give theorems which derive several substantially families of bilinear and bilateral generating functions for the multivariable Cesàro polynomials by using the similar method considered in [8], [9], [16], [17].

Lemma 3.1. *We have the following summation formula for the multivariable Cesàro polynomials:*

$$\begin{aligned} & g_n^{(s_1+s_2+1)}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k) \\ = & \sum_{k=0}^{n} g_{n-k}^{(s_1)}(\lambda_1,...,\lambda_k,x_1,...,x_k) g_k^{(s_2)}(\mu_1,...,\mu_k,x_1,...,x_k). \end{aligned} \tag{3.1}$$

Proof. If we take $s=s_1+s_2+1$, $\lambda_i=\lambda_i+\mu_i$ in Theorem 2.1 and then we use the relation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k), \tag{3.2}$$

we have

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(s_1+s_2+1)}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k)t^n \\ = & (1-t)^{-s_1-s_2-2}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j-\mu_j} \\ = & (1-t)^{-s_1-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}(1-t)^{s_2-1}\prod_{j=1}^{k}(1-x_jt)^{-\mu_j} \\ = & \sum_{n=0}^{\infty} g_n^{(s_1)}(\lambda_1,...,\lambda_k,x_1,...,x_k)t^n \sum_{k=0}^{\infty} g_k^{(s_2)}(\mu_1,...,\mu_k,x_1,...,x_k)t^k \\ = & \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_n^{(s_1)}(\lambda_1,...,\lambda_k,x_1,...,x_k) g_k^{(s_2)}(\mu_1,...,\mu_k,x_1,...,x_k)t^{n+k} \\ = & \sum_{n=0}^{\infty}\sum_{k=0}^{n} g_{n-k}^{(s_1)}(\lambda_1,...,\lambda_k,x_1,...,x_k) g_k^{(s_2)}(\mu_1,...,\mu_k,x_1,...,x_k)t^{n}. \end{aligned} \tag{3.3}$$

Equating the coefficients of the same powers of t in both sides of equation (3.3), we are led to assertion (3.1). □

Theorem 3.2. *For a non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and for $a_k\neq 0$, $\mu,\psi\in\mathbb{C}$, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) := \sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

and

$$\begin{aligned} & \Theta_{n,p}^{\mu,\psi}(\lambda_1,...,\lambda_k,x_1,...,x_k;y_1,...,y_r;\xi) \\ : & = \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\,\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k. \end{aligned}$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(\lambda_1,...,\lambda_k,x_1,...,x_k;y_1,...,y_r;\tfrac{\eta}{t^p}\right)t^n\\ &=(1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta).\end{aligned}\tag{3.4}$$

Proof. For convenience, let S denote the first member of the assertion (3.4). Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s)}\left(\lambda_1,...,\lambda_k,x_1,...,x_k\right)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n-pk}$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,g_n^{(s)}\left(\lambda_1,...,\lambda_k,x_1,...,x_k\right)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n\\ &=\sum_{n=0}^{\infty}g_n^{(s)}\left(\lambda_1,...,\lambda_k,x_1,...,x_k\right)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\\ &=(1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof. □

Theorem 3.3. *For a non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and for $a_n\neq 0$, $\mu\in\mathbb{C}$, let*

$$\begin{aligned}&\Lambda_{\mu,p,q}\left[\lambda_1,...,\lambda_k,x_1,...,x_k;y_1,...,y_r;t\right]\\ :&=\sum_{n=0}^{\infty}a_n g_{m+qn}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\Omega_{\mu+pn}(y_1,...,y_r)t^n,\end{aligned}$$

and

$$\theta_{n,p,q}(y_1,...,y_r;z):=\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k.$$

Then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}g_{m+n}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\theta_{n,p,q}(y_1,...,y_r;z)t^n\\ =\;&(1-t)^{-s-m-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}\\ &\times\Lambda_{\mu,p,q}\left(\lambda_1,...,\lambda_k,\frac{x_1(1-t)}{1-x_1t},...,\frac{x_k(1-t)}{1-x_kt};y_1,...,y_r;z\left(\frac{t}{1-t}\right)^q\right).\end{aligned}\tag{3.5}$$

Proof. By using Theorem 2.2 and similar method in proof of Theorem 3.2, we arrive at the desired result. □

