On the geometrical sensitivity of the EEG inversion algorithm

N. P. Pasiou
Department of Mathematics
National and Kapodistrian University of Athens
Panepistimiopolis
GR-15784 Zographou, Athens, Greece
pasioun@math.uoa.gr

Abstract

The relative algorithms existing in medical devices for the identification of excitation sources inside the brain using EEG data are based on the assumption that the geometry of the brain-head system is spherical. So, taking EEG measurements from a realistic ellipsoidal model and using these data in a spherical model leads to a structural error. The purpose of the present work is to estimate this geometrical error. The results show that for ellipsoids with small principal eccentricities the errors are not significant. However these errors become bigger as the eccentricities increase and this is a general result that becomes available for any related applications of this inverse problem.

Introduction

It is well known that the electric excitation of the brain is due to an equivalent electric dipole in the interior of the brain which generates an electric and a magnetic field, both in the interior and exterior of the brain [2, 11]. The electric and magnetic fields are measured on the surface and the exterior of the head via the EEG and MEG, respectively [3, 10, 19]. The bioelectromagnetism problem is solved in both spherical and ellipsoidal geometry. Given the current, the direct EEG problem consists of finding the electric potential [4, 12], while the inverse EEG problem consists of finding the current from the given EEG measurements once we know the geometry of the brain [6, 7]. Direct mathematical problems are well-posed, while the inverse problems are ill-posed, due to lack of uniqueness [15]. In order to work with the inverse and the forward EEG problems, we have to make certain assumptions, concerning the electrochemical source and the conductor that models the human brain. The most popular source model is a current dipole with fixed moment and location inside the brain. As far as the brain itself is concerned, in most of the work that has been published, it is considered to be a homogeneous or a partially homogeneous conductor [14, 16]. The most popular geometrical model is the spherical one, although the ellipsoidal geometry it best fits the anatomical model of the human brain [1, 9, 13].

In this work we use the solution of inverse problems in spherical and ellipsoidal geometry in order to develop analytic connection relations, aiming to compare the results of the present paper.

1. Statement of the problem

Let Ω be a bounded, connected and finite homogeneous conductor with a smooth boundary S=∂Ω. The domain Ω provides a simplified geometrical model of the brain as an isotropic and homogeneous conductor with conductivity σ. A neuronal current J^P with support in Ω generates the electric and magnetic activity that is generated from J^P is governed by the Quasi-Static Theory of Electromagnetism.

I. Ellipsoidal system

Let S_e denotes the triaxial ellipsoid which in rectangular coordinates is specified by

$$\frac{x_1^2}{\alpha_1^2}+\frac{x_2^2}{\alpha_2^2}+\frac{x_3^2}{\alpha_3^2}=1 \tag{1}$$

where $0<\alpha_3<\alpha_2<\alpha_1<+\infty$ are its semi-axes.

The basic ellipsoid (1) introduces an ellipsoidal system with coordinates (ρ, μ, ν) and semi-focal distances h_1 , h_2 , h_3 where

$$h_1=\sqrt{\alpha_2^2-\alpha_3^2},\quad h_2=\sqrt{\alpha_1^2-\alpha_3^2} \tag{2}$$

The ellipsoidal coordinates (ρ, μ, ν), involve the ellipsoidal variable $\rho\in[h_2,+\infty)$ and the hyperboloidal variables $\mu\in[h_3,h_2]$ and $\nu\in[-h_3,h_3]$.

The transformation from ellipsoidal to Cartesian to coordinates is given by

$$x_1^2=\frac{\rho^2\mu^2\nu^2}{h_2^2h_3^2}$$

$$x_2^2=\frac{\left(\rho^2-h_3^2\right)\left(\mu^2-h_3^2\right)\left(h_3^2-\nu^2\right)}{h_1^2h_3^2}$$

(3)

$$x_3^2=\frac{\left(\rho^2-h_2^2\right)\left(h_2^2-\mu^2\right)\left(h_2^2-\nu^2\right)}{h_1^2h_2^2}$$

The coordinate ρ plays the role of the radial variable r, while μ and ν correspond to the angular variable θ and φ in spherical coordinates.

The ellipto-spherical coordinate system $(\rho,\theta_\varepsilon,\varphi_\varepsilon)$ combines the ellipsoidal variable that specifies the family of confocal ellipsoids with the eccentric angular variables of the spherical system. It is defined by

$$\begin{aligned} x_1&=\rho\cos\theta_\varepsilon & h_2&\le\rho<+\infty \\ x_2&=\sqrt{\rho^2-h_3^2}\sin\theta_\varepsilon\cos\varphi_\varepsilon & 0&\le\theta_\varepsilon\le\pi \\ x_3&=\sqrt{\rho^2-h_2^2}\sin\theta_\varepsilon\sin\varphi_\varepsilon & 0&\le\varphi_\varepsilon<2\pi \end{aligned} \qquad (4)$$

In ellipsoidal coordinates, the surface S_e given in (1) corresponds to $\rho=\alpha_1$ and it represents the boundary of the brain. The interior space V^- is defined by the interval $\rho\in[h_2,a_1)$ and it is characterized by the conductivity σ. The exterior to S_e, non-conductive space V^+ is defined by $\rho\in(a_1,\infty)$.

