GROUND STATE AND MULTIPLE SOLUTIONS FOR SCHRÖDINGER-BOPP-PODOLSKY SYSTEM WITH CRITICAL NONLINEARITY

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Abstract. In this paper, we study the following nonlinear Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + u + l(x)\phi u = a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u + |u|^5, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $p, q \in (4, 6)$, $\mu > 0$, $l(x)$, $a(x)$ and $b(x)$ are nonnegative continuous functions. Under some certain assumptions, we prove the above system have ground state and multiple solutions by using variational.

1. Introduction

In recent years, the following Schrödinger-Bopp-Podolsky system was first introduced in [18]

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u, \phi : \mathbb{R}^3 \to \mathbb{R}$, $\omega, a > 0$, $q \neq 0$. The system is the result of the coupling of Schrödinger field $\psi = \psi(t, x)$ and its electromagnetic field in the Bopp-Podolsky electromagnetic theory and considers standing wave $\psi(t, x) = e^{i\omega t}u(x)$ in the purely electrostatic case. The Bopp-Podolsky theory is a second order gauge theory for the electromagnetic field, it was developed by Bopp [2] and then independently by Podolsky [25] and it was used to solve the "infinity problem" that appears in the Maxwell theory, we refer the readers to see [3, 4, 5, 6, 24] and the references therein.

Moreover the Bopp-Podolsky theory also was an effective theory for short distances and for large distances it is experimentally indistinguishable from the Maxwell one. For more physical details readers can see [7, 8, 9, 10, 13, 14, 19] and the references therein. For the operator $-\Delta + \Delta^2$, it appears also in different mathematical and physical domains(see [11, 20] and so on).

In [18], Pietro d'Avenia prove that problem (1.1) existence and nonexistence results depending on the parameters p, q. Moreover they also show that, in the radial case, the solutions we find tend to solutions of the classical Schrödinger-Poisson system as $a \to 0$. They also showed that, if ρ is the distribution density of the given charge, then the electrostatic potential ϕ satisfies the following equation

$$-\Delta\phi = \rho, \tag{1.2}$$

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if $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $E(x-x_0) = \frac{1}{|x-x_0|}$ and the electrostatic energy is $\mathcal{E}(E) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla E(x)|^2 = +\infty$. Thus, in the Born-Infeld theory, (1.2) is replaced by $-\mathrm{div}(\frac{\nabla\phi}{(1-|\nabla\phi|^2)^{\frac{1}{2}}}) = \rho$; In Bopp-Podolsky theory, it is replaced by $-\Delta\phi + a^2\Delta^2\phi = \rho$. Moreover, we know that $\mathcal{K}(x-x_0)$ is the fundamental solution of the equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0},$$

where

$$\mathcal{K}(x) = \frac{1-e^{-\frac{|x|}{a}}}{|x|}, \quad \lim_{x\to x_0} \mathcal{K}(x-x_0) = \frac{1}{a}.$$

Its energy is

$$\varepsilon_{BP}(\kappa) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{K}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{K}|^2 < +\infty,$$

more details in [18].

After that, Gaetano and Kaye [21] study prove, by means of the fibering approach, that the system (1.1) has no solutions at all for large values of $q's$, and has two radial solutions for small $q's$. They give also qualitative properties about the energy level of the solutions and a variational characterization of these extremals values of q.

In [15], Chen Sitong and Tang Xianhua deals with the following nonlinear Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \mu f(u) + u^5, & \text{in } \mathbb{R}^3, \\ -2\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0$, $V(x) \in C(\mathbb{R}^3, [0,\infty))$ with $V_\infty = \lim_{|y|\to\infty} V(y) \geq \sup_{x\in\mathbb{R}^3} V(x) > 0$, and $f \in C(\mathbb{R},\mathbb{R})$ satisfying $\int_0^t f(s)ds \geq t^p$ with $p \in (4,6)$ for all $t \geq 0$. By using some new analytic techniques and new inequalities, they prove that the above system admits ground state solutions for all $\mu > 0$ if $p \in (4,6)$; for all $\mu > \mu_0$ if $p \in (2,4]$ where μ_0 is a positive constant determined by a, V_∞ and p.

In [27], Zhu Yuting, Chen Chunfang and Chen Jianhua study the following nonlinear Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + V(x)u + q\phi u = f(u), & \text{in } \mathbb{R}^3, \\ -2\Delta\phi + a^2\Delta^2\phi = 4\pi u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $a > 0$, $q > 0$ and $V \in C(\mathbb{R}^3,\mathbb{R})$. By means of the variational methods, author prove the existence of infinitely many nontrivial solutions, the existence of a ground state solution for $f(x,u) = |u|^{p-2}u + h(u)u$ with $p \in [4,6)$ and the existence of at least one positive solution for $f(x,u) = p(x)|u|^5 + \mu|u|^{p-2}u$ with $p \in (4,6)$ under some certain assumptions.

Inspired by [28], in this paper, we consider the following nonlinear Schrödinger-Bopp-Podolsky system::

$$\begin{cases} -\Delta u + u + l(x)\phi u = a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u + |u|^5, & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $p,q \in (4,6)$, $a > 0$ and $\mu > 0$ is a parameter, $l(x)$, $a(x)$ and $b(x)$ satisfy the following conditions:

($l_1^\star$) $l(x) \in C(\mathbb{R}^3)$, $l(x) \geq 0$ and $\lim\limits_{|x|\to\infty} l(x) = l_\infty > 0$;

($l_2^\star$) there exist $C_0 > 0$ and $l > 0$ such that $l(x) \leq l_\infty - C_0 e^{-l|x|}$ for all $x \in \mathbb{R}^3$;

$(l_3^\star)$ there exist $C_1 > 0$ and $d > 0$ such that $l(x) \le l_\infty + C_1 e^{-d|x|}$ for all $x \in \mathbb{R}^3$;
$(a_1^\star)$ $a(x) \in C(\mathbb{R}^3)$, $a(x) \ge 0$ and $\lim\limits_{|x|\to\infty} a(x) = a_\infty > 0$;
$(a_2^\star)$ there exist $C_2 > 0$ and $m > 0$ such that $a(x) \ge a_\infty - C_2 e^{-m|x|}$ for all $x \in \mathbb{R}^3$;
$(b_1^\star)$ $b(x) \in C(\mathbb{R}^3)$, $b(x) \ge 0$ and $\lim\limits_{|x|\to\infty} b(x) = 0$;
$(b_2^\star)$ there exist $C_3 > 0$ and $b > 0$ such that $b(x) \ge C_3 e^{-b|x|}$ for all $x \in \mathbb{R}^3$.

The following is the main result of problem (1.3).

Theorem 1.1. (**A**) *Assume that $l(x)$, $a(x)$ and $b(x)$ satisfy $(l_1^\star)$, $(l_2^\star)$, $(a_1^\star)$, $(a_2^\star)$ and $(b_1^\star)$ with $0 < k < m < p$. Then problem (1.3) admits a positive ground state solution.*
(**B**) *Let $m < p$ and $d < 2$. Assume that $l(x)$, $a(x)$ and $b(x)$ satisfy $(l_1^\star)$, $(l_3^\star)$, $(a_1^\star)$-$(a_2^\star)$ and $(b_1^\star)$-$(b_2^\star)$ hold with $b < \min\{m, d\}$. Then problem (1.3) admits a positive ground state solution.*

Theorem 1.2. *Assume the assumptions of Theorem 1.1-(2) hold with $l(x) \ge l_\infty$, $a(x) \le a_\infty$ and meas$\{x \in \mathbb{R}^3 : l(x) \ge l_\infty\} > 0$. Then there exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$, problem (1.3) admits at least two nontrivial solutions.*

Remark 1.3. • *To obtain the existence of ground state solutions of (1.3), there will be several difficulties at present. On the one hand, because of the presence of the nonlocal term, we have to need analyze the influence of ϕ and $a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u + |u|^5$. On the other hand, it should be pointed out that the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ $2 \le r \le 6$(see Section 2) is not compact.*
• *To the best of our knowledge, there are few papers on the multiplicity solutions for system (1.3). Inspired by [1], we construct two mappings:*

$$\begin{cases} F_R : S^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \to \{u \in M : I(u) \le m_\infty - \varepsilon(R)\}, \\ G : \{u \in M : I(u) < m_\infty\} \to S^2, \end{cases}$$

where more precisely see Section 5. So that $G \circ F_R$ homotopic to the identity. Using the theory of Lusternik-Schnirelman category, we will establish the existence of two nontrivial solutions for system (1.3). Using the ideas in reference [1], it is essential to construct maps F_R and G using the definitions and properties of barycenter map in [16]. However, the barycenter map in [16] cannot be applied directly here because (1.3) is a different problem. In this paper, the use of the barycenter map is related with the critical term, therefore, the multiple solutions obtained of system (1.3) are different from those obtained by barycenter map in reference [17].

For simplicity, we give the following notations.

Remark 1.4. • $\|u\|_s := (\int_{\mathbb{R}^3} |u|^s dx)^{\frac{1}{s}}$, $1 \le s \le +\infty$.
• *$H^{-1}(\mathbb{R}^3)$ denotes the dual space of $H^1(\mathbb{R}^3)$.*
• *C and C_i denotes universal positive constant. (possibly different)*
• *$B_\rho(x)$ denotes the ball of radius ρ center at x.*
• *S denotes the best Sobolev constant:*

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{(\int_{\mathbb{R}^3} |u(x)|^6 dx)^{\frac{1}{3}}}$$

The paper is organized as follows. In Section 2, we're going to introduce the workspace and present some preliminaries results. In Section 3, We will establish the variational framework and consider the limit problem. In Section 4, we will prove Theorem 1.1. In Section 5, we will prove Theorem 1.2.

