LOCAL MILD SOLUTIONS TO THREE-DIMENSIONAL MAGNETOHYDRODYNAMIC SYSTEM IN MORREY SPACES

ZIRONG ZENG

ABSTRACT. In this article, the Cauchy problem of three-dimensional (3-D) incompressible magnetohydrodynamic system with infinite energy initial data is investigated. Via some elaborate analysis of the time evolution of both the vorticity $\omega := \nabla \times u$ and the current density $j =: \nabla \times b$, the local-in-time well-posedness of mild solutions with arbitrarily large initial data in Morrey spaces is established.

1. INTRODUCTION

Magnetohydrodynamic (MHD) theory discusses the motion of plasma (a conductive fluid which consist of electrons), ions and neutral particles in an electromagnetic field. In the MHD system, there is a strong interaction between the dynamic motion of fluid and magnetic field. So researchers use the equations which combined with the Navier-Stokes equations and Maxwell's equations to describe the MHD system. There is a wide range of applications of magnetohydrodynamic in many fields, for example, the magnetic drug targeting and cancer tumour treatment in biomedical engineering, the motion of liquid metals models in physics and so on. In this paper, we consider the viscous incompressible magnetohydrodynamic system, namely

$$\begin{cases} \partial_t u - \nu \triangle u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\Big(p + \frac{1}{2}|b|^2\Big) = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \partial_t b - \eta \triangle b + \nabla \times (b \times u) = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \mathrm{div} u = 0, \quad \mathrm{div} b = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \end{cases} \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \ \ x \in \mathbb{R}^3,$$

where $b(x,t)$, $u(x,t)$ and $p(x,t)$ denote the magnetic field, fluid velocity and pressure field, respectively. In the equations, ν is the kinematic viscosity coefficient and η is the magnetic diffusion. Without lose of generality, we assume that $\nu = \eta = 1$ for simplicity. Here we focus on the case which the initial vortex profiles are vortex rings and filaments. On this occasion, the initial

Date: October 6, 2020.

Key words and phrases. Incompressible magnetohydrodynamic equations, Morrey spaces, mild solutions.

2010 Mathematics Subject Classification. 35B40, 35L60, 35Q35.

data is of infinite energy (i.e. $u_0, b_0 \notin L^2$). Thus, the commonly used classical energy method and Leray's method are no longer available.

There are quite a lot of literature investigate the incompressible Navier-Stokes equations. Kato-Fujita [6] firstly established the existence of mild solutions in $H^s(\mathbb{R}^d)$ space. Considering the $L^p(\mathbb{R}^d)$ space, Kato in [11] proved a locally well-posed result for arbitrary initial data and a global well-posed result when the initial data is small enough. This result was subsequently extended to some different function spaces which have larger scale, for example, the homogeneous Besov space $B^{d/p-1}_{p,\infty}(\mathbb{R}^d)$ by Cannone [2] and Planchon [17], the BMO^{-1} space by Koch-Tataru [13] and the space χ^{-1} by Lei-Lin [16]. In order to overcome the inconvenience of the finite total variation assumption on the initial data, Giga and Miyakawa [8] introduced the Morrey space $\mathcal{M}^p(\mathbb{R}^3)$ and constructed the global existence of solutions to the N-S equations. Their result then extended by Kato [12] to a more general Morrey space $\mathcal{M}^{p,q}(\mathbb{R}^3)$. Instead of considering the vorticity, Kato in [8] directly analysed the velocity u and established a global existence and uniqueness result. For more related reference about incompressible N-S equations in measure spaces, we refer to [4, 7, 9, 14, 20].

Due to the fact that there exists a strong coupling effect between u and b, the MHD system is more complicated. For the well-posedness result of incompressible MHD system, we refer to [5, 10, 15, 18] and the reference therein. When the initial data is of finite energy, Duvaut and Lions [5] constructed the local well-posedness result of a solution in $H^s(\mathbb{R}^d)$ space. They also proved the global existence provided the initial data is sufficiently small. Sermange and Temam in [18] established the local existence and uniqueness of a strong solution to the MHD equations. Moreover, in [18] they also proved the global well-posedness of the solution in two dimensional case. Here we would like to point out that the uniqueness of the global weak solution in three dimensional case is still a challenging open problem. Considering the axisymmetric initial data, Lei in [15] provided the first example of large global solutions to the ideal MHD equations.