II. Spherical system

We assume now that the domain Ω is a sphere of radius a centered at the origin. Let S_s denotes a sphere which in rectangular coordinates is specified by

$$x_1^2+x_2^2+x_3^2=a^2 \qquad (5)$$

In spherical coordinates, the surface S_s given in (5) corresponds to $r=a$ and it represents the boundary of the brain. The interior space V^- is defined by the interval

$r \in [0, a)$ and it is characterized by the conductivity σ. The exterior to S_s, non-conductive space V^+ is defined by $r \in (a, \infty)$.

In the interior of a homogeneous ellipsoidal/spherical conductor V^- there exists a primary current dipole source given by

$$J^P(r) = Q\delta(r - r_0) \tag{6}$$

where δ stands for the Dirac measure at a fixed point $\mathbf{r}_0$ with a dipole moment equal to **Q**.

The primary current $J^P(r)$ induces an electric field **E** in the interior conductive space, which in turn generates an inductive volume current with density $J^P(r)$

$$J^v(r) = \sigma \cdot E(r) \tag{7}$$

resulting to the total current density

$$J(r) = J^P(r) + J^v(r) \tag{8}$$

The current **J** generates an electromagnetic wave, which propagates in the interior as well as in the exterior to the conducting space. Because of the values of the dielectric constant and the electric conductivity of the brain tissue, quasistatic approximation of Maxwell's equations is considered [17, 18]. Therefore the electric field **E** and the magnetic induction field **B** satisfy the following equations

$$\nabla \times E = 0 \tag{9}$$

$$\nabla \times B = \mu_0 J \tag{10}$$

$$\nabla \cdot E = 0$$

(11)

$$\nabla \cdot B = 0$$

(12)

where μ_0 is the value of the magnetic permeability in the whole space.

Since **E** is irrotational, it can be represented by an electric potential u, such that

$$E(r) = -\nabla u(r)$$

(13)

The potential u is the field recorded in any electroencephalogram. In particular, we denote the electric potential in the interior space V^- by u^- and in the exterior space V^+ by u^+ . Combining equations (8), (13) and (10) we obtain the Poisson equation

$$\Delta u^-(r) = \frac{1}{\sigma} \nabla \cdot J^p(r) \qquad r \in V^-$$

(14)

which the interior potential u- must satisfy in V- .

In the source-free space V+ the potential u+ and solve the Laplace equation

$$\Delta u^+(r) = 0 \qquad r \in V^+$$

(15)

On the surface S the following transmission conditions hold

$$u^+(r) = u^-(r) \qquad r \in S$$

(16)

$$\frac{\partial u^-(r)}{\partial \rho} = 0 \qquad r \in S$$

(17)

where the outward normal differentiation on the surface is considered. Conditions (16)-(17) state the continuity of the potential function as well as the continuity of the normal component of current density on S.

In addition the asymptotic behavior at infinity

$$u^{+}(r)=O\left(\frac{1}{r}\right) \qquad r \to \infty$$

(18)

Separation of variables for Laplace's equation in the ellipsoidal coordinate system leads to the Lamé equation

$$(x^2-h_3^2)(x^2-h_2^2)E''(x)+x(2x^2-h_3^2-h_2^2)E'(x)+[(h_3^2+h_2^2)P-n(n+1)x^2]E(x)=0$$

(19)

for each one of the factors E(ρ), E(μ) and E(ν) that form the interior harmonic function

$$\mathrm{IE}_n^m(\rho,\mu,v)==E_n^m(\rho)E_n^m(\mu)E_n^m(v)$$

(20)

In Eq. (19) the parameters P and n are constants that define, in a complicated way, the degree n and the order m of the interior ellipsoidal harmonic (20).

The corresponding exterior ellipsoidal harmonic assumes the form

$$IF_n^m(\rho,\mu,v)==(2n+1)IE_n^m(\rho,\mu,v)I_n^m(\mu)$$

(21)

The ρ-dependent functions $I_n^m(\rho)$ are elliptic integrals of the form

$$I_n^m(\rho)=\int_{\rho}^{+\infty}\frac{dt}{[E_n^m(t)]^2\sqrt{|t^2-h_3^2||t^2-h_2^2|}}$$

(22)

for each n = 0, 1, 2, …, and m = 1, 2,..., 2n + 1.

The products $E_n^m(\mu),E_n^m(v)$ defined on the surface of any specific ellipsoid, are known as surface ellipsoidal harmonics and they form a complete orthogonal set of surface eigenfunctions, with respect to the weighting function

$$l_{a_1}(\mu, \nu) = [(a_1^2 - \mu^2)(a_1^2 - \nu^2)]^{-1/2}$$
(23)

corresponding to the ellipsoid $\rho = a_1$.