2. PRELIMINARY LEMMAS

Let
$$H^1(\mathbb{R}^3) := \Big\{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\Big\},$$
in this case, the inner product and norm are defined as
$$(u,v) = \int_{\mathbb{R}^3} \Big(\nabla u \nabla v + uv\Big)dx,$$
$$\|u\| = \Big(\int_{\mathbb{R}^3} \Big(|\nabla u|^2 + u^2\Big)dx\Big)^{\frac{1}{2}},$$
$H^1(\mathbb{R}^3)$ is a Hilbert space and $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(p \in [2,6])$, $H^1(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^q_{loc}(\mathbb{R}^3)$ $(q \in [1,6))$(see Theorem 1.8 and Theorem 1.9 of [26]).

And denotes the Sobolev space
$$D^{1,2}(\mathbb{R}^3) := \Big\{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\Big\},$$
with the corresponding norm
$$\|u\|_{D^{1,2}} = \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{\frac{1}{2}},$$
is a Hilbert space.

Let $\mathcal{D}$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product
$$(u,v)_{\mathcal{D}} = \int_{\mathbb{R}^3} \nabla u \nabla v dx + a^2 \int_{\mathbb{R}^3} \Delta u \Delta v dx,$$
Clearly, $\mathcal{D}$ is a Hilbert space continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and therefore in $L^6(\mathbb{R}^3)$.

Now we present some preliminary results, they were obtained in [18].

Lemma 2.1. *(see Lemma 3.1 in* [18]*) The space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$.*

Lemma 2.2. *(see Lemma 3.2 in* [18]*) The space $\mathcal{C}_c^\infty(\mathbb{R}^3)$ is dense in*
$$\mathcal{A} := \{\phi \in D^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$
normed by $\sqrt{(\phi,\phi)_{\mathcal{D}}}$ and therefore $\mathcal{D} = \mathcal{A}$

Similar to the proof in [18], it can be proved that for every $u \in H^1(\mathbb{R}^3)$ there is a unique solution $\phi_u \in \mathcal{D}$ of the second equation in the system (1.3), that is satisfying
$$-\Delta\phi_u + a^2\Delta^2\phi_u = l(x)u^2. \tag{2.1}$$
Moreover it turns out that
$$\phi_u(x) = \mathcal{K} * \frac{lu^2}{4\pi} = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} l(y)u^2(y)dy, \tag{2.2}$$
where
$$\mathcal{K}(x) = \frac{1 - e^{-\frac{|x|}{a}}}{|x|}.$$

Lemma 2.3. *(see Lemma 3.4 in* [18]*) For every $u \in H^1(\mathbb{R}^3)$, we have:*
(i) for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;
(ii) $\phi_u \geq 0$;
(iii) $\phi_u \in \mathcal{D}$;
(iv) $\|\phi_u\|_6 \leq C\|u\|^2$;
(v) if $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, then $\phi_{v_n} \rightharpoonup \phi_v$ in $\mathcal{D}$.

In the next, we give a definition of Lusternik-Schnirelman category.

Definition 2.4. (*i*) *For a topological space X, we say a non-empty,closed subset $A \subset X$ is contractible to a point in X if and only if there exist a continuous mapping $\eta : [0,1] \times A \to X$ such that for some $x_0 \in X$,*
(a) $\eta(0,x) = x$ for all $x \in A$,
(b) $\eta(1,x) = x_0$ for all $x \in A$.
(ii) We define

$$\begin{aligned}\operatorname{cat}(X) = \min\{k \in \mathbb{N} :&\textit{there exist closed subsets } A_1,\ldots,A_k \subset X \textit{ such that}\\ &A_i \textit{ is contractible to a point in } X \textit{ for all } i \textit{ and}\\ &\bigcup_{i=1}^{k} A_i = X\}\end{aligned}$$

We say $\operatorname{cat}(X) = \infty$ if there do not exist finitely many closed subsets $A_1, \ldots, A_k \subset X$ such that A_i is contractible to a point in X for all i and $\bigcup_{i=1}^{k} A_i = X$.

We need the following two important lemmas. See Proposition 2.4 and Lemma 2.5 in [1].

Lemma 2.5. *Suppose that $\mathcal{M}$ is a Hilbert manifold and $\Psi \in C^1(\mathcal{M}, \mathbb{R})$. Assume that there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $\Psi(u)$ satisfies the Palais-Smale condition for $c \leq c_0$ and $\operatorname{cat}(\{u \in \mathcal{M} : \Psi(u) \leq c_0\}) \geq k$. Then $\Psi(u)$ has at least k critical points in $\{u \in \mathcal{M} : \Psi(u) \leq c_0\}$.*

Lemma 2.6. *Let X be a topological space. Suppose that there exist two continuous mapping*

$$F : S^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \to X, \quad G : X \to S^2,$$

such that $G \circ F$ is homotopic to identity $\mathrm{id} : S^2 \to S^2$, that is, there is a continuous mapping $\zeta : [0,1] \times S^2 \to S^2$ such that $\zeta(0,x) = (G \circ F)(x)$ for all $x \in S^2$ and $\zeta(1,x) = x$ for all $x \in S^2$. Then $\operatorname{cat}(X) \geq 2$.

3. Functional setting and limit problem

Firstly, in this Section, since $\lim_{|x|\to\infty} l(x) = l_\infty > 0$, $\lim_{|x|\to\infty} a(x) = a_\infty > 0$ and $\lim_{|x|\to\infty} b(x) = 0$. For simplicity, we assume $K_\infty = 1$ and $a_\infty = 1$. Substituting (2.2) into the problem (1.3), the first equation in the system (1.3) is reduced to an equation containing an variable u:

$$-\Delta u + u + l(x)\phi_u u = a(x)|u|^{p-2}u + \mu b(x)|u|^{p-2}u + u^5. \tag{3.1}$$

In order to find weak solutions to (3.1), we look for critical points of the functional $I(u) : H^1(\mathbb{R}^3) \to \mathbb{R}$ associated with (3.1) which is defined by

$$\begin{aligned} I(u) = &\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx \\ &-\frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6 dx. \end{aligned}$$

In order to prove the compactness, we need to consider the following problem at infinity with (3.1) is:

$$-\Delta u + u + \hat{\phi}_u u = |u|^{p-2}u + |u|^5, \quad u > 0, \tag{3.2}$$

where $\hat{\phi}_u \in \mathcal{D}$ is the unique solution to problem

$$-\Delta\phi + a^2\Delta^2\phi = u^2.$$

The functional associated with (3.2) is given by

$$\begin{aligned} I_\infty(u) = &\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}\hat{\phi}_u u^2 dx \\ &-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6 dx. \end{aligned}$$

Let

$$m = \inf_{u\in M} I(u), \quad m_\infty = \inf_{u\in M_\infty} I_\infty(u),$$

where

$$M = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : (I'(u), u) = 0\}, \quad M_\infty = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : (I'_\infty(u), u) = 0\}$$

are Nehari manifolds correspond to the functional I and I_∞, respectively.

Lemma 3.1. *m_∞ satisfies*

$$m_\infty < \frac{1}{3}S^{\frac{3}{2}}.$$

Proof. We define

$$v_\varepsilon(x) = \frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x|^2)^{\frac{1}{2}}},$$

where $\psi \in C_0^\infty(B_{2r}(0))$ such that $0 \le \psi(x) \le 1$ and $\psi(x) = 1$ on $B_r(0)$. Then for $\varepsilon > 0$ small(see [28]),

$$\int_{\mathbb{R}^3}|\nabla v_\varepsilon(x)|^2 dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \tag{3.3}$$

$$\int_{\mathbb{R}^3}|v_\varepsilon(x)|^6 dx = K_2 + O(\varepsilon^{\frac{3}{2}}), \tag{3.4}$$

where $S = \frac{K_1}{K_2^{\frac{1}{3}}}$.

$$\int_{\mathbb{R}^3}|v_\varepsilon(x)|^p dx = \begin{cases} O(\varepsilon^{\frac{6-p}{4}}), & p\in(3,6); \\ O(\varepsilon^{\frac{3}{4}}|\log\varepsilon|), & p=3; \\ O(\varepsilon^{\frac{p}{4}}), & p\in[1,3). \end{cases} \tag{3.5}$$

Moerover, by the Hardy-Littlewood-Sobolev inequality (see [23]) gives

$$\int_{\mathbb{R}^3}\widehat{\phi}_u u^2 dx \le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy \le \frac{8\sqrt[3]{\pi}}{3\sqrt[3]{\pi}}\|u\|^4_{\frac{12}{5}}. \tag{3.6}$$

And similar to Theorems 4.1-4.2 in [26], we can see that $m_\infty = c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1],H) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$ and

$$m_\infty = c_\infty = \inf_{u\in H\setminus\{0\}} \max_{t\geq 0} I_\infty(tu). \tag{3.7}$$

By (3.7), we see that $c_\infty \leq \sup_{t\geq 0} I_\infty(tv_\varepsilon)$. Thus we only need to prove $\sup_{t\geq 0} I_\infty(tv_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}}$ for $\varepsilon > 0$ small.