The goal of the present paper is to analyse the local-in-time well-posedness of mild solutions to the Cauchy problem (1.1) in Morrey space $\mathcal{M}^p(\mathbb{R}^3)$. First of all, since the commonly used Leray operator $(\mathbb{P})_{k,j}$ ($(\mathbb{P})_{k,j} = \delta_{kj} + \mathcal{R}_k\mathcal{R}_j$ with $\mathcal{R}_k = \partial_k(-\triangle)^{1/2}$, $k = 1,2,3$) is unbounded in $\mathcal{M}^p(\mathbb{R}^3)$. Here following the analogue analysis of [8], we apply the *curl* operator to (1.1) to eliminate the pressure term $\nabla\left(p + \frac{1}{2}|b|^2\right)$ and investigate the following equations of the vorticity $\omega := \nabla \times u$ and the current density $j =: \nabla \times b$:

$$
\begin{cases}
\partial_t \omega - \triangle \omega + \partial_{x_i}(u^i \omega - \omega^i u - b^i j + j^i b) = 0, \\
\partial_t j - \triangle j + \nabla \times ((u\cdot\nabla) b - (b\cdot\nabla) u) = 0, \\
u = K * \omega, \quad b = K * j, \\
\operatorname{div}\omega = \operatorname{div} j = 0, \\
\omega(x,0) = \omega_0, \quad j(x,0) = j_0.
\end{cases}
\tag{1.2}
$$

Here $K(x) = -\frac{1}{4\pi}\frac{(x_1,x_2,x_3)}{|x|^3}$ denotes the Biot-Savart kernel.

Due to the existence of strong nonlinear coupling terms between ω and j, the methods which used in [8] to consider the NS equations could not directly apply to our case. Moreover, since we can not use $\|\nabla\times u\|_p$ and $\|\nabla\times b\|_p$ to control $\|\nabla u\|_p$ and $\|\nabla b\|_p$, there arises some new critical difficulties compared with the pure NS system.

Here we take advantage of the following crux estimates to resolve these issues:

$$\|(u\cdot\nabla)b\|_p \le \|u\|_\infty\|\nabla b\|_p \le \|\omega\|_p\|\omega\|_{2p}\|\nabla b\|_p,$$

and

$$\|(b\cdot\nabla)u\|_p \le \|\nabla u\|_\infty\|b\|_p \le \|\nabla\omega\|_p\|\nabla\omega\|_{2p}\|b\|_p.$$

Therefore, in order to use the Banach fixed point Theorem, we now need to estimate $\|\nabla b\|_p$, $\|\nabla\omega\|_p$ and $\|b\|_p$.

At first, in order to deal with $\|\nabla b\|_p$, we introduce the equation of ∇b:

$$\partial_t\nabla b - \eta\triangle\nabla b + \nabla((u\cdot\nabla)b - (b\cdot\nabla)u) = 0,$$

which enables us to establish the estimate of $\|\nabla b\|_p$.

Then, the estimate on $\|\nabla\omega\|_p$ can be obtained via the following high order equations:

$$\begin{cases}\partial_t\nabla\omega - \nu\triangle\nabla\omega + \nabla((u\cdot\nabla)\omega - (\omega\cdot\nabla)u - (b\cdot\nabla)j + (j\cdot\nabla)b) = 0,\\ \partial_t\nabla j - \eta\triangle\nabla j + \nabla((u\cdot\nabla)j + \nabla u^i\times b_{x_i} - (b\cdot\nabla)\omega - \nabla b^i\times u_{x_i}) = 0.\end{cases}$$

At the end, we use the equation of b

$$\partial_t b - \eta\triangle b + \nabla\times(b\times u) = 0,$$

to estimate $\|b\|_p$.

The rest of the paper is organized as follows. In Section 2, we will give some important definitions, properties and inequalities in Morrey spaces and state the main results of this paper. Section 3 is devoted to proving our main result Theorem 2.4, which will be achieved by three steps:

(1) Step (i): in Subsection 3.1, we introduce the equation of ∇b;
(2) Step (ii): in Subsection 3.2, we make full use of the standard successive approximation scheme to establish the iteration estimates;
(3) Step (iii): in Subsection 3.3, we apply Banach's fixed point Theorem to obtain the results.

2. Preliminaries and main results

2.1. Preliminary definitions

We start with the definition of Morrey space.