We define the normalization constants γ_n^m as

$$\gamma_n^m = \int_{\rho=\alpha_1} \left[E_n^m(\mu) E_n^m(\nu) \right]^2 l_{a_1}(\mu, \nu) ds$$
(24)

In the present work the following normalization constants are to be used:

$$\gamma_0^1 = 4\pi$$
(25)

$$\gamma_1^m = \frac{4\pi}{3} \frac{h_1^2 h_2^2 h_3^2}{h_m^2} \qquad m = 1,2,3$$
(26)

$$\gamma_2^1 = -\frac{8\pi}{5} (\Lambda - \Lambda')(\Lambda - \alpha_1^2)(\Lambda - \alpha_2^2)(\Lambda - \alpha_3^2)$$
(27)

$$\gamma_2^2 = \frac{8\pi}{5} (\Lambda - \Lambda')(\Lambda' - \alpha_1^2)(\Lambda' - \alpha_2^2)(\Lambda' - \alpha_3^2)$$
(28)

$$\gamma_2^{6-m} = \frac{4\pi}{15} h_1^2 h_2^2 h_3^2 h_m^2 \qquad m = 1,2,3$$
(29)

where

$$\left.\begin{matrix} \Lambda \\ \Lambda' \end{matrix}\right\} = \frac{1}{3} \sum_{i=1}^{3} a_i^2 \pm \frac{1}{3} \left[\sum_{i=1}^{3} \left(a_i^4 - \frac{a_1^2 a_2^2 a_3^2}{a_i^2} \right) \right]^{1/2}$$
(30)

2. The Exterior Electric Potential

In order to apply the transimission conditions to the problem, one has to assume that the solution has an eigenfunction expansion in the appropriate coordinate system. The first step is to determine the coefficients of u^- by solving the interior problem (14), (17). Then the value of u^+ on the ellipsoidal surface S provides the Dirichlet data in order to solve the exterior Dirichlet problem (15), (16), (18). The basic notation for the spectral decomposition of the Laplace operator in ellipsoidal coordinates can be found in [4, 5], where the ellipsoidal harmonics $IE_n^m(\rho,\mu,v)$ and $IF_n^m(\rho,\mu,v)$ that are used in this work as well as useful relations connecting them, can be found.

The solution of (15) is an exterior harmonic function that assumes the:

- **ellipsoidal** expansion

$$u_e^+(\rho,\mu,v)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} k_n^m IF_n^m(\rho,\mu,v), \qquad \rho>\alpha_1, \qquad (31)$$

- **spherical** expansion

$$u_s^+(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \hat{k}_n^m \frac{1}{r^{n+1}} Y_n^m(\theta,\varphi), \qquad r>a, \qquad (32)$$

And satisfy automatically the asymptotic condition (18).

3. Connection between algorithms

The exterior EEG potential in **spherical coordinates** (32) at a fixed point

A(r,θ,φ), r = γ > α is,

$$u_{s,A}^{+}(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\hat{k}_n^m\frac{1}{\gamma^{n+1}}Y_n^m(\theta,\varphi)\Rightarrow$$

$$u_{s,A}^{+}(r,\theta,\varphi)=Y_0^0(\theta,\varphi)\frac{1}{\gamma}\hat{k}_0^0+Y_1^{-1}(\theta,\varphi)\frac{1}{\gamma^2}\hat{k}_1^{-1}+Y_1^1(\theta,\varphi)\frac{1}{\gamma^2}\hat{k}_1^1+Y_1^0(\theta,\varphi)\frac{1}{\gamma^2}\hat{k}_1^0$$

$$+Y_2^{-2}(\theta,\varphi)\frac{1}{\gamma^3}\hat{k}_2^{-2}+Y_2^{-1}(\theta,\varphi)\frac{1}{\gamma^3}\hat{k}_2^{-1}+Y_2^0(\theta,\varphi)\frac{1}{\gamma^3}\hat{k}_2^0+Y_2^1(\theta,\varphi)\frac{1}{\gamma^3}\hat{k}_2^1+Y_2^2(\theta,\varphi)\frac{1}{\gamma^3}\hat{k}_2^2\Rightarrow$$

$$u_{s,A}^{+}(r,\theta,\varphi)=\frac{1}{\sqrt{4\pi}}\frac{1}{\gamma}\hat{k}_0^0+\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^{-1}x_3+\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^1x_2+\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^0x_1$$

$$+\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^{-2}x_2x_3+\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^{-1}x_1x_3+\frac{\sqrt{5}}{\sqrt{16\pi}}\frac{1}{\gamma^5}\hat{k}_2^0(3x_1^2-\gamma^2)+\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^1x_2x_1+\frac{\sqrt{15}}{\sqrt{16\pi}}\frac{1}{\gamma^5}\hat{k}_2^2(x_2^2-x_3^2)$$

$$(33)$$

for n=0,1,2 and m = -2,0,1,2.

The exterior EEG potential in **ellipsoidal coordinates** (31) at a fixed point A(γ,θ,φ)=A(ρ,μ,ν) is:

$$u_{e,A}^{+}(\rho,\mu,\nu)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}k_n^m\, IF_n^m(\rho,\mu,\nu)\quad\Rightarrow$$

$$u_{e,A}^{+}(\rho,\mu,\nu)=k_0^1 IF_0^1(\rho,\mu,\nu)+k_1^1 IF_1^1(\rho,\mu,\nu)+k_1^2 IF_1^2(\rho,\mu,\nu)+k_1^3 IF_1^3(\rho,\mu,\nu)+$$

$$k_2^1 IF_2^1(\rho,\mu,\nu)+k_2^2 IF_2^2(\rho,\mu,\nu)+k_2^3 IF_2^3(\rho,\mu,\nu)+k_2^4 IF_2^4(\rho,\mu,\nu)+k_2^5 IF_2^5(\rho,\mu,\nu)\Rightarrow$$