By (3.6) and $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(p \in [2,6])$, we obtain that

$$\begin{aligned} I_\infty(tv_\varepsilon) &= \frac{1}{2}t^2\|v_\varepsilon\|^2 + \frac{1}{4}t^4\int_{\mathbb{R}^3} \widehat{\phi}_{v_\varepsilon} v_\varepsilon^2 dx - \frac{t^p}{p}\int_{\mathbb{R}^3} |v_\varepsilon|^p dx - \frac{t^6}{6}\int_{\mathbb{R}^3} |v_\varepsilon|^6 dx \\ &\leq \frac{1}{2}t^2\|v_\varepsilon\|^2 + Ct^4\|v_\varepsilon\|^4 - \frac{1}{6}t^6\|v_\varepsilon\|_6^6. \end{aligned} \tag{3.8}$$

Form (3.3)-(3.5), there exists $\varepsilon_1 > 0$ small enough such that

$$\|v_\varepsilon\|^2 := \int_{\mathbb{R}^3} (|\nabla v_\varepsilon|^2 + v_\varepsilon^2) dx \leq K_1 + O(\varepsilon^{\frac{1}{2}}) + O(\varepsilon^{\frac{1}{2}}) \leq \frac{3}{2}K_1, \tag{3.9}$$

$$\|v_\varepsilon\|_6^6 = K_2 + O(\varepsilon^{\frac{3}{2}}) \geq \frac{1}{2}K_2, \tag{3.10}$$

for $\varepsilon \in (0,\varepsilon_1)$. Thus, there exist a small $t_1 > 0$ and a large $t_2 > 0$ independent of $\varepsilon \in (0,\varepsilon_1)$ such that

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)} I_\infty(tv_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}}. \tag{3.11}$$

Form (3.3)-(3.6), we get

$$\begin{aligned} \sup_{t\in[t_1,t_2]} I_\infty(tv_\varepsilon) &\leq \sup_{t\geq 0}[\frac{1}{2}t^2\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{1}{6}t^6\int_{\mathbb{R}^3} |v_\varepsilon(x)|^6 dx] \\ &\quad + C\|v_\varepsilon\|_2^2 + C\|v_\varepsilon\|_{\frac{12}{5}}^4 - C\|v_\varepsilon\|_p^p \\ &= \frac{1}{3}S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) - C\varepsilon^{\frac{6-P}{4}}. \end{aligned} \tag{3.12}$$

In view of $p \in (4,6)$, so we see that $\frac{6-p}{4} < \frac{1}{2}$. By choosing $\varepsilon \in (0,\varepsilon_1)$ small, we get

$$\sup_{t\in[t_1,t_2]} I_\infty(tv_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}}. \tag{3.13}$$

By (3.11) and (3.13), we have

$$m_\infty < \frac{1}{3}S^{\frac{3}{2}}. \tag{3.14}$$

$\square$

Lemma 3.2. *The problem* (3.2) *admits a positive ground state solution* $u_\infty \in H^1(\mathbb{R}^3)$ *such that* $I'_\infty(u_\infty) = 0$ *and* m_∞ *is attained by* u_∞.

Proof. Firstly, it is easy check that I_∞ possesses a mountain pass geometry. Since the proof is standard, we omit it here. Applying the mountain pass theorem, there exists a sequence $u_n \in H^1(\mathbb{R}^3)$ such that

$$I_\infty(u_n) \to m_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t)), \tag{3.15}$$

where $\Gamma = \{\gamma \in C([0,1],H) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$. And

$$\|I'_\infty(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.16}$$

Then, we have

$$I_\infty(u_n) - \frac{1}{4}(I'_\infty(u_n), u_n) = \frac{1}{4}\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} |u_n|^p dx + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx$$
$$\geq \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\|u_n\|_6^6, \tag{3.17}$$

which yields the boundedness of $\{u_n\}$. Next, we claim that

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx > 0. \tag{3.18}$$

If not, then Lions' concentration compactness principle [[26], Lemma 1.21] implies that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 6)$. Thus, by the Hardy-Littlewood-Sobolev inequality (see [23]) show that

$$\int_{\mathbb{R}^3} \widehat{\phi}_u u^2 dx \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \leq \frac{8\sqrt[3]{\pi}}{3\sqrt[3]{\pi}}\|u\|^4_{\frac{12}{5}} = o(1). \tag{3.19}$$

Then, we have

$$o(1) = (I'_\infty(u_n), u_n) = \|u_n\|^2 - \|u_n\|_6^6 + o(1), \tag{3.20}$$

and

$$m_\infty + o(1) = I_\infty(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{6}\|u_n\|_6^6 + o(1). \tag{3.21}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $m_\infty > 0$, then up to a subsequence, we may assume that

$$\|u_n\|^2 \to l > 0, \quad \|u_n\|_6^6 \to l > 0. \tag{3.22}$$

By (3.22) and the Sobolev inequality, we can see that

$$l = \lim_{n\to\infty} \|u_n\|^2 \geq \lim_{n\to\infty} \|\nabla u_n\|_2^2 \geq \lim_{n\to\infty} S\|u_n\|_6^2 = Sl^{\frac{1}{3}}. \tag{3.23}$$

Then, by Lemma 3.1 and (3.21), we get that

$$\frac{1}{3}S^{\frac{3}{2}} > m_\infty = \frac{1}{3}l \geq \frac{1}{3}S^{\frac{3}{2}}, \tag{3.24}$$

which is a contradiction, so (3.18) holds. Thus there exist $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Set $\bar{u}_n(x) = u_n(x + y_n)$, then

$$I_\infty(\bar{u}_n) \to m_\infty, \quad I'_\infty(\bar{u}_n) \to 0. \tag{3.25}$$

And

$$\int_{B_1(0)} |\bar{u}_n|^2 dx > \delta, \tag{3.26}$$

for all $n \in \mathbb{N}$. Therefore, there exists $u_\infty \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that, passing eventually to a further subsequence,

$$\bar{u}_n \rightharpoonup u_\infty \quad in H^1(\mathbb{R}^3), \tag{3.27}$$

$$\bar{u}_n \to u_\infty \quad in L^r_{loc}(\mathbb{R}^3) \quad r \in [1, 6), \tag{3.28}$$

$$\bar{u}_n \to u_\infty \quad a.e. in \mathbb{R}^3. \tag{3.29}$$

A standard argument shows that $I'_\infty(u_\infty)=0$ and $I_\infty(u_\infty)\ge m_\infty$. Then, by Fatous lemma, we can see that

$$\begin{aligned} m_\infty &= \lim_{n\to\infty}[I_\infty(\bar u_n)-\frac{1}{4}(I'_\infty(\bar u_n),\bar u_n)] \\ &= \lim_{n\to\infty}\{\frac{1}{4}\|\bar u_n\|^2+(\frac{1}{4}-\frac{1}{p})\|\bar u_n\|_p^p+\frac{1}{12}\|\bar u_n\|_6^6\} \\ &\ge \frac{1}{4}\|u_\infty\|^2+(\frac{1}{4}-\frac{1}{p})\|u_\infty\|_p^p+\frac{1}{12}\|u_\infty\|_6^6 \\ &= I_\infty(u_\infty)-\frac{1}{4}(I'_\infty(u_\infty),u_\infty)=I_\infty(u_\infty)\ge m_\infty. \end{aligned} \tag{3.30}$$

Hence, we have $I'_\infty(u_\infty)\to 0$ and $I_\infty(u_\infty)=m_\infty$. Moreover, If u_∞ sign-changing, then $I_\infty(u_\infty)\ge 2m_\infty$, a contradiction. Thus, we may assume $u_\infty\ge 0$ in $H^1(\mathbb{R}^3)$, then the maximum principle implies u_∞ is positive.

□

Lemma 3.3. *For any $\delta>0$, there exists $C_\delta>0$ such that*

$$u_\infty(x)\le C_\delta e^{-(1-\delta)|x|}.$$

Proof. By elliptic estimates(see [22]), we have $u_\infty(x)\in L^\infty(\mathbb{R}^3)$ and

$$u_\infty(x)\to 0,\quad |x|\to\infty,$$

so, for any $\delta>0$, there exists $D_\delta>0$ such that

$$1-u_\infty^{p-2}(x)-u_\infty^4(x)\ge(1-\delta)^2,\quad |x|\ge D_\delta. \tag{3.31}$$

In addition, $u_\infty(x)$ satisfies that

$$-\Delta u_\infty(x)+u_\infty(x)+\hat\phi_{u_\infty(x)}u_\infty(x)=|u_\infty(x)|^{p-2}u_\infty(x)+|u_\infty(x)|^5, \tag{3.32}$$

in view of $\hat\phi_{u_\infty(x)}\ge 0$, $\lim_{|x|\to\infty}u_\infty(x)=0$, (3.31) and (3.32), we can see that

$$-\Delta u_\infty(x)+(1-\delta)^2u_\infty(x)\le 0,\quad |x|\ge D_\delta,$$

and there exists $R_\delta>0$ such that

$$u_\infty(x)\le R_\delta,\quad |x|=D_\delta.$$

Let $v(x)=R_\delta e^{-(1-\delta)(|x|-D_\delta)}$, directly computation, we deduce that

$$-\Delta v(x)+(1-\delta)^2v(x)\ge 0,\quad |x|\ne 0.$$

Therefore the maximum principle implies that $u_\infty(x)\le R_\delta e^{-(1-\delta)(|x|-D_\delta)}$ for $|x|\ge D_\delta$. Hence $u_\infty(x)\le\max\{R_\delta,\|u_\infty\|_\infty\}e^{-(1-\delta)(|x|-D_\delta)}=C_\delta e^{-(1-\delta)|x|}$.

