The τ-symmetries and Lie algebra structure of the Blaszak-Marciniak lattice equation

Jian-bing Zhang* Qi Chen

School of Mathematics and Statistics , Jiangsu Normal University, Xuzhou 221116, Jiangsu, P.R.China

October 29, 2023

Abstract

A general approach is developed for discriminating strong and hereditary symmetric operators. The recursion operator of the Blaszak-Marciniak (BM) equation hierarchy is proved to be strong and hereditary symmetric. As an example of discrete soliton equations related to 3×3 matrix spectral problems, the τ-symmetries and Lie algebra structure of the BM equation are built firstly.

Keywords: 3×3 matrix spectral problem, τ-symmetry, Lie algebraic structure

Mathematical Sub Classification Numbers: 15A03, 15A03

1 Introduction

It is well known that through a matrix spectral problem we can obtain isospectral and non-isospectral soliton equation hierachies [1, 2]. These soliton hierarchies not only can be solved by the inverse scattering transformation (IST) [1, 2], but also possess remarkably rich algebraic characteristics, including the existence of conservation laws (CLs) and infinite many symmetries [3].

For continue soliton equations, there is a mature method to construct nonisospectral soliton equation hierarchies [4, 5, 6]. We always suppose their solutions, i.e., the potentials of the matrix spectral problems go to zero as $x \to \pm\infty$ because of the properties of solitary waves. Next, we

*The corresponding author. Email address: jbzmath@jsnu.edu.cn (J.B. Zhang)

usually can assume the relation of the spectral parameter λ depending on the time variable t is $\lambda_t = \lambda^k, k \in \mathbb{N}$. Through the so-called zero-curvature equation, we can obtain nonisospectral soliton equation hierarchies. But that is not the case for discrete soliton equations. Du to physical backgrounds, some potentials do not go to zero as the variable n go to infinity. So suitable time-dependence of the spectral parameter and correct initial conditions are the name of game if one wants to construct discrete nonisospectral soliton equation hierarchies. In [2], we gave a relation between the spectral parameter λ and t and solve the generating nonisospectral Toda lattice equations by the IST method. Later, Zhu et al. presented another relation and discussed the Lie algebraic structure between K-symmetries and the corresponding τ-symmetries [7].

The so-called K-symmetries actually are isospectral soliton equation hierarchies which do not depend explicitly on space and time variables. How to find the τ-symmetries was a difficult problem once a time. In 1987, Y.S. Li *et al.* found a general way to construct τ-symmetries [6, 8]. These symmetries often constitute a Lie algebra together with K-symmetries. Li and Cheng found that there also exist new sets of symmetries for the evolution equations which take τ-symmetries as vector fields [9, 10]. Tu showed that these τ-symmetries may be generated by the generators of the first degree [11]. On the basis of Tu's work, Ma established a more general skeleton on K-symmetries and τ-symmetries of evolution equations and their Lie algebraic structures [12, 13]. Chen and Zhang introduced the implicit representations of the flows of some continue soliton equations and derived the Lie algebraic structures of these flows [14, 15]. Furthermore, they expanded this method to the Ablowitz–Ladik hierarchy [16, 17]. In [7], they also discussed the Lie algebraic symmetries of the Toda lattice equation. Actually, there are also many other methods to consider the Lie algebra of soliton equations such as semi-direct sums [18, 19] and adjoined symmetries approach [20, 21].

To the authors' knowledge, the discussion on Lie algebraic structures tends to focus on soliton equations owning 2×2 matrix spectral problems. The higher order matrix spectral problems especially the discrete ones have not been researched yet. In this paper, we obtain the isospectral and nonisospectral equation hierarchy by the 3×3 matrix spectral problem of the BM lattice equation. In the course of constructing the nonisospectral equation hierarchy we take $\lambda_{t_k} = -4\lambda^{k+1} - 27\lambda^{k-2} \quad (k \geq 2)$. The recursion operator of the BM equation hierarchy is proved to be strong and hereditary symmetric by a general approach. The Lie algebraic relations of the BM equation are revealed.

The rest of the paper is organized as follows. In section 2, we will discuss basic notions and notations. In section 3, we will obtain the isospectral and nonisospectral the BM lattice equation hierarchies. In section 4, two types of symmetries will be constructed and proved to constitute an infinite-dimensional Lie algebra. We conclude the paper in section 5.

2 Basic notations

For the sake of simplicity, we first bring in some needful definitions and foundational notations. They are very important to the discussion of symmetries for discrete soltion equations.

Let the functions $v_i = v_i(t,n), 1 \leq i \leq s$ defined over $\mathbb{R}\times\mathbb{Z}$ satisfy $v_i \to 0$ when $|n| \to \infty$. We define a s-dimensional vector field as $\mathbf{v}_n = \mathbf{v}(t,n) = (v_1, v_2, \cdots, v_s)^T$. Suppose that f_i is a C^∞ differentiable function with the variables t and n defined over $\mathbf{v}_n$ satisfying $f_i(\mathbf{v}(t,n))|_{\mathbf{v}_n=\mathbf{0}} = 0$. We define the linear space by $\mathcal{V}_l$ consisting of all vector fields $\mathbf{f} = (f_1, f_2, \cdots, f_l)^T$. Assume that p_{ij} is Laurent polynomial of λ and $P(t,n,\mathbf{v}_n,\lambda) = (p_{ij}(t,n,\mathbf{v}_n,\lambda))_{m\times m}$ is $m\times m$ matrix, then we define a Laurent matrix polynomials space $\Lambda_m(\lambda)$ composed by all matrices $P(t,n,\mathbf{v}_n,\lambda)$.

Now, let us define the shift operator E as

$$Ef_n = f_{n+1}, \quad E^{-1}f_n = f_{n-1}, \quad n \in \mathbb{Z},$$

where $f_n = f(n)$ is a function of variable n.