$$u_{e,A}^{+}(\rho,\mu,\nu)=k_0^1 I_0^1(\rho)+3k_1^1 I_1^1(\rho)h_2h_3x_1+3k_1^2 I_1^2(\rho)h_1h_3x_2+3k_1^3 I_1^3(\rho)h_2h_1x_3+$$

$$5k_2^1 I_2^1(\rho)IL\left(\sum_{n=1}^{3}\frac{x_n^2}{\Lambda-a_n^2}+1\right)+5k_2^2 I_2^2(\rho)IL'\left(\sum_{n=1}^{3}\frac{x_n^2}{\Lambda'-a_n^2}+1\right)$$

$$+5k_2^3 I_2^3(\rho)h_1h_2h_3^2x_1x_2+5k_2^4 I_2^4(\rho)h_1h_3h_2^2x_1x_3+5k_2^5 I_2^5(\rho)h_3h_2h_1^2x_3x_2 \qquad (34)$$

for n=0,1,2 and m=1, 2, …, 2n+1.

If we assume at the point A:

$$u_{s,A}^{+}(r,\theta,\varphi)=u_{e,A}^{+}(\rho,\mu,v) \qquad (35)$$

We obtain the following connected relations:

$$\frac{1}{\sqrt{4\pi}}\frac{1}{\gamma}\hat{k}_0^0-\frac{\sqrt{5}}{\sqrt{16\pi}}\frac{1}{\gamma^3}\hat{k}_2^0=k_0^1 I_0^1(\rho)+5k_2^1 I_2^1(\rho)IL+5k_2^2 I_2^2(\rho)IL', \qquad (36)$$

$$\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^0=3k_1^1 I_1^1(\rho)h_2 h_3, \qquad (37)$$

$$\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^1=3k_1^2 I_1^2(\rho)h_1 h_3, \qquad (38)$$

$$\frac{\sqrt{3}}{\sqrt{4\pi}}\frac{1}{\gamma^3}\hat{k}_1^{-1}=3k_1^3 I_1^3(\rho)h_2 h_1, \qquad (39)$$

$$\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^{-2}=5k_2^5 I_2^5(\rho)h_3 h_2 h_1^2, \qquad (40)$$

$$\Rightarrow ¿¿\hat{k}_2^2=\frac{10\sqrt{\pi}}{\sqrt{15}}h_1^2\gamma^5\left(k_2^1 I_2^1(\rho)\left(\Lambda-\alpha_1^2\right)+k_2^2 I_2^2(\rho)\left(\Lambda'-\alpha_1^2\right)\right)¿¿$$

$$\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^{-1}=5k_2^4 I_2^4(\rho)h_1 h_3 h_2^2, \qquad (41)$$

$$\frac{\sqrt{15}}{\sqrt{4\pi}}\frac{1}{\gamma^5}\hat{k}_2^1=5k_2^3 I_2^3(\rho)h_1 h_2 h_3^2, \qquad (42)$$

$$3\frac{\sqrt{5}}{\sqrt{16\pi}}\frac{1}{\gamma^5}\hat{k}_2^0=5k_2^1 I_2^1(\rho)IL\frac{1}{\Lambda-\alpha_1^2}+5k_2^2 I_2^2(\rho)IL'\frac{1}{\Lambda'-\alpha_1^2}, \qquad (43)$$

$$\frac{\sqrt{15}}{\sqrt{16\pi}}\frac{1}{\gamma^5}\hat{k}_2^2=5k_2^1 I_2^1(\rho)IL\frac{1}{\Lambda-\alpha_2^2}+5k_2^2 I_2^2(\rho)IL'\frac{1}{\Lambda'-\alpha_2^2}, \qquad (44) ¿ \Big\} ¿ \Big| ¿$$

¿

4. **The inverse EEG problem for a dipole**

- Let the brain be represented by the reference ellipsoid (1) and let a localized neuronal current which is represented by a dipole located at the point $\boldsymbol{r}_0$, inside the brain, having moment $\boldsymbol{Q}$. The unique solution for the inverse EEG problem of a current dipole inside an ellipsoid (1) is given by

$$Q_e=\left(\frac{4\pi\sigma\, a_2 a_3 h_2 h_3}{3} k_1^1, \frac{4\pi\sigma\, a_1 a_3 h_1 h_3}{3} k_1^2, \frac{4\pi\sigma\, a_2 a_1 h_2 h_1}{3} k_1^3\right) \qquad (46)$$

$$r_{0,e}=\frac{h_1 h_2 h_3}{5}\left(\frac{a_1^2+a_3^2}{a_1 h_1}\frac{k_2^4}{k_1^3}, \frac{a_1^2+a_2^2}{a_2 h_2}\frac{k_2^3}{k_1^1}, \frac{a_2^2+a_3^2}{a_3 h_3}\frac{k_2^5}{k_1^2}\right) \qquad (47)$$

Now we use the inversion algorithm for the ellipsoid without using the intrinsic ellipsoidal data k_n^m but the data $\hat{k}_n^m$ as they are expressed in terms of k_n^m