□

Lemma 3.4. *Let $\{u_n\}\subset H^1(\mathbb{R}^3)$ be a bounded sequence such that $I(u_n)\to c\in(0,m_\infty)$ and $I'(u_n)\to 0$. If $m_\infty<\frac{1}{3}S^{\frac{3}{2}}$, then $\{u_n\}$ admits a strongly convergent subsequence in $H^1(\mathbb{R}^3)$.*

Proof. Since the sequence $\{u_n\}$ is bounded, there exists $u\in H^1(\mathbb{R}^3)\backslash\{0\}$ such that

$$u_n\rightharpoonup u\quad in H^1(\mathbb{R}^3),$$

$$u_n\to u\quad in L^s_{loc}(\mathbb{R}^3)\quad(2<s<6),$$

$$u_n\to u\quad a.e. in\mathbb{R}^3.$$

Then $I'(u) = 0$. Set $v_n = u_n - u$. By Brezis-Lieb lemma in [12],

$$\|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o(1), \tag{3.33}$$

$$\|v_n\|_6^6 = \|u_n\|_6^6 - \|u\|_6^6 + o(1), \tag{3.34}$$

Similarly, in view of $\lim_{|x|\to\infty} a(x) = 1$, $\lim_{|x|\to\infty} b(x) = 0$ and $v_n \to 0$ in $L^s_{loc}(\mathbb{R}^3)$(2¡s¡6), we have

$$\int_{\mathbb{R}^3} a(x)|u_n|^p dx - \int_{\mathbb{R}^3} a(x)|u|^p dx = \int_{\mathbb{R}^3} |v_n|^p dx + o(1), \tag{3.35}$$

$$\int_{\mathbb{R}^3} b(x)|u_n|^q dx - \int_{\mathbb{R}^3} b(x)|u|^q dx = o(1). \tag{3.36}$$

By Lemma 5.1 in [27], $\lim_{|x|\to\infty} l(x) = 1$ and Hölder's inequality, it is easy to deduce that

$$\int_{\mathbb{R}^3} \widehat{\phi}_{v_n} v_n^2 dx = \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx + o(1). \tag{3.37}$$

In view of (3.33)-(3.37), we have that

$$c - I(u) = I_\infty(v_n) + o(1). \tag{3.38}$$

By using Proposition 5.1.1 in [22], we see that $u \in L^\infty(\mathbb{R}^3)$. Then by Lemma 8.9 and Lemma 8.1 in [26] with $\lim\limits_{|x|\to\infty} a(x) = 1$, $\lim\limits_{|x|\to\infty} b(x) = 0$, we have that

$$\begin{aligned} |\int_{\mathbb{R}^3} (u_n^5 - u^5 - v_n^5)\varphi dx| &= o(1)\|\varphi\|, \quad \forall \varphi \in H^1(\mathbb{R}^3), \\ |\int_{\mathbb{R}^3} [a(x)(|u_n|^{p-2}u_n - |u|^{p-2}u) - |v_n|^{p-2}v_n]\varphi dx| &= o(1)\|\varphi\|, \quad \forall \varphi \in H^1(\mathbb{R}^3), \\ |\int_{\mathbb{R}^3} b(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)\varphi dx| = o(1) &= o(1)\|\varphi\|, \quad \forall \varphi \in H^1(\mathbb{R}^3). \end{aligned} \tag{3.39}$$

By Lemma 5.1 in [27] and similar to the of proof of (3.37), we can see that

$$|\int_{\mathbb{R}^3} K(x)(\phi_{u_n} u_n - \phi_u u)\varphi dx - \int_{\mathbb{R}^3} \hat{\phi}_{v_n} v_n \varphi dx| = o(1)\|\varphi\|, \quad \forall \varphi \in H^1(\mathbb{R}^3). \tag{3.40}$$

Hence, by (3.39)-(3.40), there holds

$$I'_\infty(v_n) = o(1). \tag{3.41}$$

We claim $v_n \to 0$ in $H^1(\mathbb{R}^3)$. Two cases occur: either

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}_3} \int_{B_1(y)} |v_n|^2 dx = 0,$$

or there exists $\gamma > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}_3} \int_{B_1(y)} |v_n|^2 dx \geq \gamma.$$

Thus, either $\|v_n\|_r \to 0$ for any $r \in (2, 6)$ through using vanishing Lemma, or there $y_n \in \mathbb{R}^3$ with $|y_n| \to \infty$ such that $v_n(. + y_n) \rightharpoonup v \neq 0$ weakly in $H^1(\mathbb{R}^3)$.

If $v_n(.+y_n) \rightharpoonup v \neq 0$ weakly in $H^1(\mathbb{R}^3)$, from (3.38) and (3.41), it follows that $c - I(u) = I_\infty(v_n(.+y_n)) + o(1)$ and $I'_\infty(v_n(.+y_n)) = o(1)$. Thus $I'_\infty(v) = 0$ and

$$\begin{aligned}c - I(u) &= I_\infty(v_n(.+y_n)) - \frac{1}{4}[I'_\infty(v_n(.+y_n)), v_n(.+y_n)]\\ &= \frac{1}{4}\|(v_n(.+y_n)\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3}|(v_n(.+y_n)|^p dx\\ &+ \frac{1}{12}\int_{\mathbb{R}^3}|(v_n(.+y_n)|^6 dx + o(1),\end{aligned}$$

form which we get

$$\begin{aligned}c \geq & I(u) + \frac{1}{4}\|v\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3}|v|^p dx + \frac{1}{12}\int_{\mathbb{R}^3}|v|^6 dx\\ &= I(u) + I_\infty(v) - \frac{1}{4}(I'_\infty(v), v)) = I(u) + I_\infty(v).\end{aligned}$$

By the definition of m_∞, we have $I_\infty(v) \geq m_\infty$. Since $I'(u) = 0$, we have

$$\begin{aligned}I(u) &= I(u) - \frac{1}{4}(I'(u), u)\\ &= \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} a(x)|u|^p dx\\ &+ (\frac{1}{4} - \frac{1}{q})\mu\int_{\mathbb{R}^3} b(x)|u|^q dx + \frac{1}{12}\int_{\mathbb{R}^3}|u|^6 dx \geq 0\end{aligned}$$

which leads to a contradiction with $c < m_\infty$. Thus $\|v_n\|_{L^r} \to 0$ for any $r \in (2,6)$. By (3.38) and (3.41), we have

$$\begin{aligned}&c - I(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{6}\|v_n\|_6^6 + o(1),\\ &\|v_n\|^2 - \|v_n\|_6^6 = o(1).\end{aligned}$$

Up to a subsequence, we may assume that $\|v_n\|^2 \to l$. Thus $\|v_n\|_6^6 \to l$. If $l > 0$, by the definition of S, we get $l \geq S^{\frac{3}{2}}$. Hence,

$$c = I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{6}\|v_n\|_6^6 = I(u) + \frac{1}{3}l \geq \frac{s}{3}S^{\frac{3}{2}},$$

which contradicts with $c < m_\infty < \frac{1}{3}S^{\frac{3}{2}}$. Thus $l = 0$ and we complete the proof. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1-(A). Let $\{u_n\} \subset M$ be a minimizing sequence for functional I, that is, $\{u_n\} \subset M$ and $I(u_n) \to m$, where

$$M = \{u \in H\backslash\{0\} : G(u) = (I'(u), u) = 0\}.$$

We claim $I'(u_n) \to 0$. By the Lagrange multiplier Theorem, there exists $\lambda_n \in \mathbb{R}$ such that

$$I'(u_n) - \lambda_n G'(u_n) \to 0.$$

Since $u_n \subset M$, we have that

$$m + o(1) = I(u_n) - \frac{1}{4}(I'(u_n), u_n) \geq \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3$. Hence

$$\lambda_n(G'(u_n),u_n)\to 0. \tag{4.1}$$

By $(a_1^{\star})$ and $(b_1^{\star})$, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$a(x)|u|^p+b(x)|u|^q+|u|^6\le \varepsilon|u|^2+C_\varepsilon|u|^6.$$

Taking $\varepsilon=\frac{1}{2}$ and recalling the definition of S, we have that

$$\begin{aligned}\|u_n\|^2&\le\int_{\mathbb{R}^3}a(x)|u_n|^pdx+\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx+\int_{\mathbb{R}^3}|u_n|^6dx\\&\le\frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2dx+C_{\frac{1}{2}}\int_{\mathbb{R}^3}|u_n|^6dx\\&\le\frac{1}{2}\int_{\mathbb{R}^3}|u_n|^2dx+C_{\frac{1}{2}}\frac{\|u_n\|^6}{S^3},\end{aligned}$$

which implies that

$$\|u_n\|^2\ge\frac{S^{\frac{3}{2}}}{(2C_{\frac{1}{2}})^{\frac{1}{2}}}. \tag{4.2}$$

By (4.2), we get

$$\begin{aligned}(G'(u_n),u_n)&=(G'(u_n),u_n)-4(I'(u_n),u_n)\\&=2\|u_n\|^2+4\int_{\mathbb{R}^3}l(x)\phi_{u_n}u_n^2dx-p\int_{\mathbb{R}^3}a(x)|u_n|^pdx-q\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx\\&-6\int_{\mathbb{R}^3}|u_n|^6dx-4[\|u_n\|^2+\int_{\mathbb{R}^3}l(x)\phi_{u_n}u_n^2dx-\int_{\mathbb{R}^3}a(x)|u_n|^pdx\\&-\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx-\int_{\mathbb{R}^3}|u_n|^6dx]=-2\|u_n\|^2+(4-p)\int_{\mathbb{R}^3}a(x)|u_n|^pdx\\&(4-q)\int_{\mathbb{R}^3}\mu b(x)|u_n|^qdx+(-2)\int_{\mathbb{R}^3}|u_n|^6dx\le-2\|u_n\|^2\le-2\frac{S^{\frac{3}{2}}}{(2C_{\frac{1}{2}})^{\frac{1}{2}}}.\end{aligned}$$

From (4.1), we have $\lambda_n\to 0$. Thus $I'(u_n)\to 0$. This means that $\{u_n\}$ is a $(PS)_m$ sequence for I, that is, $I(u_n)\to m$ and $I'(u_n)\to 0$. By Lemma 3.4, if $m\in(0,m_\infty)$, then $u_n\to u$ in $H^1(\mathbb{R}^3)$ and thus $I(u)=m$ and $I'(u)=0$. Hence, m is attained by $u\in H^1(\mathbb{R}^3)\backslash\{0\}$. For this purpose, it is sufficient to prove $m<m_\infty$.