Definition 1. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *belong to* $\mathcal{V}_l$*, then the Gâteaux derivative of* $\mathbf{f}$ *in direction* $\mathbf{g}$ *is*

$$\mathbf{f}'[\mathbf{g}] = \mathbf{f}(\mathbf{v}_n)'[\mathbf{g}] = \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathbf{f}(\mathbf{v}_n + \varepsilon\mathbf{g}), \varepsilon \in \mathbb{R}.$$

Furthermore, we can define the Lie product for any $\mathbf{f}, \mathbf{g} \in \mathcal{V}_l$ *as*

$$[\![\mathbf{f}, \mathbf{g}]\!] = \mathbf{f}'[\mathbf{g}] - \mathbf{g}'[\mathbf{f}],$$

by the Gâteaux derivative.

Note: If $\mathbf{f}$ *is an operator on* $\mathcal{V}_l$*, the definition of Gâteaux derivative stays the same.*

Definition 2. *(Ref.22). A function* $\sigma(\mathbf{v}(t,n))$ *defined over* $\mathcal{V}_l$ *is called the symmetry of a discrete evolution equation*

$$\mathbf{v}_{n,t} = K(t,n,\mathbf{v}_n),$$

if $\sigma_t(\mathbf{v}(t,n)) = K'[\sigma(\mathbf{v}(t,n))]$. *Similarly, a function* $\gamma(\mathbf{v}(t,n))$ *satisfying* $-\gamma_t(\mathbf{v}(t,n)) = K'^{*}\gamma(\mathbf{v}(t,n))$ *is called conserved covariant. We mark the linear space constituted by all symmetries of Eq.*(2) *as* S. *Its adjoint space* S^* *is formed by all conserved covariate.*

Definition 3. *(Ref.22). An operator* Φ *on* $\mathcal{V}_l$ *is called the strong symmetry of Eq.*(2) *if* $\Phi : S \longmapsto S^*$, *i.e.,*

$$\frac{d\Phi}{dt} = [K', \Phi] = K'\Phi - \Phi K'.$$

Specially, (3) *can be simplified as*

$$\Phi'[K] = [K', \Phi]$$

if the operator Φ *does not contain* t *explicitly. Finally,* Φ *is called a hereditary symmetry operator if it meets*

$$\Phi'[\Phi\mathbf{f}]\mathbf{g} - \Phi'[\Phi\mathbf{g}]\mathbf{f} = \Phi(\Phi'[\mathbf{f}]\mathbf{g} - \Phi'[\mathbf{g}]\mathbf{f}), \quad \forall \mathbf{f}, \mathbf{g} \in \mathcal{V}_l.$$

3 Isospectral and non-isospectral hierarchies of the BM lattice equation

In this section, we will construct the nonisospectral BM lattice equation hierarchy by introducing a new time-dependence spectral parameter λ. On the isospectral BM lattice equation hierarchy, there already have been a lot of researches. In [23], the authors studied the BM equation through the r-matrix approach. In [24], the symplectic map of the BM lattice equation was presented. In [25], its Hamiltonian structure was shown.

For the sake of simplicity, we introduce some denotes:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

First of all, let us discuss the BM spectral problem [23, 24]

$$E\vec{\phi}_n = M\vec{\phi}_n, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ p_n - \lambda & q_n & 1 \\ r_n & 0 & 0 \end{pmatrix}, \quad \vec{\phi}_n = \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \\ \phi_{3,n} \end{pmatrix} \tag{3.1a}$$

and its time evolution equation

$$\vec{\phi}_{n,t} = N\vec{\phi}_n, \quad N = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & I_n \end{pmatrix}, \tag{3.1b}$$

where E is a shift operator and $p_n = p_n(t), q_n = q_n(t), \ln r_n = \ln r_n(t)$ are potential functions defined over $\mathbb{R} \times \mathbb{Z}$, and rapidly near to 0 as $|n| \to \infty$.

By the compatibility of the equations (3.1), the matrices M and N satisfy the discrete zero curvature equation

$$M_t = (EN)M - MN, \tag{3.2}$$

i.e.,

$$(p_n - \lambda)B_{n+1} + r_n C_{n+1} - D_n = 0,$$

$$A_{n+1} + q_n B_{n+1} - E_n = 0,$$

$$B_{n+1} - F_n = 0,$$

$$(p_n - \lambda)E_{n+1} + r_n F_{n+1} - (p_n - \lambda)A_n - q_n D_n - G_n = p_{n,t} - \lambda_t,$$

$$H_{n+1} - r_n C_n = 0,$$

$$D_{n+1} + q_n E_{n+1} - (p_n - \lambda)B_n - q_n E_n - H_n = q_{n,t},$$

$$G_{n+1} + q_n H_{n+1} - r_n B_n = 0,$$

$$E_{n+1} - (p_n - \lambda)C_n - q_n F_n - I_n = 0,$$

$$(p_n - \lambda)H_{n+1} - r_n A_n + r_n I_{n+1} = r_{n,t}.$$

Direct calculation gives

$$\mathbf{u}_{n,t} = L_1 \begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix} - \lambda L_2 \begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix} + \lambda_t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.4}$$

where

$$L_1 = \begin{pmatrix} Ep_nE - p_n & Er_nE - E^{-1}r_n & q_n(E-1) \\ q_{n-1} - Eq_nE & (1-E)p_n & E^2 - E^{-1} \\ r_nE^2 - E^{-1}r_n - p_n(E-1)q_{n-1} & E^{-1}q_nr_n - q_nr_nE & p_n(E - E^{-1}) \end{pmatrix},$$

$$L_2 = \begin{pmatrix} E^2-1 & 0 & 0 \\ 0 & 1-E & 0 \\ q_{n-1}-q_nE & 0 & E-E^{-1} \end{pmatrix}, \quad \mathbf{u}_n = \begin{pmatrix} q_n \\ \ln r_n \\ p_n \end{pmatrix}.$$