From (37)-(39) and (40)-(42) we have,

$$\hat{Q}_{eap}=\left(\frac{4\pi\sigma\, a_2 a_3\sqrt{3}}{9\gamma^3\sqrt{4\pi}}\frac{\hat{k}_1^0}{I_1^1(\rho)}, \frac{4\pi\sigma\, a_1 a_3\sqrt{3}}{9\gamma^3\sqrt{4\pi}}\frac{\hat{k}_1^1}{I_1^2(\rho)}, \frac{4\pi\sigma\, a_1 a_2\sqrt{3}}{9\gamma^3\sqrt{4\pi}}\frac{\hat{k}_1^{-1}}{I_1^3(\rho)}\right) \qquad (48)$$

$$\hat{r}_{0,eap}=\left(\frac{a_1^2+a_3^2}{25a_1}\frac{3\sqrt{5}I_1^3(\rho)}{\gamma^2 I_2^4(\rho)}\frac{\hat{k}_2^{-1}}{\hat{k}_1^{-1}}, \frac{a_1^2+a_2^2}{25a_2}\frac{3\sqrt{5}I_1^1(\rho)}{\gamma^2 I_2^3(\rho)}\frac{\hat{k}_2^1}{\hat{k}_1^0}, \frac{a_2^2+a_3^2}{25a_3}\frac{3\sqrt{5}I_1^2(\rho)}{\gamma^2 I_2^5(\rho)}\frac{\hat{k}_2^{-2}}{\hat{k}_1^1}\right) \qquad (49)$$

- Let the brain be represented by the reference sphere (5) and let a localized neuronal current which is represented by a dipole located at the point $\boldsymbol{r}_0$, inside the brain, having moment $\boldsymbol{Q}$. The unique solution for the inverse EEG problem of a current dipole inside a sphere (5) is given by

$$Q_s=\left(\frac{\sqrt{2\pi}}{\sqrt{3}}\sigma\alpha^2(\hat{k}_1^{-1}+\hat{k}_1^1), i\frac{\sqrt{2\pi}}{\sqrt{3}}\sigma\alpha^2(\hat{k}_1^{-1}-\hat{k}_1^1), \frac{\sqrt{4\pi}}{\sqrt{3}}\sigma\alpha^2\hat{k}_1^0\right) \qquad (50)$$

$$r_{0,s}=\frac{a}{\sqrt{5}}\left(\frac{\hat{k}_2^{-2}}{\hat{k}_1^{-1}}+\frac{\hat{k}_2^2}{\hat{k}_1^1}, i\left(\frac{\hat{k}_2^{-2}}{\hat{k}_1^{-1}}-\frac{\hat{k}_2^2}{\hat{k}_1^1}\right), 2\frac{\hat{k}_2^1}{\hat{k}_1^1}-\sqrt{2}\frac{\hat{k}_1^0}{\hat{k}_1^1}\frac{\hat{k}_2^2}{\hat{k}_1^1}\right) \qquad (51)$$

Now we use the inversion algorithm for the sphere without using the intrinsic spherical data $\hat{k}_n^m$ but the data k_n^m as they are expressed in terms of $\hat{k}_n^m$,

From (37)-(39) and (40)-(42), (44), (45) we have,

$$\hat{Q}_{sap} = \text{¿} \left(2\sqrt{2\pi}\sigma\alpha^2\gamma^3 \left(k_1^3 I_1^3(\rho) h_2 h_1 + k_1^2 I_1^2(\rho) h_1 h_3 \right), \text{¿} \right) \left(i2\sqrt{2\pi}\sigma\alpha^2\gamma^3 \left(k_1^3 I_1^3(\rho) h_2 h_1 - k_1^2 I_1^2(\rho) h_1 h_3 \right), \text{¿} \text{¿} \right) \text{¿}\text{¿}\text{¿}$$

$$\text{¿}$$

$$\text{¿}$$

5. **Numerical implementation**

I. We assume that there is a dipole (6) inside the ellipsoid (1) with semi-axes $\alpha_1 = 9\,cm, a_2 = 6.5\,cm, a_3 = 6\,cm$ and inside the sphere (5) of radius $r = \alpha = (\alpha_1 \cdot \alpha_2 \cdot \alpha_3)^{1/3} = 7.054\,cm$. Now at the point A(r,θ,φ)=A(15,30,30) or equivalently A(ρ, μ, ν)=A(ρ, θ_ε ,φ_ε)=(15.3808, 32.85, 30.32) we assume that

$$u_{s,A}^{+}(r,\theta,\varphi) = u_{e,A}^{+}(\rho,\mu,v) = c_1$$

From (46), (47) we have,

$$Q = (Q_1 = 104.44\,c_1, \quad Q_2 = 79.07\,c_1, \quad Q_3 = 50.26\,c_1)$$

$$r_0 = (r_{01} = 10.98, \quad r_{02} = 8.76, \quad r_{03} = 3.53)$$

From (48), (49) we have,

$$\hat{Q} = (\hat{Q}_1 = 96.10\,c_1, \quad \hat{Q}_2 = 59.72\,c_1, \quad \hat{Q}_3 = 36.65\,c_1)$$

$$\hat{r}_0 = (\hat{r}_{01} = 11.29, \quad \hat{r}_{02} = 7.43, \quad \hat{r}_{03} = 2.82)$$