Similar argument as (3.7), we can obtain the equivalent characterization of the least energy m:

$$m=\inf_{u\in H\backslash\{0\}}\max_{t\ge 0}I(tu). \tag{4.3}$$

Let $R>0$ and $\gamma\in\mathbb{R}^3$ with $|\gamma|=1$. By (4.3), clearly, we have

$$m\le\sup_{t\ge 0}I(tu_\infty(x-R\gamma)),$$

where u_∞ is a positive ground state solution for limit problem (3.2). Since

$$\begin{aligned}I(tu_\infty(x-R\gamma))&\le\frac{t^2}{2}\|u_\infty(x-R\gamma)\|^2+Ct^4\|u_\infty(x-R\gamma)\|^4-\frac{t^6}{6}\|u_\infty(x-R\gamma)\|_6^6\\&=\frac{t^2}{2}\|u_\infty\|^2+Ct^4\|u_\infty\|^4-\frac{t^6}{6}\|u_\infty\|_6^6,\end{aligned}$$

there exist a small $t' > 0$ and a large $t'' > 0$ independent of R and γ such that

$$\sup_{t\in[0,t']\cup[t'',+\infty)} I(tu_\infty(x-R\gamma)) < m_\infty. \tag{4.4}$$

On the other hand, by $(b_1^\star)$, for any $u \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} I(tu) &\le I_\infty(tu) + \frac{t^4}{4}\int_{\mathbb{R}^3}(l(x)-1)\phi_u u^2 dx - \frac{1}{p}t^p\int_{\mathbb{R}^3}(a(x)-1)|u|^p dx \\ &+ \frac{t^4}{4}\int_{\mathbb{R}^3}(\phi_u - \hat{\phi}_u)u^2\,\mathrm{d}x \\ &= I_\infty(tu) + \frac{t^4}{4}\int_{\mathbb{R}^3}(l(x)-1)\phi_u u^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3}(l(x)-1)\hat{\phi}_u u^2\,\mathrm{d}x \\ &- \frac{1}{p}t^p\int_{\mathbb{R}^3}(a(x)-1)|u|^p dx. \end{aligned}$$

Thus, choosing $u = u_\infty(x-R\gamma)$ in the inequality above and using $(K_2^\star)$, $(a_1^\star)$ and Lemma 3.3 we get

$$\begin{aligned} I(tu_\infty(x-R\gamma)) &\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\phi_{u_\infty}|u_\infty(x)|^2 dx \\ &- \frac{t^4}{4}C_0\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2 dx \\ &+ \frac{1}{p}t^p C_2\int_{\mathbb{R}^3} e^{-m|x+R\gamma|}|u_\infty(x)|^p dx \\ &\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\phi_{u_\infty}|u_\infty(x)|^2 dx \\ &- \frac{t^4}{4}C_0\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2 dx \\ &+ \frac{1}{p}t^p C_2 C_\sigma^p\int_{\mathbb{R}^3} e^{-mR}e^{m|x|-p(1-\delta)|x|}dx \\ &\le I_\infty(tu_\infty) - \frac{t^4}{4}C_0\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2 dx \\ &+ \frac{1}{p}t^p C_2 C_\sigma^p\int_{\mathbb{R}^3} e^{-mR}e^{m|x|-p(1-\delta)|x|}dx. \end{aligned}$$

Set $l(t) = I_\infty(tu_\infty)$, $t \in (0,\infty)$. It is easy to verify that $\sup\limits_{t\ge 0} l(t) = I_\infty(u_\infty) = m_\infty$. Let $\delta_0 \in (0, 1-\frac{m}{p})$, we have

$$\int_{\mathbb{R}^3} e^{-mR}e^{m|x|-p(1-\delta)|x|}dx \le Ce^{-mR}.$$

In addition,

$$\int_{\mathbb{R}^3} e^{-l|x+R\gamma|}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2 dx \ge e^{-lR}\int_{|x|\le 1} e^{-l|x|}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2 dx \ge Ce^{-lR}.$$

Hence, we have

$$\sup_{t'\le t\le t''} I(tu_\infty(x-R\gamma)) \le m_\infty - \frac{(t')^4}{4}Ce^{-lR} + \frac{1}{p}(t'')^p Ce^{-mR},$$

where C represent different constants. Since $0 < l < m$, so there exists $\hat{R} > 0$ such that $R > \hat{R}$, we get

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\gamma)) < m_\infty, \quad \forall\, |\gamma| = 1.$$

which yields $m < m_\infty$. Thus, the proof of Theorem 1.1-(**A**) is completed.

Proof of Theorem 1.1-(B). Similar to the argument of Theorem 1.1-(**A**), we only need to prove for $R > 0$ large, $\sup_{t\ge 0} I(tu_\infty(x - R\gamma)) < m_\infty$ uniformly in γ. Clearly, there exist $0 < t' < t''$ independent of R and γ such that

$$\sup_{t\in[0,t']\cup[t'',+\infty)} I(tu_\infty(x - R\gamma)) < m_\infty.$$

On the other hand, by (K_3), (a_2), (b_2), Lemma 3.3 and Hölder's inequality, we have for any $\sigma > 0$, there exist $C_\sigma > 0$ such that

$$\begin{aligned}
&\sup_{t\in[t',t'']} I(tu_\infty(x - R\gamma))\\
&\le \sup_{t\ge 0} I_\infty(tu_\infty) + \frac{C_1(t'')^4}{4}\int_{\mathbb{R}^3} e^{-d|x+R\gamma|}\phi_{u_\infty}|u_\infty(x)|^2dx\\
&+ \frac{C_1(t'')^4}{4}\int_{\mathbb{R}^3} e^{-d|x+R\gamma|}\hat{\phi}_{u_\infty}|u_\infty(x)|^2dx\\
&+ \frac{1}{p}(t'')^pC_2\int_{\mathbb{R}^3} e^{-m|x+R\gamma|}|u_\infty(x)|^pdx\\
&- \mu\frac{C_3(t')^q}{q}\int_{\mathbb{R}^3} e^{-b|x+R\gamma|}|u_\infty(x)|^qdx\\
&\le \sup_{t\ge 0} I_\infty(tu_\infty) - \mu\frac{C_3(t')^q}{q}\int_{\mathbb{R}^3} e^{-bR}e^{-b|x|}|u_\infty(x)|^qdx\\
&+ \frac{C_1C_\delta^2(t'')^4}{4}(\|\phi_{u_\infty}\|_6 + \|\hat{\phi}_{u_\infty}\|_6)\int_{\mathbb{R}^3} e^{-\frac{6}{5}dR}e^{\frac{6}{5}d|x|-\frac{12}{5}(1-\delta)|x|}dx\\
&+ \frac{C_2C_\delta^p(t'')^p}{p}\int_{\mathbb{R}^3} e^{-mR}e^{m|x|-p(1-\delta)|x|}dx.
\end{aligned}$$

Let $\delta \in (0, \min\{1 - \frac{m}{p}, 1 - \frac{d}{2}\})$, then

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\gamma)) < m_\infty - Ce^{-bR} + Ce^{-dR} + Ce^{-mR},$$

where C represent different constants. Since $b < \min\{m, d\}$, so there exists $\hat{R} > 0$ such that $R > \hat{R}$, we get

$$\sup_{t' \le t \le t''} I(tu_\infty(x - R\gamma)) < m_\infty, \quad \forall\, |\gamma| = 1.$$

which yields $m < m_\infty$. Thus, the proof of Theorem 1.1-(**B**) is completed.

5. Proof of Theorem 1.2

In the next, we consider the existence of multiple solutions of problem (1.3).

Let $h(t) = I(tu_\infty(x - R\gamma))$, $t \in (0, \infty)$, $\gamma \in \mathbb{R}^3$ with $|\gamma| = 1$. Form the proof of Theorem 1.1-(**B**), we know there exists $R_0 > 0$ such that for $R > R_0$, there exists $\varepsilon(R) > 0$ satisfying

$$\sup_{t\ge 0} h(t) \le m_\infty - \varepsilon(R) < m_\infty \quad \text{uniformly in } \gamma.$$

For any fixing R and γ, it is easy to check that $h(t)$ attains its maximum at a unique point $t = t_\infty$. Hence, we define a mapping $F_R : S^2 = \{\gamma \in \mathbb{R}^3 : |\gamma| = 1\} \to M$ by

$$F_R(\gamma) = t_\infty u_\infty(x - R\gamma).$$

Immediately we have the following Lemma.