Let λ be independent of time t, B_n , C_n , E_n can be expanded as polynomials of λ

$$\begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} b_{n,j} \\ c_{n,j} \\ e_{n,j} \end{pmatrix} \lambda^{k-j} \tag{3.5a}$$

meeting the initial conditions

$$\begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix} \Bigg|_{\mathbf{u}_n=0} = \begin{pmatrix} 0 \\ \lambda^k \\ 0 \end{pmatrix}. \tag{3.5b}$$

Setting $G_k = (b_{n,k}, c_{n,k}, e_{n,k})^T, k \in \mathbb{N}$, we obtain the following relations

$$\mathbf{u}_{n,t_k} = L_1 G_k,$$

$$L_1 G_j = L_2 G_{j+1} \quad (j = 0, 1, 2, \cdots, k-1),$$

$$L_2 G_0 = 0,$$

through comparing the coefficients of the same power of λ in (3.4). Then, we can obtain the isospectral BM lattice hierarchy

$$\mathbf{u}_{n,t_k} = L_1 G_k = L^k K_0 \quad (k = 0, 1, 2, \cdots),$$

where

$$K_0 = (r_{n+1} - r_{n-1}, p_n - p_{n+1}, q_{n-1}r_{n-1} - q_n r_n)^T$$

and

$$L \triangleq L_1 L_2^{-1} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$$

with

$$\begin{aligned}
&L_{11} = [\bar{\triangle}^{-1}(p_n) + q_n \triangle_- \triangle_+^{-1} q_n]\triangle^{-1}, && L_{12} = \hat{\triangle}(r_n)\triangle_-^{-1}, && L_{13} = q_n \triangle_+^{-1} E,\\
&L_{21} = - \triangle_- \triangle_+^{-1} q_n \triangle^{-1}, && L_{22} = \triangle_- p_n \triangle_-^{-1}, && L_{23} = (E^2 - E^{-1})\triangle^{-1},\\
&L_{31} = \hat{\triangle}^{-1}(r_n)\triangle^{-1}, && L_{32} = \bar{\triangle}(q_n r_n)\triangle_-^{-1}, && L_{33} = p_n.
\end{aligned}$$

by noting $\triangle = E - E^{-1}, \triangle_- = E - 1, \triangle_+ = E + 1$, $\hat{\triangle}(r_n) = E^{-1} r_n - E r_n E$, $\hat{\triangle}^{-1}(r_n) = r_n E - E^{-1} r_n E^{-1}$, $\bar{\triangle}^{-1}(p_n) = E p_n - p_n E^{-1}$, $\bar{\triangle}(q_n r_n) = q_n r_n E - E^{-1} q_n r_n$.

In particular, when $k = 0, 1$, the first two isospectral equations are

$$u_{n,t_0} = K_0, \tag{3.7}$$

$$u_{n,t_1} = K_1 = LK_0 = (k_{11}, k_{12}, k_{13})^T, \tag{3.8}$$

where $k_{11} = \triangle_- \triangle_+ r_{n-1} \triangle_+ p_{n-1} - \triangle_- p_n$, $k_{12} = - \triangle_- [\triangle_+ q_{n-1} r_{n-1} + p_n^2] + \triangle q_n$, and $k_{13} = \triangle_-(\triangle_+ r_{n-2} r_{n-1} - q_{n-1} r_{n-1} \triangle_+ p_{n-1} - r_{n-1}) + p_n \triangle_- q_{n-1}$.

Their corresponding matrices N related to time-development are

$$N_0 = \begin{pmatrix} 0 & 0 & 1 \\ r_n & 0 & 0 \\ -q_{n-1} r_{n-1} & r_{n-1} & \lambda - p_n \end{pmatrix}$$

and

$$N_1 = \begin{pmatrix} -q_{n-1}(r_{n-1} - 1) & r_{n-1} - 1 & \lambda + p_n \\ \lambda + p_{n+1} r_n + p_n(r_n - 1) & 0 & r_n - 1 \\ r_{n-1}(r_{n-2} - 1 - q_{n-1} p_{n-1} - \lambda q_{n-1}) & r_{n-1}(\lambda + p_{n-1}) & \lambda^2 - p_n^2 + q_n(1 - r_n) \end{pmatrix},$$

respectively.

In the nonisospectral case, we suppose

$$\lambda_{t_k} = -4\lambda^{k+1} - 27\lambda^{k-2} \quad (k \geq 2), \tag{3.9}$$

and B_n, C_n, E_n still are polynomials of λ satisfying

$$B_n = 9n\lambda^{k-2}, \quad C_n = 6n\lambda^{k-1}, \quad E_n = -2n\lambda^k, \tag{3.10}$$

when $q_n = 0, r_n = 1, p_n = 0$.

For convenience, we only consider the cases of $k = 2$ and $k > 2$.

When $k=2$, $\lambda_{t_2}=-4\lambda^3-27$, substituting (3.5a) into (3.4) and comparing the coefficients of same power of λ yield

$$\mathbf{u}_{n,t_2}=L_1G_2-27e_3,$$

$$L_1G_1=L_2G_2,$$

$$L_1G_0=L_2G_1,$$

$$L_2G_0=-4e_3,$$

where $e_3=(0,0,1)^T$. It follows that $b_{n,0}=0, c_{n,0}=0, e_{n,0}=-2n$ from the initial value condition (3.10). In this case, we obtain

$$\mathbf{u}_{n,t_2}\triangleq\sigma_2=(u_{21},u_{22},u_{23})^T,$$

where

$$\begin{aligned}
u_{21}=&-2[p_{n+1}+q_n\triangle_-\triangle_+^{-1}q_n]\triangle^{-1}\{[p_{n+1}+q_n\triangle_-\triangle_+^{-1}q_n]\triangle^{-1}q_n+2q_n\triangle_+^{-1}p_{n+1}\\
&+3\hat{\triangle}(r_n)n\}+2\hat{\triangle}(r_n)[\triangle_+^{-1}q_n\triangle^{-1}q_n-3np_n-2(\triangle_++E^{-1})\triangle^{-1}p_n]\\
&-2q_n\triangle_+^{-1}[\tilde{\triangle}(r_n)\triangle^{-1}q_n+3\bar{\triangle}(q_nr_n)n+2p_n^2],
\end{aligned}$$