Theorem 3.4. *For non-vanishing function $\Omega_{\mu}(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and for $a_k\neq 0$, $\mu,\psi\in\mathbb{C}$, let*

$$\begin{aligned}&\Lambda^{n,p}_{\mu,\psi}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k;y_1,...,y_r;z)\\ :&=\sum_{k=0}^{[n/p]}a_k g^{(s_1+s_2+1)}_{n-pk}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k)\Omega_{\mu+\psi k}(y_1,...,y_r)z^k.\end{aligned}$$

Then, for $p\in\mathbb{N}$, we have

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g^{(s_1)}_{n-k}(\lambda_1,...,\lambda_k,x_1,...,x_k)g^{(s_2)}_{k-pl}(\mu_1,...,\mu_k,x_1,...,x_k)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l\\ =\;&\Lambda^{n,p}_{\mu,\psi}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k;y_1,...,y_r;z).\end{aligned}\tag{3.6}$$

Proof. Applying the well-known equality

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}A(k,l)=\sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}A(k+pl,l)$$

and then using Lemma 3.1, we get

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g^{(s_1)}_{n-k}(\lambda_1,...,\lambda_k,x_1,...,x_k)g^{(s_2)}_{k-pl}(\mu_1,...,\mu_k,x_1,...,x_k)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l\\ =\;&\sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l g^{(s_1)}_{n-k-pl}(\lambda_1,...,\lambda_k,x_1,...,x_k)g^{(s_2)}_{k}(\mu_1,...,\mu_k,x_1,...,x_k)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l\\ =\;&\sum_{l=0}^{[n/p]}a_l\left(\sum_{k=0}^{n-pl}g^{(s_1)}_{n-k-pl}(\lambda_1,...,\lambda_k,x_1,...,x_k)g^{(s_2)}_{k}(\mu_1,...,\mu_k,x_1,...,x_k)\right)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l\\ =\;&\sum_{l=0}^{[n/p]}a_l g^{(s_1+s_2+1)}_{n-pl}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k)\Omega_{\mu+\psi l}(y_1,...,y_r)z^l\\ =\;&\Lambda^{n,p}_{\mu,\psi}(\lambda_1+\mu_1,...,\lambda_k+\mu_k,x_1,...,x_k;y_1,...,y_r;z)\end{aligned}$$

which completes the proof. □

Notice that, when the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $r\in\mathbb{N}$ is expressed as an appropriate product of several simpler functions of one or more variables, then each suitable choice of the coefficients a_k $(k\in\mathbb{N}_0)$ in Theorems 3.1, 3.2 and 3.3 can be shown to yield various families of multilinear and multilateral generating functions for the multivariable Cesàro polynomials defined by (2.1).

4. The multivariable Cesàro polynomials and the generalized Lauricella functions

Now, for a suitable bounded non-vanishing multiple sequence $\{\Omega(m_1,...,m_s)\}_{m_1,...,m_s\in\mathbb{N}_0}$ (real or complex) parameters, a function $\phi_n(u_1;u_2,...,u_s)$ of s (real or complex) variables $u_1,...,u_s$ is defined by [11]

$$\begin{aligned} & \phi_n(u_1;u_2,...,u_s) \\ :&= \sum_{m_1=0}^{n}\sum_{m_2,...,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,...,m_s),m_2,...,m_s\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_s^{m_s}}{m_s!}. \end{aligned}$$

Here for convenience $((b))_{m_1\phi}=\prod_{j=1}^{B}(b_j)_{m_1\phi_j}$ and $((d))_{m_1\delta}=\prod_{j=1}^{D}(d_j)_{m_1\delta_j}$.