Lemma 5.1. *There exists $R_0 > 0$ such that for $R > R_0$, there exists $\varepsilon(R) > 0$ satisfying $F_R(S^2) \subset \{u \in M : I(u) \le m_\infty - \varepsilon(R)\}$ uniformly in $\gamma \in S^2$.*

For $u \in H$, define a map $\Phi : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$:

$$\Phi(u)(x) := \frac{1}{|B_1(x)|} \int_{B_1(x)} |u(y)| dy, \quad \forall x \in \mathbb{R}^3,$$

where $|B_1(x)|$ is the Lebesgue measure of $B_1(x)$. Let

$$\hat{u}(x) = [\Phi(u)(x) - \frac{1}{2} \max_{x\in\mathbb{R}^3} \Phi(u)(x)]^+,$$

and $\beta : H^1(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}^3$ given by

$$\beta(u) = \frac{1}{\|\hat{u}\|_1} \int_{\mathbb{R}^3} x\hat{u}(x)dx.$$

Obviously, $\beta(u)$ is well defined for all $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and $\beta(u)$ has a compact support in $\mathbb{R}^3$. Moreover, $\beta(u)$ is continuous in $H^1(\mathbb{R}^3)\backslash\{0\}$ and satisfies the following properties.

Lemma 5.2. *(i) For any $t \neq 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, $\beta(tu) = \beta(u)$.*
(ii) For any $z \in \mathbb{R}^3$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, $\beta(u(x - z)) = \beta(u) + z$.

Define a functional $J : H^1(\mathbb{R}^3) \to \mathbb{R}$ given as follows

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} a(x)|u|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx, \quad u \in H^1(\mathbb{R}^3).$$

Lemma 5.3. *$m_0 := \inf_{u\in M_0} J(u) = m_\infty$ is not attained, where*

$$M_0 = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : (J'(u), u) = 0\}.$$

Proof. Firstly we show that for any $u \in M_0$, there exists a unique $0 < \tau \le 1$ such that $\tau u \in M_\infty$. Indeed, by virtue of $u \in M_0$ and $\tau u \in M_\infty$, we have

$$\|u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx = \int_{\mathbb{R}^3} a(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^6 dx, \tag{5.1}$$

and then

$$\begin{aligned} &\tau^p \int_{\mathbb{R}^3} a(x)|u|^p dx + \tau^6 \int_{\mathbb{R}^3} |u|^6 dx \\ &\le \tau^p \int_{\mathbb{R}^3} |u|^p dx + \tau^6 \int_{\mathbb{R}^3} |u|^6 dx = \tau^2\|u\|^2 + \tau^4 \int_{\mathbb{R}^3} \hat{\phi}_u u^2 dx. \end{aligned} \tag{5.2}$$

From $(l_3^\star)$ and $l(x) \ge 1$ for any $x \in \mathbb{R}^3$, it follows that

$$\int_{\mathbb{R}^3} \hat{\phi}_u u^2 dx \le \int_{\mathbb{R}^3} l(x)\hat{\phi}_u u^2 dx \le \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx. \tag{5.3}$$

If $\tau > 1$, by (5.1), (5.2) and (5.3), we deduce that

$$\begin{aligned}&\tau^4(\|u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx) \geq \tau^4(\|u\|^2 + \int_{\mathbb{R}^3} \hat{\phi}_u u^2 dx)\\ &\geq \tau^p(\int_{\mathbb{R}^3} a(x)|u|^p dx + \int_{\mathbb{R}^3} |u|^6 dx) = \tau^p(\|u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx),\end{aligned}$$

which yields $\tau \leq 1$, this achieves a contradiction. Hence $\tau \leq 1$ and the claim holds true.

For $u \in M_0$, using (5.3), we have that

$$\begin{aligned}J(u) &= J(u) - \frac{1}{p}(J'(u), u)\\ &= (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx + (\frac{1}{p} - \frac{1}{6})\int_{\mathbb{R}^3} |u|^6 dx\\ &\geq (\frac{1}{2} - \frac{1}{p})\|\tau u\|^2 + (\frac{1}{4} - \frac{1}{p})\tau^4 \int_{\mathbb{R}^3} \hat{\phi}_u u^2 dx + (\frac{1}{p} - \frac{1}{6})\int_{\mathbb{R}^3} |\tau u|^6 dx\\ &= I_\infty(\tau u) - \frac{1}{p}(I'_\infty(\tau u), \tau u) = I_\infty(\tau u) \geq m_\infty,\end{aligned}$$

which implies that $m_0 \geq m_\infty$.

Next we prove $m_0 \leq m_\infty$. Let $w_n = u_\infty(. - z_n)$, where $z_n \in \mathbb{R}^3$ with $|z_n| \to \infty$. We claim that for $w_n \in M_\infty$, there exists $\tau_n \geq 1$ such that $\tau_n w_n \in M_0$. In fact, from $w_n \in M_\infty$ and $\tau_n w_n \in M_0$, there holds

$$\|w_n\|^2 + \int_{\mathbb{R}^3} \hat{\phi}_{w_n} w_n^2 dx = \int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^6 dx,$$

and then

$$\begin{aligned}&\tau_n^p \int_{\mathbb{R}^3} |w_n|^p dx + \tau_n^6 \int_{\mathbb{R}^3} |w_n|^6 dx\\ &\geq \tau_n^p \int_{\mathbb{R}^3} a(x)|w_n|^p dx + \tau_n^6 \int_{\mathbb{R}^3} |w_n|^6 dx\\ &= \tau_n^2\|w_n\|^2 + \tau_n^4 \int_{\mathbb{R}^3} l(x)\phi_{w_n} w_n^2 dx.\end{aligned}$$

If $\tau_n < 1$, we have

$$\begin{aligned}&\tau_n^p(\int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^6 dx) \geq \tau_n^4(\|w_n\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{w_n} w_n^2 dx)\\ &\geq \tau_n^4(\|w_n\|^2 + \int_{\mathbb{R}^3} \hat{\phi}_{w_n} w_n^2 dx) = \tau_n^4(\int_{\mathbb{R}^3} |w_n|^p dx + \int_{\mathbb{R}^3} |w_n|^6 dx),\end{aligned}$$

which leads to a contradiction with $\tau_n < 1$. Hence $\tau_n \geq 1$ and the claim holds true.

By the definition of m_0 and $\tau_n u_n \in M_0$, we have

$$\begin{aligned}m_0 \leq J(\tau_n w_n) = &\frac{1}{2}\tau_n^2\|u_\infty\|^2 + \frac{1}{4}\tau_n^4 \int_{\mathbb{R}^3} l(x)\phi_{w_n} w_n^2 dx\\ &- \frac{1}{p}\tau_n^p \int_{\mathbb{R}^3} a(x)|w_n|^p dx - \frac{1}{6}\tau_n^6 \int_{\mathbb{R}^3} |u_\infty(x)|^6 dx.\end{aligned}$$

By Lebesgue dominated convergence Theorem, we deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^3} l(x)\phi_{w_n}w_n^2dx &= \lim_{n\to\infty}\int_{\mathbb{R}^3} l(x+z_n)\phi_{u_\infty}|u_\infty(x)|^2dx\\ &= \int_{\mathbb{R}^3}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2dx,\\ \lim_{n\to\infty}\int_{\mathbb{R}^3} a(x)|w_n|^pdx &= \lim_{n\to\infty}\int_{\mathbb{R}^3} a(x+z_n)|u_\infty(x)|^pdx\\ &= \int_{\mathbb{R}^3}|u_\infty(x)|^pdx.\end{aligned} \tag{5.4}$$

If $\tau_n\to 1$, we get $m_0\le \lim_{n\to\infty} J(t_nw_n) = I_\infty(u_\infty) = m_\infty$, form which we see that $m_0 = m_\infty$. Thus we only need to prove $\tau_n\to 1$. By $\tau_nw_n\in M_0$, with $\tau_n\ge 1$, we have

$$\begin{aligned}&\tau_n^4(\|w_n\|^2+\int_{\mathbb{R}^3} l(x)\phi_{w_n}w_n^2dx)\\ &\ge \tau_n^2\|w_n\|^2+\tau_n^4\int_{\mathbb{R}^3} l(x)\phi_{w_n}w_n^2dx = \tau_n^p\int_{\mathbb{R}^3} a(x)|w_n|^pdx+\tau_n^6\int_{\mathbb{R}^3}|w_n|^6dx\\ &\ge \tau_n^p(\int_{\mathbb{R}^3} a(x)|w_n|^pdx+\int_{\mathbb{R}^3}|w_n|^6dx).\end{aligned}$$

Thus, by (5.4), we deduce that

$$1\le \tau_n^{p-4}\le \frac{\|w_n\|^2+\int_{\mathbb{R}^3} l(x)\phi_{w_n}w_n^2dx}{\int_{\mathbb{R}^3} a(x)|w_n|^pdx+\int_{\mathbb{R}^3}|w_n|^6dx} = \frac{\|u_\infty\|^2+\int_{\mathbb{R}^3}\hat{\phi}_{u_\infty}(x)|u_\infty(x)|^2dx+o(1)}{\int_{\mathbb{R}^3}|u_\infty(x)|^pdx+o(1)+\int_{\mathbb{R}^3}|u_\infty(x)|^6dx},$$

which yields $\tau_n\to 1$ by using $u_\infty\in M_\infty$.