$$\begin{aligned}
u_{22}=&2[\triangle_-\triangle_+^{-1}q_n\triangle^{-1}]\{[p_{n+1}+q_n\triangle_-\triangle_+^{-1}q_n]\triangle^{-1}q_n+3\hat{\triangle}(r_n)n+2q_n\triangle_+^{-1}p_{n+1}\}\\
&+2\triangle_-p_n[\triangle_+^{-1}q_n\triangle^{-1}q_n-3np_n-2(\triangle_++E^{-1})\triangle^{-1}p_n]\\
&-2(E^2-E^{-1})\triangle^{-1}[\hat{\triangle}^{-1}(r_n)\triangle^{-1}q_n+3\bar{\triangle}(q_nr_n)n+2p_n^2],
\end{aligned}$$

$$\begin{aligned}
u_{23}=&-2\hat{\triangle}^{-1}(r_n)\triangle^{-1}\{[Ep_n+q_n\triangle_-\triangle_+^{-1}q_n]\triangle^{-1}q_n+3\hat{\triangle}(r_n)n+2q_n\triangle_+^{-1}p_{n+1}\}\\
&+2\bar{\triangle}(q_nr_n)[\triangle_+^{-1}q_n\triangle^{-1}q_n-3np_n-2(\triangle_++E^{-1})\triangle^{-1}p_n]\\
&-2p_n[\hat{\triangle}^{-1}(r_n)\triangle^{-1}q_n+3\bar{\triangle}(q_nr_n)n+2p_n^2]-27.
\end{aligned}$$

where $\tilde{\triangle}(r_n)=Er_nE-r_nE^{-1}$.

When $k>2$, we have

$$u_{n,t_k}=L_1G_k,$$

$$L_1G_j = L_2G_{j+1} \quad (j = 0, 1, 3, \cdots, k-1),$$

$$L_1G_2 - L_2G_3 - 27e_3 = 0,$$

$$L_2G_0 = -4e_3.$$

Then, we get the following nonisospectral BM lattice hierarchy

$$\mathbf{u}_{n,t_k} \triangleq \sigma_k = L^k \begin{pmatrix} -2q_n \\ -6 \\ -4p_n \end{pmatrix} - 27L^{k-2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (k > 2). \tag{3.11}$$

The right side of the equation (3.11) is the so-called k-order BM lattice nonisospectral flows, which are vector fields with three components. In a word, the isospectral and nonisospectral flows possess the following general representations

$$K_m = L^m K_0 (m = 0, 1, \cdots), \quad \sigma_l = L^{l-2}\sigma_2 (l = 2, 3, \cdots).$$

Obviously, we find that the nonisospectral flows (3.11) include two parts. This is because we assume the time relation of spectral parameter (3.9). In [7], the time relation is $\lambda_t = \lambda^{k+1} - 4\lambda^{k-1}$. Actually, to meet the initial condition

$$\sigma_k|_{\mathbf{u}_n=\mathbf{0}} = \mathbf{0},$$

we always suppose that the difference in order of λ in the evolution relation is the order of the spectral matrix. It is a general relationship and can be applied in other discrete soliton equations.

4 Two sets of symmetries for a hierarchy of the BM lattice equation and their Lie algebra structure

In this section, general theories of distinguishing strong and heredity symmetry operators are developed. The two sets of symmetries named K-symmetry $\{K_m\}$ and τ-symmetry $\{\sigma_l\}$ of the BM lattice equation are constructed and proved to satisfy the Lie algebra relations.

4.1 Zero curvature representation

In this subsection, we will develop general theories to prove the recursion operator L of the BM equation hierarchy is strong and heredity symmetric by the implicit expression of the zero representations.

In the isospectral case, for $\lambda_t = 0$, we have $M_t = M'[\mathbf{u}_{n,t}]$, then the discrete zero curvature equation (3.2) can be rewritten as

$$M'[\mathbf{u}_{n,t}] = (EN)M - MN.$$

In non-isospectral cases, because $\lambda_{t_n} \neq 0$, we get $M_t = M'[\mathbf{u}_{n,t}] + \lambda_{t_n} M_\lambda$. So (3.2) can be transformed into

$$M'[\mathbf{u}_{n,t}] = (EP)M - MP - \lambda_{t_n} M_\lambda.$$

With a view to the corresponding the isospectral flow K_l and the non-isospectral flow σ_k, the above two equations have the following expression

$$M'[K_l] = (EN_l)M - MN_l, \tag{4.1a}$$

$$M'[\sigma_k] = (EP_k)M - MP_k + \lambda^{k-2}(4\lambda^3 + 27)M_\lambda. \tag{4.1b}$$

Eq.(4.1a) and (4.1b) are called the zero-curvature representations of the isospectral flow K_l and the non-isospectral flow σ_k of the BM lattice equations, separately.

Theorem 1. *Let U is a $l \times l$ matrix containing the potential $\mathbf{u}_n$ and $\Phi : \mathcal{V}_l \mapsto \mathcal{V}_l$ is a linear map. If for any given $\mathbf{f} \in \mathcal{V}_l$, there are solutions $V = V(\mathbf{u}_n) \in \Lambda_l(\lambda)$ meeting*

$$U'[\Phi f - \lambda f] = (EV)U - UV, \quad V|_{\mathbf{u}_n=\mathbf{0}} = \mathbf{0}, \tag{4.2}$$

where $U'[\Phi f - \lambda f]$ refers to the Gâteaux derivative of U with respect to $\mathbf{u}_n$, then Φ is a hereditary symmetric operator.