From (50), (51) we have,

$$Q=\left(Q_1=71.15\,c_1,\quad Q_2=-19.06\,ic_1,\quad Q_3=127.59\,c_1\right)$$

$$r_0=\left(r_{01}=4.07,\quad r_{02}=2.03\,i,\quad r_{03}=9.34\right)$$

From (52), (53) we have,

$$\hat{Q}=\left(\hat{Q}_1=95.45\,c_1,\quad \hat{Q}_2=-24.01\,ic_1,\quad \hat{Q}_3=138.66\,c_1\right)$$

$$\hat{r}_0=\left(\hat{r}_{01}=4.99,\quad \hat{r}_{02}=2.39\,i,\quad \hat{r}_{03}=8.80\right)$$

II. We assume that there is a dipole (6) inside the ellipsoid (1) with semi-axes $\alpha_1=25\,cm, a_2=4\,cm, a_3=3.51\,cm$ and inside the sphere (5) of radius $r=\alpha=\left(\alpha_1\cdot\alpha_2\cdot\alpha_3\right)^{1/3}=7.054\,cm$. Now at the point A(r,θ,φ)=A(15,30,30) or equivalently A(ρ, μ, ν) =A(ρ, θ_{ε}, φ_{ε}) =(26.1652, 60.26, 30.54) we assume that

$$\overset{+}{u}_{s,A}(r,\theta,\varphi)=\overset{+}{u}_{e,A}(\rho,\mu,\nu)=c_1$$

From (46), (47) we have,

$$Q=\left(Q_1=35.83\,c_1,\quad Q_2=336.91\,c_1,\quad Q_3=227.28\,c_1\right)$$

$$r_0=\left(r_{01}=12.66,\quad r_{02}=119.70,\quad r_{03}=3.57\right)$$

From (48), (49) we have,

$$\hat{Q}=\left(\hat{Q}_1=64.33\,c_1,\quad \hat{Q}_2=48.66\,c_1,\quad \hat{Q}_3=31.17\,c_1\right)$$

$$\hat{r}_0=\left(\hat{r}_{01}=47.08,\quad \hat{r}_{02}=35.99\quad \hat{r}_{03}=0.73\right)$$

From (50), (51) we have,

$$Q=\left(Q_1=71.15\,c_1,\quad Q_2=-19.06\,ic_1,\quad Q_3=127.59\,c_1\right)$$

$$r_0=\left(r_{01}=4.07,\quad r_{02}=2.03\,i,\quad r_{03}=9.34\right)$$

From (52), (53) we have,

$$\hat{Q}=\left(\hat{Q}_1=502.26\,c_1,\quad \hat{Q}_2=-122.44\,ic_1,\quad \hat{Q}_3=71.06\,c_1\right)$$

$$\hat{r}_0=\left(\hat{r}_{01}=18.91,\quad \hat{r}_{02}=9.46\,i,\quad \hat{r}_{03}=2.19\right)$$

III. We assume that there is a dipole (6) inside the ellipsoid (1) with semi-axes $\alpha_1 = 12\,cm, a_2 = 11.7\,cm, a_3 = 2.5\,cm$ and inside the sphere (5) of radius $r = \alpha = (\alpha_1 \cdot \alpha_2 \cdot \alpha_3)^{1/3} = 7.054\,cm$. Now at the point A(r,θ,φ)=A(15,30,30) or equivalently A(ρ, μ, ν) =A(ρ, θ_{ε}, φ_{ε}) =(15.634, 33.9, 40.76) we assume that

$$u_{s,A}^{+}(r,\theta,\varphi) = u_{e,A}^{+}(\rho,\mu,\nu) = c_1$$

From (46), (47) we have,

$$Q = (Q_1 = 80.09\,c_1, \quad Q_2 = 41.68\,c_1, \quad Q_3 = 167.98\,c_1)$$
$$r_0 = (r_{01} = 10.40, \quad r_{02} = 10.12, \quad r_{03} = 20.79)$$

From (48), (49) we have,

$$\hat{Q} = (\hat{Q}_1 = 67.20\,c_1, \quad \hat{Q}_2 = 33.82\,c_1, \quad \hat{Q}_3 = 57.21\,c_1)$$
$$\hat{r}_0 = (\hat{r}_{01} = 10.30, \quad \hat{r}_{02} = 10.53, \quad \hat{r}_{03} = 8.32)$$

From (50), (51) we have,

$$Q = (Q_1 = 71.15\,c_1, \quad Q_2 = -19.06\,ic_1, \quad Q_3 = 127.59\,c_1)$$
$$r_0 = (r_{01} = 4.07, \quad r_{02} = 2.03\,i, \quad r_{03} = 9.34)$$

From (52), (53) we have,

$$\hat{Q} = (\hat{Q}_1 = 132.06\,c_1, \quad \hat{Q}_2 = 20.89\,ic_1, \quad \hat{Q}_3 = 152.06\,c_1)$$
$$\hat{r}_0 = (\hat{r}_{01} = 5.84, \quad \hat{r}_{02} = 0.55\,i, \quad \hat{r}_{03} = 4.12)$$

We observe (tables 1-8) that for ellipsoids with small principal eccentricities, approximate algorithms (48), (49), (52), (53) for Q and r0, give relatively good results. On the contrary as the eccentricities increase, approximate algorithms have a large deviation from the real data.