Next we prove m_0 is not attained. Assume by contradiction that there exists $u_0\in M_0$ such that $m_0 = J(u_0)$. We claim $J'(u_0) = 0$. Set $\widetilde{G}(u) = (J'(u), u)$, By the Lagrange multipliers Theorem, we obtain $\lambda\in\mathbb{R}$ such that $J'(u_0)-\lambda\widetilde{G}'(u_0)\to 0$, similar to the of proof of Theorem 1.1-(**B**), we have $J'(u_0) = 0$. Note that if u_0 is sing-changing, we see that $J(u_0)\ge 2m_0$, a contradiction. Thus we may assume that $u_0\ge 0$ in $H^1(\mathbb{R}^3)$ and the maximum principle implies that u_0 is positive.

From the above proof, we see that for $u_0\in M_0$, there exists a unique $\tau_0\le 1$ such that $\tau_0u_0\in M_\infty$. Thus,

$$\begin{aligned}m_\infty &\le I_\infty(\tau_0u_0)\\ &= I_\infty(\tau_0u_0)-\frac{1}{p}(I'_\infty(\tau_0u_0),\tau_0u_0)\\ &= (\frac{1}{2}-\frac{1}{p})\|\tau_0u_0\|^2+(\frac{1}{4}-\frac{1}{p})\tau_0^4\int_{\mathbb{R}^3}\hat{\phi}_{u_0}u_0^2dx+(\frac{1}{p}-\frac{1}{6})\int_{\mathbb{R}^3}|\tau_0u_0|^6dx\\ &\le (\frac{1}{2}-\frac{1}{p})\|u_0\|^2+(\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3} l(x)\phi_{u_0}^tu_0^2dx+(\frac{1}{p}-\frac{1}{6})\int_{\mathbb{R}^3}|u_0|^6dx\\ &= J(u_0)-\frac{1}{p}(J'(u_0),u_0) = J(u_0) = m_0.\end{aligned}$$

From $m_0 = m_\infty$, it follows that

$$(\frac{1}{2}-\frac{1}{p})\|\tau_0 u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\tau_0^4\int_{\mathbb{R}^3}\hat{\phi}_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{6})\int_{\mathbb{R}^3}|\tau_0 u_0|^6dx$$
$$= (\frac{1}{2}-\frac{1}{p})\|u_0\|^2 + (\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3}l(x)\phi_{u_0}u_0^2dx + (\frac{1}{p}-\frac{1}{6})\int_{\mathbb{R}^3}|u_0|^6dx,$$

that is

$$\tau_0^2\|u_0\|^2 + \tau_0^4\int_{\mathbb{R}^3}\hat{\phi}_{u_0}u_0^2dx + \tau_0^6\int_{\mathbb{R}^3}|u_0|^6dx$$
$$= \|u_0\|^2 + \int_{\mathbb{R}^3}l(x)\phi_{u_0}u_0^2dx + \int_{\mathbb{R}^3}|u_0|^6dx.$$

Thus

$$(1-\tau_0^2)\|u_0\|^2 + \int_{\mathbb{R}^3}(l(x)-1)\phi_{u_0}u_0^2dx + \int_{\mathbb{R}^3}(\phi_{u_0}-\hat{\phi}_{u_0})u_0^2dx$$
$$+ (1-\tau_0^4)\int_{\mathbb{R}^3}\hat{\phi}_{u_0}u_0^2dx + (1-\tau_0^6)\int_{\mathbb{R}^3}|u_0|^6dx = (1-\tau_0^2)\|u_0\|^2$$
$$+ \int_{\mathbb{R}^3}(l(x)-1)(\phi_{u_0}+\hat{\phi}_{u_0})u_0^2dx + (1-\tau_0^4)\int_{\mathbb{R}^3}\hat{\phi}_{u_0}u_0^2dx$$
$$+ (1-\tau_0^6)\int_{\mathbb{R}^3}|u_0|^6dx = 0,$$

by $\tau_0 \leq 1$, so

$$\int_{\mathbb{R}^3}(l(x)-1)(\phi_{u_0}+\hat{\phi}_{u_0})u_0^2dx = 0,$$

this is a contradiction with u_0 is positive, $l(x) \geq 1$ and meas$\{x \in \mathbb{R}^3 : l(x) > 1\} > 0$. □

Lemma 5.4. *There exists $\rho_0 > 0$ such that for $u \in M_0$ satisfying $J(u) \leq m_\infty + \rho_0$, there holds $|\beta(u)| > 0$.*

Proof. Assume by the contrary that there exists $\{u_n\} \subset M_0$ such that $J(u_n) \to m_\infty = m_0$ and $|\beta(u)| = 0$. Similar to the proof Theorem 1.1, we can derive by the Lagrange multipliers Theorem that $J'(u_n) \to 0$. We omit the proof here. Similar to the proof Lemma 3.4, we obtain $u_n \rightharpoonup u$ weakly in H, $J'(u) = 0$, and

$$m_\infty - J(u) = I_\infty(v_n) + o(1) \text{ and } I'_\infty(v_n) = o(1), \tag{5.5}$$

where $v_n = u_n - u$.

For the sequence $\{v_n\}$, two cases may occur: $\|v_n\|_r \to 0$ for any $r \in (2,6)$, or there $y_n \in \mathbb{R}^3$ with $|y_n| \to \infty$ such that $v_n(.+y_n) \rightharpoonup v \neq 0$ weakly in $H^1(\mathbb{R}^3)$. By virtue of $J'(u) = 0$, we can deduce that $J(u) \geq 0$. From Lemma 3.4, we see that $m_\infty < \frac{1}{3}S^{\frac{3}{2}}$. Thus $m_\infty - J(u) < \frac{1}{3}S^{\frac{3}{2}}$.

If $\|v_n\|_r \to 0$ for any $r \in (2,6)$, by (3.33), we have $m_\infty - J(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{6}\|v_n\|_6^6 + o(1)$ and $\|v_n\|^2 - \|v_n\|_6^6 = o(1)$. Up to a subsequence, we may assume that $\|v_n\|^2 \to l$ and then $\|v_n\|_6^6 \to l$. If $l > 0$, by the definition of S, we get $l \geq S^{\frac{3}{2}}$. So $m_\infty - J(u) = \frac{1}{2}\|v_n\|^2 - \frac{1}{6}\|v_n\|_6^6 = \frac{1}{3}l \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction with $m_\infty - J(u) < \frac{1}{3}S^{\frac{3}{2}}$. Thus, $l = 0$ and then $u_n \to u$ in $H^1(\mathbb{R}^3)$, we get $m_0 = J(u)$, a contradiction with m_0 is not attained. Therefore, $v_n(.+y_n) \rightharpoonup v \neq 0$ weakly in $H^1(\mathbb{R}^3)$. Similar to the proof Lemma 3.4, we can deduce that

$$\begin{aligned} &m_\infty - J(u) = I_\infty(v_n(.+y_n)) + o(1), \\ &I'_\infty(v_n(.+y_n)) = o(1). \end{aligned}$$

Hence, $I'_\infty(v) = 0$ and by using Fatou's Lemma, we have that

$$\begin{aligned} m_\infty - J(u) &= I_\infty(v_n(.+y_n)) - \frac{1}{4}[I'_\infty(v_n(.+y_n)), v_n(.+y_n)] + o(1) \\ &= \frac{1}{4}\|(v_n(.+y_n)\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} |(v_n(.+y_n)|^p dx \\ &+ \frac{1}{12}\int_{\mathbb{R}^3} |(v_n(.+y_n)|^6 dx + o(1) \\ &\geq \frac{1}{4}\|v\|^2 + (\frac{1}{4} - \frac{1}{p})\int_{\mathbb{R}^3} |v|^p dx + \frac{1}{12}\int_{\mathbb{R}^3} |v|^6 dx \\ &= I_\infty(v) - \frac{1}{4}(I'_\infty(v), v)) = I_\infty(v) \geq m_\infty. \end{aligned}$$

Combining with $J(u) \geq 0$, we get $J(u) = 0$ and then $v_n(.+y_n) = u_n(.+y_n) \to v$ in $H^1(\mathbb{R}^3)$. By Lemma 5.2, we have

$$\beta(v(x)) + o(1) = \beta(u_n(x+y_n)) = \beta(u_n) - y_n = -y_n.$$

Which yields $|\beta(v(x))| = \infty$, this leads to a contradiction. □

Lemma 5.5. *There exists $\mu_0 > 0$ small such that for $\mu \in (0, \mu_0)$, we have $|\beta(u)| > 0$ for $u \in \{u \in M : I(u) < m_\infty\}$.*

Proof. Let $u \in M$ be such that $I(u) < m_\infty$, then we have

$$m_\infty > I(u) = I(u) - \frac{1}{4}(I'(u), u) \geq \frac{1}{4}\|u\|^2. \tag{5.6}$$

Using the conditions $(a_1^\star)$, $(b_1^\star)$ and $u \in M$, we get for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \|u\|^2 &\leq \int_{\mathbb{R}^3} a(x)|u|^p dx + \mu\int_{\mathbb{R}^3} b(x)|u|^q dx + \int_{\mathbb{R}^3} |u|^6 dx \\ &\leq (1+\mu)\Big[\varepsilon\int_{\mathbb{R}^3} |u|^2 dx + C_\varepsilon\int_{\mathbb{R}^3} |u|^6 dx\Big]. \end{aligned} \tag{5.7}$$