Proof. For arbitrary $\mathbf{f}, \mathbf{g} \in \mathcal{V}_\mathbf{3}$, we can suppose that

$$U'[\Phi\mathbf{f} - \lambda\mathbf{f}] = (E\bar{N})U - U\bar{N}, \tag{4.3a}$$

$$U'[\Phi\mathbf{g} - \lambda\mathbf{g}] = (ER)U - MU, \tag{4.3b}$$

$$U'[\Phi(\Phi'[\mathbf{g}]\mathbf{f}) - \lambda(\Phi'[\mathbf{g}])\mathbf{f}] = (EV)U - UV,$$

$$U'[\Phi(\Phi'[\mathbf{f}]\mathbf{g}) - \lambda(\Phi'[\mathbf{f}])\mathbf{g}] = (EW)U - UW,$$

by taking advantage of (4.2). One more derivative of (4.3a) and (4.3b) in direction of $\Phi\mathbf{g} - \lambda\mathbf{g}$ and $\Phi\mathbf{f} - \lambda\mathbf{f}$ respectively gives

$$U'[\Phi'[\Phi\mathbf{g}]\mathbf{f} - \Phi'[\Phi\mathbf{f}]\mathbf{g} - \Phi(\Phi'[\mathbf{g}]\mathbf{f} - \Phi'[\mathbf{f}]\mathbf{g})] = (EP)U - UP, P|_{\mathbf{u}_n=\mathbf{0}} = \mathbf{0},$$

where

$$P = \bar{N}'[\Phi\mathbf{g} - \lambda\mathbf{g}] - R'[\Phi\mathbf{f} - \lambda\mathbf{f}] + [\bar{N}, R] - V + W.$$

Obviously, we obtain

$$\Phi'[\Phi\mathbf{g}]\mathbf{f} - \Phi'[\Phi\mathbf{f}]\mathbf{g} - \Phi(\Phi'[\mathbf{g}]\mathbf{f} - \Phi'[\mathbf{f}]\mathbf{g}) = \mathbf{0},$$

i.e., the operator Φ is hereditary. □

Theorem 2. *Let $K \in \mathcal{V}_l$, U and V are $l \times l$ matrices comprising potential $\mathbf{v}_n$, which satisfy $U'[K] = (EV_0)U - UV_0$, where $U'[K]$ represents the Gâteaux derivation with respect to $\mathbf{v}_n$. For any given $\mathbf{f} \in \mathcal{V}_l$, there are solutions $V = V(\mathbf{v}_n, \lambda) \in \Lambda_l(\lambda)$ meeting*

$$U'[\Phi\mathbf{f} - \lambda\mathbf{f}] = (EV)U - UV, \quad V|_{\mathbf{v}_n=\mathbf{0}} = \mathbf{0}, \tag{4.4}$$

where $\Phi : \mathcal{V}_l \mapsto \mathcal{V}_l$ is a linear map. Then Φ is a strong symtric operator of equation $\mathbf{v}_{n,t} = K$.

Proof. For any $\mathbf{f}, \mathbf{g} \in \mathcal{V}_l$, we obtain

$$\Phi'[\![\mathbf{f}, \mathbf{g}]\!] = (\Phi'[\mathbf{f}])'[\mathbf{g}] - (\Phi'[\mathbf{g}])'[\mathbf{f}],$$

by directly calculation. Through (4.4), we get

$$U'[\![\Phi\mathbf{f} - \lambda\mathbf{f}, K]\!] = (E\tilde{V})U - U\tilde{V},$$

where

$$\tilde{V} = V'[K] + [V, V_0] - V_0'[\Phi\mathbf{f} - \lambda\mathbf{f}].$$

On the other hand, it is easy to acquire

$$[\![\Phi\mathbf{f} - \lambda\mathbf{f}, K]\!] = (\Phi'[K] - [K', \Phi])\mathbf{f} + \Phi[\![\mathbf{f}, K]\!] - \lambda[\![\mathbf{f}, K]\!].$$

Assume that

$$U'[\Phi[\![\mathbf{f},K]\!]-\lambda[\![\mathbf{f},K]\!]]=(E\hat{V})U-U\hat{N},\quad \hat{V}|_{\mathbf{u}_n=0}=0,$$

then we can have

$$U'[(\Phi'[K]-[K',\Phi])\mathbf{f}]=(EQ)U-UQ,\quad Q|_{\mathbf{u}_n=\mathbf{0}}=\mathbf{0}.$$

So we obtain

$$(\Phi'[K]-[K',\Phi])\mathbf{f}=\mathbf{0},$$

which means the operator Φ is a strong symmetry of the equation $\mathbf{u}_{n,t}=K$. □

Corollary 1. *The recursion operator of the BM hierarchy L is a strong and hereditary operator of the equation* (3.7).

Proof. It is easy to verify that

$$M'[K_0]=(EN_0)M-MN_0.$$

For any $\mathbf{f}\in\mathcal{V}_3$, the equation

$$M'[L\mathbf{f}-\lambda\mathbf{f}]=(EV)M-MV$$

gives

$$(p_n-\lambda)v_{n,12}+r_nv_{n+1,13}=v_{n,21},\qquad v_{n+1,11}+q_nv_{n+1,12}=v_{n,22},\qquad v_{n+1,12}=v_{n,23},$$
$$v_{n+1,22}-(p_n-\lambda)v_{n,13}-q_nv_{n,23}=v_{n,33},\quad v_{n+1,31}+q_nv_{n+1,32}=r_nv_{n,12},\quad v_{n+1,32}=r_nv_{n,13},$$

and

$$L\mathbf{f}-\lambda\mathbf{f}=L_1\begin{pmatrix}v_{n,12}\\ v_{n,13}\\ v_{n,22}\end{pmatrix}-\lambda L_2\begin{pmatrix}v_{n,12}\\ v_{n,13}\\ v_{n,22}\end{pmatrix},\tag{4.5}$$

where $V=(v_{n,ij})$ is a 3×3 matrix. Obviously, the equation (4.5) has solutions

$$L_2^{-1}\mathbf{f}=\begin{pmatrix}v_{n,12}\\ v_{n,13}\\ v_{n,22}\end{pmatrix}.$$