Theorem 4.1. *The following bilateral generating function relationship applies:*

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\phi_n(u_1;u_2,...,u_r)t^n \\ =\ & (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j} \\ & \times\sum_{k_1,...,k_k,m_1,...,m_r=0}^{\infty}\frac{((b))_{(k_1+...+k_k+m_1)\phi}}{((d))_{(k_1+...+k_k+m_1)\delta}} \\ & \times\Omega\left(f\left(k_1+...+k_k+m_1\right),m_2,...,m_r\right)(\lambda_1)_{k_1}...(\lambda_k)_{k_k}(s+1)_{m_1} \\ & \times\frac{\left(\frac{u_1x_1t}{x_1t-1}\right)^{k_1}}{k_1!}...\frac{\left(\frac{u_1x_kt}{x_kt-1}\right)^{k_k}}{k_k!}\frac{\left(\frac{u_1t}{t-1}\right)^{m_1}}{m_1!}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}. \end{aligned}$$

Proof. By using the relation in the statement of Theorem 2.2, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\phi_n(u_1;u_2,...,u_r)t^n \\ =\ & \sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\sum_{m_1=0}^{n}\sum_{m_2,...,m_r=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ & \times\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)\frac{u_1^{m_1}}{m_1!}...\frac{u_r^{m_r}}{m_r!}t^n \\ =\ & \sum_{m_1,...,m_k=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right) \\ & \times(-u_1t)^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}(1-t)^{-s-m_1-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j} \\ & \times g_{m_1}^{(s)}\left(\lambda_1,...,\lambda_k,\frac{x_1(1-t)}{1-x_1t},...,\frac{x_k(1-t)}{1-x_kt}\right). \end{aligned}$$

Using the definition (2.1), it is easily observed that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\phi_n(u_1;u_2,...,u_r)t^n \\
= \quad & (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}\sum_{m_1,...,m_r=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right) \\
&\times(\frac{-u_1t}{1-t})^{m_1}\frac{u_2^{m_2}}{m_2!}...\frac{u_r^{m_r}}{m_r!}\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_1-k_1}\sum_{k_3=0}^{m_1-k_1-k_2}...\sum_{k_k=0}^{m_1-k_1-k_2-...-k_{k-1}}\binom{s+m_1}{m_1} \\
&\times\frac{(-m_1)_{k_1+...+k_k}\prod\limits_{j=1}^{k}(\lambda_j)_{k_j}}{(-s-m_1)_{k_1+...+k_k}\prod\limits_{j=1}^{k}(k_j)!}\prod_{j=1}^{k}\left(\frac{x_j(1-t)}{1-x_jt}\right)^{k_j}
\end{aligned}
$$

whence the result. □

By appropriately choosing the multiple sequence $\Omega(m_1,...,m_r)$ in Theorem 4.1, we obtain several results including, for example, the following bilateral generating functions:

Example 4.1. *By letting*

$$
\begin{aligned}
&\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right) \\
= \quad &\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+...+m_r\theta_j^{(r)}}}{\prod\limits_{j=1}^{E}(c_j)_{m_1\psi_j^{(1)}+...+m_r\psi_j^{(r)}}}\frac{\prod\limits_{j=1}^{B^{(2)}}(b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod\limits_{j=1}^{D^{(2)}}(d_j^{(2)})_{m_2\delta_j^{(2)}}}...\frac{\prod\limits_{j=1}^{B^{(r)}}(b_j^{(r)})_{m_r\phi_j^{(r)}}}{\prod\limits_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}}
\end{aligned}
$$

in Theorem 4.1,we obtain the following bilateral generating function:

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k) \\
&\times F^{A:B+1;B^{(2)};...;B^{(r)}}_{E:D;D^{(2)};...;D^{(r)}}\left(\begin{array}{l} \left[(a):\theta^{(1)},...,\theta^{(r)}\right]:\quad [-n:1],\quad [(b):\phi]; \\ \left[(c):\psi^{(1)},...,\psi^{(r)}\right]: \qquad\qquad\qquad [(d):\delta]; \end{array}\right. \\
&\qquad\qquad \left.\begin{array}{l} [(b^{(2)}):\phi^{(2)}];\quad ...;\quad [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(2)}):\delta^{(2)}];\quad ...;\quad [(d^{(r)}):\delta^{(r)}]; \end{array} u_1,...,u_r\right) t^n \\
= \quad & (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j} \\
&\times F^{A+B:1;...;1;B^{(2)};...;B^{(r)}}_{E+D:0;...;0;D^{(2)};...;D^{(r)}}\left(\begin{array}{l} \left[(e):\varphi^{(1)},...,\varphi^{(k+r)}\right]:[\lambda_1:1];...;[\lambda_k:1];\quad [s+1:1], \\ \left[(f):\xi^{(1)},...,\xi^{(k+r)}\right]: \quad - \quad ; \quad ... \quad ; \quad -; \qquad - \end{array}\right. \\
&\left.\begin{array}{l} [(b^{(2)}):\phi^{(2)}];\quad ...;\quad [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(2)}):\delta^{(2)}];\quad ...;\quad [(d^{(r)}):\delta^{(r)}]; \end{array} (\frac{u_1x_1t}{x_1t-1}),...,(\frac{u_1x_kt}{x_kt-1}),\left(\frac{u_1t}{t-1}\right),u_2,...,u_r\right).
\end{aligned}$$