In the following graphs we observe that when the semi-axes approaches the realistic shape of the brain i.e. α_1 = 9cm, α_2 = 6.5cm, α_3 = 6cm, the errors calculated from algorithms (46)-(53) for Q and r_0 are not significant so the approximate solutions of those algorithms meet a certain quality guarantee. As the semi-axes grow the errors become bigger and there is a large difference between exact and approximate data (fig.1, fig.2, fig.3, fig.4). On the other hand, when we assume that the semi-axes diverge from the standard measurements of the brain i.e. α_1 = 25cm, α_2 = 4cm, α_3 = 3.51cm, the errors increase dramatically. In this case the approximate solutions through the algorithms (46)-(53) give bad results (fig.5, fig.6, fig.7, fig.8).

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Errors of Q_e as the semi-axes a1,a2,a3, increase.			
α1	α2	α3	$\|\|Q-Q_e\|\|_2$
9.0000	**6.5000**	**6.0000**	**25.0893c1**
9.5263	**7.0263**	**5.2439**	**30.6161c1**
10.0526	**7.5526**	**4.6231**	**38.0411c1**
10.5789	**8.0789**	**4.1069**	**47.7091c1**
11.1053	**8.6053**	**3.6729**	**60.0433c1**
11.6316	**9.1316**	**3.3046**	**75.6453c1**
12.1579	**9.6579**	**2.9893**	**95.4316c1**
12.6842	**10.1842**	**2.7172**	**120.8857c1**
13.2105	**10.7105**	**2.4807**	**154.6112c1**
13.7368	**11.2368**	**2.2739**	**201.7507c1**
14.2632	**11.7632**	**2.0920**	**274.5000c1**
14.7895	**12.2895**	**1.9312**	**413.1682c1**
15.3158	**12.8158**	**1.7882**	**970.9220c1**
15.8421	**13.3421**	**1.6606**	**738.7033c1**
16.3684	**13.8684**	**1.5462**	**504.8524c1**
16.8947	**14.3947**	**1.4433**	**426.8974c1**
17.4211	**14.9211**	**1.3503**	**387.8817c1**
17.9474	**15.4474**	**1.2661**	**365.3463c1**
18.4737	**15.9737**	**1.1895**	**351.5686c1**
19.0000	**16.5000**	**1.1196**	**343.1108c1**

Table 1: shows how the error of Q_e, grows as the semi-axes increase

Errors of r_e as the semi-axes a1,a2,a3, increase.			
α1	α2	α3	$\|\|r-r_e\|\|_2$
9.000	**6.500**	**6.000**	**1.535**
9.526	**7.026**	**5.244**	**1.932**
10.053	**7.553**	**4.623**	**2.496**
10.579	**8.079**	**4.107**	**3.293**
11.105	**8.605**	**3.673**	**4.420**
11.632	**9.132**	**3.305**	**6.012**
12.158	**9.658**	**2.989**	**8.257**
12.684	**10.184**	**2.717**	**11.445**
13.211	**10.711**	**2.481**	**16.051**
13.737	**11.237**	**2.274**	**22.972**
14.263	**11.763**	**2.092**	**34.249**
14.789	**12.289**	**1.931**	**56.404**
15.316	**12.816**	**1.788**	**144.950**
15.842	**13.342**	**1.661**	**120.070**
16.368	**13.868**	**1.546**	**88.468**
16.895	**14.395**	**1.443**	**80.449**

17.421	14.921	1.350	78.423
17.947	15.447	1.266	79.071
18.474	15.974	1.190	81.273
19.000	16.500	1.120	84.544

Table 2: shows how the error of r_e, grows as the semi-axes increase.

Errors of Q_s as the semi-axes a1,a2,a3 increase.			
α1	α2	α3	$\|\|Q-Q_s\|\|_2$
9.0000	6.5000	6.0000	27.1538c1
9.5263	7.0263	5.2439	33.6629c1
10.0526	7.5526	4.6231	40.7378c1
10.5789	8.0789	4.1069	48.6183c1
11.1053	8.6053	3.6729	57.6545c1
11.6316	9.1316	3.3046	68.3387c1
12.1579	9.6579	2.9893	81.3913c1
12.6842	10.1842	2.7172	97.9331c1
13.2105	10.7105	2.4807	119.8607c1
13.7368	11.2368	2.2739	150.7978c1
14.2632	11.7632	2.0920	199.1315c1
14.7895	12.2895	1.9312	292.3014c1
15.3158	12.8158	1.7882	671.4706c1
15.8421	13.3421	1.6606	502.2536c1
16.3684	13.8684	1.5462	335.9783c1
16.8947	14.3947	1.4433	275.7967c1
17.4211	14.9211	1.3503	241.4188c1
17.9474	15.4474	1.2661	217.6606c1
18.4737	15.9737	1.1895	199.4899c1
19.0000	16.5000	1.1196	184.7909c1

Table 3 : shows how the error of Qs, grows as the semi-axes increase.