Furthermore, we can deduce the other elements of the matrix V. From the Theorem 1 and Theorem 2, we conclude that L is a strong and hereditary operator of the equation (3.7). □

4.2 Symmetry and Lie algebra structure

The two sets of symmetries called $K-$ and $\tau-$symmetries are formed by the isospectral and nonisospectral flows of the BM lattice equation. Next, the Lie algebra structure of them will be discussed. Since the time evolution matrices of the isospectral flow K_m and the nonisospectral flow σ_l are N_m and P_l respectively, we define:

Definition 4.

$$< N_m, N_l >= N_m'[K_l] - N_l'[K_m] + [N_m, N_l],$$

$$< N_m, P_l >= N_m'[\sigma_l] - P_l'[K_m] + [N_m, P_l] - \lambda^{l-2}(4\lambda^3 + 27)N_{m,\lambda},$$

$$< P_m, P_l >= P_m'[\sigma_l] - P_l'[\sigma_m] + [P_m, P_l] + \lambda^{m-2}(4\lambda^3 + 27)P_{l,\lambda} - \lambda^{l-2}(4\lambda^3 + 27)P_{m,\lambda}.$$

Theorem 3. *The Lie products of the flows of the BM equation satisfy*

$$M'[\![K_m, K_l]\!] = (E < N_m, N_l >)M - M < N_m, N_l >, \tag{4.7a}$$

$$M'[\![K_m, \sigma_l]\!] = (E < N_m, P_l >)M - M < N_m, P_l >, \tag{4.7b}$$

$$M'[\![\sigma_m, \sigma_l]\!] = (E < P_m, P_l >)M - M < P_m, P_l > +(l-m)\lambda^{m+l-5}(4\lambda^3 + 27)^2 M_\lambda, \tag{4.7c}$$

with boundary conditions

$$< N_m, N_l > |_{\mathbf{u}_n=\mathbf{0}} = \mathbf{0}, \tag{4.8a}$$

$$< P_m, P_l > |_{\mathbf{u}_n=\mathbf{0}} = (l-m)(4P_{m+l} + 27P_{m+l-3}), \tag{4.8b}$$

$$< N_m, P_l > |_{\mathbf{u}_n=\mathbf{0}} = 6\lambda^{m+l}(E_1 - E_2\lambda^{-1} - \frac{3}{2}E_3\lambda^{-2}) - 4(m+1)N_{m+l} - 27mN_{m+l-3}. \tag{4.8c}$$

Proof. We only prove (4.7c) and (4.8b), the others can be obtained similarly. By the Gâteaux derivatives in the direction σ_l with respect to $\mathbf{u}_n$, we have

$$\begin{aligned}(M'[\sigma_m])'[\sigma_l] &= (EP_m)'[\sigma_l]M + (EP_m)(EP_l)M - (EP_m)MP_l - (EP_l)MP_m \\ &+ \lambda^{l-2}(4\lambda^3 + 27)[(EP_m)M_\lambda - M_\lambda P_m] + \lambda^{l+m-4}(4\lambda^3 + 27)^2 M_{\lambda,\lambda} \\ &+ \lambda^{m-2}(4\lambda^3 + 27)\{(EP_l)_\lambda M + (EP_l)M_\lambda - M_\lambda P_l - MP_{l,\lambda} \\ &+ [12\lambda^l + (4\lambda^3 + 27)(l-2)\lambda^{l-3}]M_\lambda\} + (MP_l)P_m - MP_m'[\sigma_l].\end{aligned}$$

Similarly,

$$\begin{aligned}(M'[\sigma_l])'[\sigma_m] = &(EP_l)'[\sigma_m]M + (EP_l)(EP_m)M - (EP_l)MP_m - (EP_m)MP_l \\ &+ \lambda^{m-2}(4\lambda^3+27)[(EP_l)M_\lambda - M_\lambda P_l] + \lambda^{l+m-4}(4\lambda^3+27)^2 M_{\lambda,\lambda} \\ &+ \lambda^{l-2}(4\lambda^3+27)\{(EP_m)_\lambda M + (EP_m)M_\lambda - M_\lambda P_m - MP_{m,\lambda} \\ &+ [12\lambda^m + (4\lambda^3+27)(m-2)\lambda^{m-3}]M_\lambda\} + (MP_m)P_l - MP_l'[\sigma_m].\end{aligned}$$

So

$$\begin{aligned}&M'[[\![\sigma_m, \sigma_l]\!]] \\ =&(M'[\sigma_m])'[\sigma_l] - (M'[\sigma_l])'[\sigma_m] \\ =&(EP_m)'[\sigma_l]M - (EP_l)'[\sigma_m]M + MP_l'[\sigma_m] - MP_m'[\sigma_l] + (l-m)(4\lambda^3+27)^2\lambda^{m+l-5}M_\lambda \\ &- \lambda^{l-2}(4\lambda^3+27)[(EP_m)_\lambda M - MP_{m,\lambda}] + \lambda^{m-2}(4\lambda^3+27)[(EP_l)_\lambda M - MP_{l,\lambda}].\end{aligned}$$