Choose $\varepsilon \in (0, \frac{1}{4})$, we have $\frac{1}{2}\|u\|^2 \leq (1+\mu)C_\varepsilon \int_{\mathbb{R}^3} |u|^6 dx$ for $\mu \in (0,1)$. In fact, if $\varepsilon \in (0, \frac{1}{4})$ and $\mu \in (0,1)$, we get $0 < (1+\mu)\varepsilon < \frac{1}{2}$, and then $\frac{1}{2} - (1+\mu)\varepsilon > 0$. Thus, there holds true

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + (\frac{1}{2} - (1+\mu)\varepsilon)\int_{\mathbb{R}^3} |u|^2 dx \geq 0,$$

that is

$$\frac{1}{2}\|u\|^2 \leq \|u\|^2 - (1+\mu)\varepsilon\int_{\mathbb{R}^3} |u|^2 dx,$$

by (5.7), we have

$$\|u\|^2 - (1+\mu)\varepsilon\int_{\mathbb{R}^3} |u|^2 dx \leq (1+\mu)C_\varepsilon\int_{\mathbb{R}^3} |u|^6 dx,$$

so

$$\frac{1}{2}\|u\|^2 \le (1+\mu)C_\varepsilon \int_{\mathbb{R}^3} |u|^6 dx.$$

Thus, by the definition of S, there exists $L_0 > 0$ independent of $\mu \in (0,1)$ such that

$$\int_{\mathbb{R}^3} |u|^6 dx \ge \frac{L_0}{(1+\mu)^{\frac{3}{2}}}. \tag{5.8}$$

Similar to the argument of Lemma 5.3, we can deduce that for any $u \in M$, there exists a unique $\tau(u) \ge 1$ such that $\tau(u)u \in M_0$. Then

$$\begin{aligned}&\tau^4(u)(\|u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx)\\ &\ge \tau^2(u)\|u\|^2 + \tau^4(u)\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx = \tau^p(u)\int_{\mathbb{R}^3} a(x)|u|^p dx + \tau^6(u)\int_{\mathbb{R}^3}|u|^6 dx\\ &\ge \tau^6(u)\int_{\mathbb{R}^3}|u|^6 dx,\end{aligned}$$

which implies that $\tau^2(u) \le \frac{\|u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_u^t u^2 dx}{\int_{\mathbb{R}^3}|u|^6 dx}$. Together with (5.6) and (5.8), we derive there exists $\overline{C} > 0$ independent of $\mu \in (0,1)$ such that

$$1 \le \tau^2(u) \le \overline{C}(1+\mu)^{\frac{3}{2}}. \tag{5.9}$$

Note that for $u \in M$ with $I(u) < m_\infty$, thus

$$m_\infty > I(u) = \sup_{t\ge 0} I(tu) \ge I(t(u)u) = J(t(u)u) - \mu\frac{t^q(u)}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx.$$

By (5.6) and (5.9), there exists a small $\mu_0 \in (0,1)$ such that $\mu \in (0,\mu_0)$,

$$J(t(u)u) < m_\infty + \mu\frac{t^q(u)}{q}\int_{\mathbb{R}^3} b(x)|u|^q dx \le m_\infty + \rho_0.$$

Form Lemma 5.4, we have $|\beta(t(u)u)| > 0$. Hence, Lemma 5.2 implies that $|\beta(u)| > 0$. □

Lemma 5.6. *For $\mu \in (0,\mu_0)$, define $G : \{u \in M : I(u) < m_\infty\} \to S^2$ by $G(u) = \frac{\beta(u)}{|\beta(u)|}$. Then for $R > R_0$ and $\mu \in (0,\mu_0)$, the map*

$$G \circ F_R : S^2 \to S^2; y \to G\circ(F_R(y))$$

is homotopic to the identity.

Proof. Similar to the argument of Proposition 2.9 in [1], define the map $\zeta(\theta,y) : [0,1]\times S^2 \to S^2$ by

$$\zeta(\theta,y) = \begin{cases} G((1-2\theta)F_R(y) + 2\theta u_\infty(x-Ry)), & \theta \in [0,\frac{1}{2}),\\ G(u_\infty(x-\frac{R}{2(1-\theta)}y), & \theta\in[\frac{1}{2},1),\\ y, & \theta = 1.\end{cases}$$

By the definition of G and Lemma 2.6, it is not difficult to check that $\zeta(\theta,y) \in C([0,1]\times S^2, S^2)$, $\zeta(0,y) = G\circ(F_R(y))$ for $y\in S^2$ and $\zeta(1,y) = y$ for $y \in S^2$. The proof is completed. □

Proof of Theorem 1.2. Form Lemma 2.6, Lemma 5.1 and Lemma 5.6, we have that for $R > R_0$ and $\mu \in (0, \mu_0)$, there holds

$$\mathrm{cat}(\{u \in M : I(u) \leq m_\infty - \varepsilon(R)\}) \geq 2.$$

Then by Lemma 3.4 and Lemma 2.5, we see that I admits at least two nontrivial critical point in $\{u \in M : I(u) < m_\infty\}$.

References

[1] S. Adachi and K. Tanaka, Four positive solutions for the semilinear elliptic equation:$-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations., **11** (2000), 63–95.

[2] F. Bopp, Eine lineare theorie des elektrons, Ann. Phys., **430** (1940), 345–384.

[3] M. Born, Modified field equations with a finite radius of the electron, Nature., **132** (1933), 282.

[4] M. Born, On the quantum theory of the electromagnetic field, Proc. R. Soc. Lond. Ser., **143** (1934), 410–437.

[5] M. Born and L. Infeld, Foundations of the new field theory, Nature., **143** (1933), 1004.

[6] M. Born and L. Infeld, Foundations of the new field theory, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng., **144** (1934), 425–451.

[7] C.A. Bonin, B.M. Pimentel and P.H. Ortega, Multipole Expansion in Generalized Electrodynamics, preprint, arXiv: 1608.00902.

[8] M.C. Bertin, B.M. Pimentel, C.E. Valc*á*rcel and G.E.R. Zambrano, Hamilton-Jacobi formalism for Podolsky's electromagnetic theory on the null-plane, J. Math. Phys., **58** (2017), 082902.

[9] R. Bufalo, B.M. Pimentel and D.E. Soto, Causal approach for the electronCpositron scattering in generalized quantum electrodynamics, Phys. Rev. D., **90** (2014), 085012.

[10] R. Bufalo, B.M. Pimentel and D.E. Soto, Normalizability analysis of the generalized quantum electrodynamics from the causal point of view, Internat. J. Modern Phys. A., **32** (2017), 1750165.

[11] D. Bonheure, J.B. Cast*é*ras, E. Moreira dos Santos and R. Nascimento, Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation, SIAM J. Math. Anal., **50** (2018), 5027–5071.

[12] H. Brezis and E. Lieb, Arelation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc., **8** (1983), 486–490.

[13] R.R. Cuzinatto, C.A.M. de Melo, L.G. Medeiros, B.M. Pimentel and P.J. Pompeia, Bopp-Podolsky black holes and the no-hair theorem, Eur. Phys. J. C., **78** (2018), 43.

[14] R.R. Cuzinatto, E.M. De Morais, L.G. Medeiros, C. Naldoni De Souza and B.M. Pimentel, De Broglie-Proca and Bopp-Podolsky massive photon gases in cosmology, EPL., **118** (2017), 19001.

[15] S. Chen and X. Tang, On the critical Schrödinger-Bopp-Podolsky system with general nonlinearities, Nonlinear Anal., **195** (2020), 111734.

[16] G. Cerami and D. Passaseo, The effect of concentrating potentials in some singularly perturbed problems, Calc. Var. Partial. Differential. Equations., **17** (2003), 257–281.

[17] G. Cerami and G. Vaira, Positive solutions for some non-autonomous SchrödingerCPoisson systems, J. Differ. Equ., **248** (2010), 521–543.

[18] P. d'Avenia and G. Siciliano, Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: solutions in the electrostatic case, J. Differ. Equ., **267** (2019), 1025–1065.

[19] J. Frenkel, 4/3 problem in classical electrodynamics, Phys. Rev. E., **54** (1996), 5859–5862.

[20] G. Fibich, B. Ilan and G. Papanicolaou, Self-focusing with fourth-order dispersion, SIAM J. Appl. Math., **62** (2002), 1437–1462.

[21] S. Gaetano and S. Kaye, The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field, Publ. Mat., **64** (2020), 373–390.

[22] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equentions of Second order," Springer-Verlag, New York, 1998.

[23] E.H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev inequalitie, Ann. Math., **118** (1983), 349–374.

[24] G. Mie, Grundlagen einer theorie der materie, Ann. Phys., **345** (1913), 1–66.

[25] B. Podolsky, A generalized electrodynamics, Ann. Phys., **62** (1942), 68–71.

[26] M. Willem, "Minimax Theorems," Birkhäuser, Boston, 1996.
[27] Y. Zhu, C. Chen and J. Chen, The Schrödinger-Bopp-Podolsky equation under the effect of nonlinearities, The Bulletin of the Malaysian Mathematical Society Series., **2** (2020).
[28] J. Zhang, Ground state and multiple solutions for SchrödingerCPoisson equations with critical nonlinearity, J. Math. Anal. Appl., **440** (2016), 466–482.

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