Here, the coefficients are given by

$$e_j=\begin{cases} a_j & (1\le j\le A) \\ b_{j-A} & (A<j\le A+B) \end{cases}\quad,\quad f_j=\begin{cases} c_j & (1\le j\le E) \\ d_{j-E} & (E<j\le E+D) \end{cases}\quad,$$

$$\varphi_j^{(s)}=\begin{cases} \theta_j^{(1)} & (1\le j\le A;\ 1\le s\le k+1) \\ \theta_j^{(s-k)} & (1\le j\le A;\ k+1<s\le k+r) \\ \phi_{j-A} & (A<j\le A+B;\ 1\le s\le k+1) \\ 0 & (A<j\le A+B;\ k+1<s\le k+r) \end{cases}\quad,$$

$$\xi_j^{(s)}=\begin{cases} \psi_j^{(1)} & (1\le j\le E;\ 1\le s\le k+1) \\ \psi_j^{(s-k)} & (1\le j\le E;\ k+1<s\le k+r) \\ \delta_{j-E} & (E<j\le E+D;\ 1\le s\le k+1) \\ 0 & (E<j\le E+D;\ k+1<s\le k+r) \end{cases}\quad.$$

Example 4.2. *If we set*

$$\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)=\frac{(a)_{m_1+...+m_r}(b_2)_{m_2}...(b_r)_{m_r}}{(c_1)_{m_1}...(c_r)_{m_r}}$$

and

$$\phi=\delta=0\quad (\textit{that is, } \phi_1=...=\phi_B=\delta_1=...=\delta_D=0)$$

in Theorem 4.1, the following bilateral generating function holds true:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)F_A^{(r)}\left[a,-n,b_2,...,b_r;c_1,...,c_r;u_1,...,u_r\right]t^n$$

$$= (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}$$

$$\times F^{1:1;1;...;1;1;...;1}_{1:0;0;...;0;1;...;1}\left(\begin{array}{l} [(a):1,...,1]: \quad [\lambda_1:1];...;[\lambda_k:1];[s+1:1]; \\ \left[(c_1):\psi^{(1)},...,\psi^{(k+r)}\right]: - \quad ; \quad ... \quad ; \quad - \quad ; \quad - \quad ; \end{array}\right.$$

$$\left.\begin{array}{l} [b_2:1]; \quad ...; \quad [b_r:1]; \\ [c_2:1]; \quad ...; \quad [c_r:1]; \quad (\frac{u_1x_1t}{x_1t-1}),...,(\frac{u_1x_kt}{x_kt-1}),\left(\frac{u_1t}{t-1}\right),u_2,...,u_r \end{array}\right).$$

Here $F_A^{(r)}$ *is the Lauricella function and the coefficients* $\psi^{(\eta)}$ *are given by*

$$\psi^{(\eta)}=\begin{cases} 1 & (1\leq\eta\leq k+1) \\ 0 & k+1<\eta\leq k+r \end{cases}.$$

Example 4.3. *By letting*

$$\Omega\left(f(m_1,...,m_r),m_2,...,m_r\right)=\frac{(a)_{m_1+...+m_r}(b_2)_{m_2}...(b_r)_{m_r}}{(c)_{m_1+...+m_r}}$$

and

$$\phi=\delta=0,$$

in Theorem 4.1,we obtain the following bilateral generating function:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)F_D^{(r)}\left[a,-n,b_2,...,b_r;c;u_1,...,u_r\right]t^n$$

$$= (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}F_D^{(k+r)}\left[a,\lambda_1,...,\lambda_k,s+1,b_2,...,b_r;c;\right.$$

$$\left.(\frac{u_1x_1t}{x_1t-1}),...,(\frac{u_1x_kt}{x_kt-1}),\left(\frac{u_1t}{t-1}\right),u_2,...,u_r\right].$$

Here $F_D^{(r)}$ *is the Lauricella function.*

5. Miscellaneous Properties

In this section, we give a theorem which derive an integral representation for the multivariable Cesàro polynomials defined by (2.1) and then we obtain some recurrence relations for these polynomials.