On the other side,

$$\begin{aligned}&(E < P_m, P_l >)M - M < P_m, P_l > + (l-m)\lambda^{m+l-5}(4\lambda^3+27)^2 M_\lambda \\ =&\{E[P_m'[\sigma_l] - P_l'[\sigma_m] + [P_m, P_l] + \lambda^{m-2}(4\lambda^3+27)P_{l,\lambda} - \lambda^{l-2}(4\lambda^3+27)P_{m,\lambda}]\}M \\ &- M[P_m'[\sigma_l] - P_l'[\sigma_m] + [P_m, P_l] + \lambda^{m-2}(4\lambda^3+27)P_{l,\lambda} - \lambda^{l-2}(4\lambda^3+27)P_{m,\lambda}] \\ &+ (l-m)\lambda^{m+l-5}(4^3+27)^2 M_\lambda \\ =&(EP_m)'[\sigma_l]M - (EP_l)'[\sigma_m]M + MP_l'[\sigma_m] - MP_m'[\sigma_l] + (l-m)(4\lambda^3+27)^2\lambda^{m+l-5}M_\lambda \\ &- \lambda^{l-2}(4\lambda^3+27)((EP_m)_\lambda M - MP_{m,\lambda}) + \lambda^{m-2}(4\lambda^3+27)((EP_l)_\lambda M - MP_{l,\lambda}).\end{aligned}$$

That is to say (4.7c) is correct.

Now, let us verify (4.8b). For

$$P_l|_{\mathbf{u}_n=\mathbf{0}} = \begin{pmatrix} -2(n-1)\lambda^2 & 9(n-1) & 6n\lambda \\ (-3n+6)\lambda & -2n\lambda^2 & 9n \\ 9(n-2) & 6(n-1)\lambda & (4n-2)\lambda^2 \end{pmatrix}\lambda^{l-2},$$

then

$$
\begin{aligned}
&< P_m, P_l > |_{\mathbf{u}_n=\mathbf{0}}\\
=&[P'_m[\sigma_l] - P'_l[\sigma_m] + [P_m, P_l] + \lambda^{m-2}(4\lambda^3+27)P_{l,\lambda} - \lambda^{l-2}(4\lambda^3+27)P_{m,\lambda}]|_{\mathbf{u}_n=\mathbf{0}}\\
=&[\lambda^{m-2}(4\lambda^3+27)P_{l,\lambda} - \lambda^{l-2}(4\lambda^3+27)P_{m,\lambda}]|_{\mathbf{u}_n=\mathbf{0}}\\
=&(l-m)\lambda^{m+l-5}(4\lambda^3+27)\begin{pmatrix} -2(n-1)\lambda^2 & 9(n-1) & 6n\lambda \\ (-3n+6)\lambda & -2n\lambda^2 & 9n \\ 9(n-2) & 6(n-1) & (4n-2)\lambda^2 \end{pmatrix}\\
=&(l-m)(4P_{m+l} + 27P_{m+l-3})|_{\mathbf{u}_n=\mathbf{0}}.
\end{aligned}
$$

□

The Lie algebra relation of two flows K_m and σ_l of the BM lattice equations is discussed below.

Theorem 4. *For the isospectral flow K_m and the non-isospectral flow σ_l of the BM lattice equation, we have the following relations:*

$$[\![K_m, K_l]\!]|_{\mathbf{u}_n=0} = \mathbf{0}, \tag{4.9a}$$

$$[\![K_m, \sigma_l]\!]|_{\mathbf{u}_n} = -4(m+1)K_{m+l} - 27mK_{m+l-3}, \tag{4.9b}$$

$$[\![\sigma_m, \sigma_l]\!]|_{\mathbf{u}_n} = (l-m)(4\sigma_{m+l} + 27\sigma_{m+l-3}), \tag{4.9c}$$

and also have the following form:

$$< N_m, N_l >= 0, \tag{4.10a}$$

$$< N_m, P_l >= -4(m+1)N_{m+l} - 27mN_{m+l-3} + 6\lambda^{m+l}(E_1 - E_2\lambda^{-1} - \frac{3}{2}E_3\lambda^{-2}), \tag{4.10b}$$

$$< P_m, P_l >= (l-m)(4P_{m+l} + 27P_{m+l-3}), \tag{4.10c}$$

in which $m = 0, 1, 2, \cdots$ and also $K_{-1} = 0$, but $l = 2, 3, \cdots$.

Proof. In the light of Theorem 1, only admitting zero solution, comparing (4.7a) and (4.8a), we can get (4.9a) and (4.10a) are right.

Setting

$$\widetilde{X} = [\![K_m, \sigma_l]\!] + 4(m+1)K_{m+l} + 27mK_{m+l-3},$$

$$\widetilde{A} =< N_m, P_l > +4(m+1)N_{m+l} + 27mN_{m+l-3},$$

then we find that

$$\begin{aligned}&M'[[\![K_m, \sigma_l]\!] + 4(m+1)K_{m+l} + 27mK_{m+l-3}]\\ =&[E(< N_m, P_l > +4(m+1)N_{m+l} + 27mN_{m+l-3})]M - M[< N_m, P_l >\\ &+ 4(m+1)N_{m+l} + 27mN_{m+l-3}]\end{aligned}$$

and satisfy

$$(< N_m, P_l > +4(m+1)N_{m+l} + 27mN_{m+l-3})|_{\mathbf{u}_n=\mathbf{0}} = 6\lambda^{m+l}(E_1 - E_2\lambda^{-1} - \frac{3}{2}E_3\lambda^{-2}).$$

which have only zero solutions $\widetilde{X} = 0$ and

$$\widetilde{A} = -9E_3\lambda^{m+l-2} - 6E_2\lambda^{m+l-1} + 6E_1\lambda^{m+l}$$

that means (4.9b) and (4.10b) are valid. In the similarly way, we can prove (4.9c) and (4.10c) are correct. □

By virtue of the above results, it is very easy to obtain two sets of symmetries and their Lie algebra for any equation in the isospectral BM equation hierarchy.