Errors of r_s as the semi-axes a1,a2,a3 increase.			
α1	α2	α3	$\|\|r-r_s\|\|_2$
9.000	6.500	6.000	1.124
9.526	7.026	5.244	1.798
10.053	7.553	4.623	2.556
10.579	8.079	4.107	3.362
11.105	8.605	3.673	4.221
11.632	9.132	3.305	5.142
12.158	9.658	2.990	6.137
12.684	10.184	2.717	7.212
13.211	10.711	2.481	8.369
13.737	11.237	2.274	9.597
14.263	11.763	2.092	10.873
14.789	12.289	1.931	12.161
15.316	12.816	1.788	13.414
15.842	13.342	1.661	14.583
16.368	13.868	1.546	15.627
16.895	14.395	1.443	16.521

17.421	14.921	1.350	17.255
17.947	15.447	1.266	17.835
18.474	15.974	1.190	18.275
19.000	16.500	1.120	18.593

Table 4 : shows how the error of r_s, grows as the semi-axes increase

Errors of Q_e as the semi-axes a1,a2,a3, increase.			
α1	α2	α3	$\|\|Q-Q_e\|\|_2$
25.0000	4.0000	3.5100	349.8009c1
25.5263	4.5263	3.0379	438.7780c1
26.0526	5.0526	2.6665	765.6380c1
26.5789	5.5789	2.3671	965.6468c1
27.1053	6.1053	2.1210	717.6692c1
27.6316	6.6316	1.9155	509.4681c1
28.1579	7.1579	1.7415	441.5583c1
28.6842	7.6842	1.5924	409.0694c1
29.2105	8.2105	1.4635	392.1303c1
29.7368	8.7368	1.3510	383.6136c1
30.2632	9.2632	1.2521	380.2329c1
30.7895	9.7895	1.1645	380.2447c1
31.3158	10.3158	1.0865	382.6367c1
31.8421	10.8421	1.0167	386.7827c1
32.3684	11.3684	0.9539	392.2780c1
32.8947	11.8947	0.8971	398.8523c1
33.4211	12.4211	0.8455	406.3213c1
33.9474	12.9474	0.7986	414.5579c1
34.4737	13.4737	0.7557	423.4747c1
35.0000	14.0000	0.7163	433.0119c1

Table 5: shows how the error of Q_e, grows as the semi-axes increase

Errors of r_e as the semi-axes a1,a2,a3, increase.			
α1	α2	α3	$\|\|r-r_e\|\|_2$
25.000	4.000	3.510	90.553
25.526	4.526	3.038	104.680
26.053	5.053	2.667	135.440
26.579	5.579	2.367	254.190
27.105	6.105	2.121	235.550
27.632	6.632	1.916	144.530
28.158	7.158	1.742	116.130
28.684	7.684	1.592	102.450
29.211	8.211	1.464	95.129
29.737	8.737	1.351	91.316
30.263	9.263	1.252	89.743
30.789	9.790	1.165	89.740
31.316	10.316	1.087	90.917
31.842	10.842	1.017	93.026
32.368	11.368	0.954	95.905
32.895	11.895	0.897	99.444
33.421	12.421	0.846	103.570
33.947	12.947	0.799	108.230
34.474	13.474	0.756	113.380

35.000	14.000	0.716	119.020

Table 6: shows how the error of r_e, grows as the semi-axes increase

Errors of Q_s as the semi-axes a1,a2,a3, increase			
α1	α2	α3	$\|\|Q-Q_s\|\|_2$
25.000	4.000	3.510	446.910c1
25.526	4.526	3.038	569.000c1
26.053	5.053	2.667	902.670c1
26.579	5.579	2.367	1332.800c1
27.105	6.105	2.121	1104.900c1
27.632	6.632	1.916	695.430c1
28.158	7.158	1.742	551.020c1
28.684	7.684	1.592	468.520c1
29.211	8.211	1.464	412.110c1
29.737	8.737	1.351	369.680c1
30.263	9.263	1.252	335.920c1
30.789	9.790	1.164	308.150c1
31.316	10.316	1.087	284.840c1
31.842	10.842	1.017	265.010c1
32.368	11.368	0.954	248.030c1
32.895	11.895	0.897	233.430c1
33.421	12.421	0.846	220.830c1
33.947	12.947	0.799	209.940c1
34.474	13.474	0.756	200.530c1
35.000	14.000	0.716	192.390c1

Table 7: shows how the error of Q_s, grows as the semi-axes increase

Errors of r_s as the semi-axes a1,a2,a3 increase.			
α1	α2	α3	$\|\|r-r_s\|\|_2$
25.000	4.000	3.510	18.073
25.526	4.526	3.038	24.390
26.053	5.053	2.667	37.299
26.579	5.579	2.367	81.835
27.105	6.105	2.121	84.467
27.632	6.632	1.916	59.531
28.158	7.158	1.742	53.559
28.684	7.684	1.592	51.527
29.211	8.211	1.464	50.842
29.737	8.737	1.351	50.634
30.263	9.263	1.252	50.540
30.789	9.790	1.165	50.404
31.316	10.316	1.087	50.167
31.842	10.842	1.017	49.822
32.368	11.368	0.954	49.381
32.895	11.895	0.897	48.865
33.421	12.421	0.846	48.297
33.947	12.947	0.799	47.695
34.474	13.474	0.756	47.077
35.000	14.000	0.716	46.455

Table 8: shows how the error of r_s, grows as the semi-axes increase