Theorem 5.1. *The multivariable Cesàro polynomials have the following integral notation:*

$$g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k) = \frac{1}{\Gamma(s+1)\Gamma(\lambda_1)...\Gamma(\lambda_k)}\int_0^{\infty}...\int_0^{\infty}e^{-(m+u_1+...+u_k)}$$

$$\times\frac{(m+u_1x_1+...+u_kx_k)^n}{n!}m^s u_1^{\lambda_1-1}...u_k^{\lambda_k-1}dmdu_1...du_k,$$

where $\mathrm{Re}\,(s+1)>0$*,* $\mathrm{Re}\,(\lambda_i)>0$*,* $i=1,...,k$.

Proof. If we use the identity

$$a^{-v}=\frac{1}{\Gamma(v)}\int\limits_0^\infty e^{-at}t^{v-1}dt \qquad (\mathrm{Re}\,(\nu)>0)$$

on the right hand-side of the generating function (2.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)t^n\\
=\;& (1-t)^{-s-1}\prod_{j=1}^{k}(1-x_jt)^{-\lambda_j}\\
=\;& \frac{1}{\Gamma(s+1)\Gamma(\lambda_1)...\Gamma(\lambda_k)}\int\limits_0^\infty ...\int\limits_0^\infty e^{-(m+u_1+...+u_k)}\\
&\times e^{(m+u_1x_1+...+u_kx_k)t}m^s u_1^{\lambda_1-1}...u_k^{\lambda_k-1}dmdu_1...du_k\\
=\;& \frac{1}{\Gamma(s+1)\Gamma(\lambda_1)...\Gamma(\lambda_k)}\int\limits_0^\infty ...\int\limits_0^\infty e^{-(m+u_1+...+u_k)}\\
&\times\sum_{n=0}^{\infty}\frac{(m+u_1x_1+...+u_kx_k)^n}{n!}t^n m^s u_1^{\lambda_1-1}...u_k^{\lambda_k-1}dmdu_1...du_k.
\end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result. □

We now discuss some miscellaneous recurrence relations for the multivariable Cesàro polynomials. By differentiating each member of the generating function relation (2.3) with respect to x_j $(j=1,...,k)$ and using (3.2), we arrive at the following (differential) recurrence relation for the multivariable Cesàro polynomials:

$$\begin{aligned}
&\frac{\partial}{\partial x_j}g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)-x_j\frac{\partial}{\partial x_j}g_{n-1}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\\
&\qquad=\lambda_j g_{n-1}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k), \qquad (n\geq 1).
\end{aligned}\tag{5.1}$$

Furthermore, using the similar idea, one can obtain another recurrence relation as follows:

$$\frac{\partial}{\partial x_j}g_n^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)=\lambda_j\sum_{k=0}^{n-1}x_j^{n-k-1}g_k^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k).$$

If we compare this equality and (5.1), we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1}x_j^{n-k-1}g_k^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)-\sum_{k=0}^{n-2}x_j^{n-k-1}g_k^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\\
&\qquad=g_{n-1}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k).
\end{aligned}$$

Besides, by differentiating each member of the generating function relation (2.3) with respect to t, we have the following recurrence relation for the multivariable

Cesàro polynomials:

$$(n+1)g_{n+1}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)=\sum_{m=0}^{n}\Bigg[(s+1)\,g_{n-m}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)$$
$$+\sum_{j=1}^{k}\lambda_j x_j^{m+1}g_{n-m}^{(s)}(\lambda_1,...,\lambda_k,x_1,...,x_k)\Bigg],\qquad (j=1,...,k).$$

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Nejla Özmen
Düzce University
Faculty of Art and Science
Department of Mathematics
Konuralp TR-81620, Düzce, Turkey
E-mail addresses: nejlaozmen06@gmail.com, nejlaozmen@duzce.edu.tr