Theorem 5. *There are two sets of symmetries* $\{K_m\}$ *and*

$$\tau_m^l = -4(l+1)t_lK_{l+m} - 27lt_lK_{l+m-3} + \sigma_m,$$

where $l = 0, 1, 2, \cdots, m = 2, 3, \cdots, K_{-1} = \mathbf{0}$*, which we called* K*-symmetries and* τ*-symmetries, respectively. They construct a Lie algebra which satisfy the following Lie product relations:*

$$[\![K_m, K_k]\!]|_{\mathbf{u}_n} = 0,$$

$$[\![K_m, T_k^l]\!]|_{\mathbf{u}_n} = -4(m+1)K_{m+k} - 27mK_{m+k-3},$$

$$[\![T_m^l, T_k^l]\!]|_{\mathbf{u}_n} = (k-m)(4T_{m+k}^l + 27T_{m+k-3}^l).$$

5 Conclusion

In this paper, the nonisospectral soliton equation hierarchy of a discrete 3×3 spectral problem is first presented by taking the BM spectral problem as example. To achieve this goal, we select the relation of the spectral parameter λ and t as $\lambda_{t_k} = -4\lambda^{k+1} - 27\lambda^{k-2} \quad (k \geq 2)$. The nonisospectral soliton equation hierarchy together with isospectral one are formed the K-symmetries and τ-symmetries of the BM equation respectively. The infinite-dimensional Lie algebra are constructed by the two sets of symmetries.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments This work is supported by the National Natural Science Foundation of China (No.12171209 and 11771186) and Practice Innovation Program of Jiangsu Province (Grant No. KYCX22_2778).

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. **53**, 249-315 (1974).

[2] J.B. Zhang, D.J. Zhang, and D.Y. Chen, Exact solutions to a mixed Toda lattice hierarchy through the inverse scattering transformation, J. Phys. A Math. Gen. **44**, 115201 (2011).

[3] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. **19**, 1156-1162 (1978).

[4] J.B. Zhang, J. Ji, and D.Y. Chen, Conservation laws and symmetries of the Levi equation, Commun. Theor. Phys. **50**, 1261-1264 (2008).

[5] J.B. Zhang, Y.Y. Gongye, and W.X. Ma, Conservation laws and τ-symmetry algebra of the Gerdjikov–Ivanov soliton hierarchy, Bull. Malays. Math. Sci. Soc. **43**, 111–123 (2020).

[6] Y.S. Li, and G.C. Zhu, New set of symmetries of the integrable equations, Lie algebra and non-isospectral evolution equations. II. AKNS system, J. Phys. A Math. Gen. **19**, 3713-3725 (1986).

[7] X.Y. Zhu, D.J. Zhang, and Z.C. Li, New set of symmetries and Lie algebraic structures of the Toda lattice hierarchy, J. Maths. Phys. **56**, 023508 (2015).

[8] Y.S. Li, and G.C. Zhu, New Set of symmetries of the integrable equations, Lie algebra and non-isospectral evolution equations(I), Sci. Sin. Ser. A **30**, 1243-1250 (1987).

[9] Y. Cheng, and Y.S. Li, Symmetries and constants of motion for new AKNS hierarchies, J. Phys. A: Math. Gen. **20**, 1951-1959 (1987).

[10] Y.S Li,.and Y. Cheng, Symmetries and constants of motion for new KdV hierarchies, Scientia Sinica A **31**, 769-776 (1988).

[11] G.Z. Tu, The Lie algebraic structure of symmetries generated by hereditary symmetries, J. Phys. A Math. Gen. **21**, 1951-1957 (1988).

[12] W.X. Ma, K-symmetries and τ-symmetries of evolution equations and their Lie algebras, J. Phys. A Math. Gen. **23**, 2707-2716 (1990).

[13] W.X. Ma, Generators of vector fields and time dependent symmetries of evolution equations, Science in China A **34(7)**, 769-782 (1991).

[14] D.Y. Chen, and D.J. Zhang, Lie algebraic structure of (1+1)-dimensional Lax integrable systems, J. Math. Phys. **37**, 5524-5538 (1996).

[15] D.Y. Chen, H.W. Xin, and D.J. Zhang, Lie algebraic structure of some (1+2)-dimensional Lax integrable systems, Chaos, Solitons & Fractals **15**, 761-770 (2003).

[16] D.J. Zhang, and S.T. Chen, Symmetries for the Ablowitz–Ladik hierarchy: Part I. Four-potential case, Stud. Appl. Math. **125**, 393-418 (2010).

[17] D.J. Zhang, and S.T. Chen, Symmetries for the Ablowitz-Ladik hierarchy: Part II. Integrable discrete nonlinear Schrödinger equations and discrete AKNS hierarchy, Stud. Appl. Math. **125**, 419-443 (2010).

[18] W.X. Ma, and M. Chen, Hamiltonian and quasi-Hamiltonian structrues associated with semi-direct sums of Lie algebras, J. Phys. A Math. Gen. **39**, 10787-10801 (2006).

[19] W.X. Ma, A discrete variational identity on semi-direct sums of Lie algebras, J. Phys. A Math. Gen. **40**, 15055-15070 (2007).

[20] W.X. Ma, Conservation laws of discrete evolution equations by symmetries and adjoint symmetries, Symmetry **7**, 714-725 (2015).

[21] W.X. Ma, Conservation laws by symmetries and adjoint symmetries, Disc. Cont. Dyn. Syst. Ser. S **11**, 707-721 (2018).

[22] B. Fuchssteiner, and A.S. Fokas, Symplectic structures, their Bac̆klund transformations and hereditary symmetries, Phys. D **4**, 47-66 (1981).

[23] M. Blaszak, and K. Marciniak, R -matrix approach to lattice integrable systems, J. Math. Phys. **35**, 4661-4682 (1994).

[24] Y.T. Wu, and X. Geng, A new integrable symplectic map associated with lattice soliton equations, J. Math. Phys. **37**, 2338-2345 (1996).

[25] D.J. Zhang, and D.Y. Chen, Hamiltonian structure of discrete soliton systems, J. Phys. A: Math. Gen. **35**, 7225–7241 (2002) .