Definition 2.1. For $1 \leq p \leq \infty$, we define the Morrey space $\overline{\mathcal{M}}^p(\mathbb{R}^3)$ be the space of random measures μ such that

$$\|\mu\|_{\overline{\mathcal{M}}^p(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3, r>0} r^{-\frac{3}{p'}} |\mu|\left(B(x,r)\right) < \infty.$$

Here $B(x,r)$ is the open ball in $\mathbb{R}^3$ with radius r centered at x and $|\mu|(B(x,r))$ is the total variation of μ.

Let us denote $\mathcal{M}^p(\mathbb{R}^3) = \overline{\mathcal{M}}^p(\mathbb{R}^3) \bigcap L^1_{loc}(\mathbb{R}^3)$ be the closed subspace of $\overline{\mathcal{M}}^p(\mathbb{R}^3)$. Here and after, for convenience, we adopt the notation $\|\cdot\|_p$ to represent the norm of the Morrey space $\mathcal{M}^p(\mathbb{R}^3)$. It can be proved that $\mathcal{M}^p(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_p$ is a Banach space. We also denote by $\mathcal{C}(x,y)$ the beta function

$$\mathcal{C}(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt,$$

with $x > 0$, $y > 0$.

Next we give the definition of the mild solution to (1.2).

Definition 2.2. (Mild Solution). Let $S(x,t) = (4\pi t)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4t})$ be the heat kernel and $i \in \{1,2,3\}$. We say that (ω, j) is the mild solution to (1.2) if for all $0 < t < \infty$ and $x \in \mathbb{R}^3$, it satisfies $\omega(x,t) \xrightarrow{weak^*} \omega_0(x)$, $j(x,t) \xrightarrow{weak^*} j_0(x)$ as $t \to 0$ and

$$\begin{cases} \omega(x,t) = S(\cdot,t) * \omega_0(x) + \displaystyle\int_0^t \int_{\mathbb{R}^3} S(x-y,t-s) \partial_{y_i}(u^i \omega - u\omega^i - b^i j + b j^i)(y,s) \mathrm{d}y\mathrm{d}s, \\ j(x,t) = S(\cdot,t) * j_0(x) + \displaystyle\int_0^t \int_{\mathbb{R}^3} S(x-y,t-s) \nabla_y \times ((u\cdot\nabla)b - (b\cdot\nabla)u)(y,s)\mathrm{d}y\mathrm{d}s. \end{cases} \tag{2.1}$$

2.2. Important Propositions

In this part, we introduce some propositions which play an important role in the proof of the main results. The details of the proof of these propositions are given by [8]. Here we begin with the basic properties of Morrey space.

Proposition 2.1. *(Basic properties of Morrey space).*

(i) $\overline{\mathcal{M}}^1(\mathbb{R}^3)$ *is the Banach space of finite measures and* $\|\mu\|_1 = |\mu|$.

(ii) $L^p(\mathbb{R}^3) \subset \mathcal{M}^p(\mathbb{R}^3)$ *for* $1 < p < \infty$.

(iii) *When* $p = \infty$, *the Morrey space* $\mathcal{M}^\infty(\mathbb{R}^3)$ *is equivalent to* $L^\infty(\mathbb{R}^3)$.

(iv) *(Interpolation) Let* $k \in (0,1)$*, if* $1 \le p, p_1, p_2 < \infty$ *and* $\frac{1}{p} = \frac{1-k}{p_1} + \frac{k}{p_2}$*, then for all* $\mu \in \mathcal{M}^{p_1}(\mathbb{R}^3) \cap \mathcal{M}^{p_2}(\mathbb{R}^3)$ *we have*

$$\|\mu\|_p \le \|\mu\|_{p_1}^{1-k} \|\mu\|_{p_2}^{k}.$$

Next, we discuss about the estimates of Biot-Savart kernel in Morrey space.

Proposition 2.2. *(Estimates of the Biot-Savart kernel)*

(i) *Denote* $K(\cdot)$ *be the Biot-Savart kernel, if* $\frac{1}{p} = \frac{1}{q} + \frac{1}{3}$ *and* $\mu \in \mathcal{M}^p(\mathbb{R}^3)$*, then* $K * \mu \in \mathcal{M}^q(\mathbb{R}^3)$ *and we have the following estimate*

$$\|K * \mu\|_q \le C\|\mu\|_p,$$

with C *independent of* μ*.*

(ii) *If* $\mu \in \mathcal{M}^p(\mathbb{R}^3) \cap \mathcal{M}^q(\mathbb{R}^3)$ *for some* p, q *satisfy* $1 \le p < 3 < q$*, then* $K * \mu \in L^\infty(\mathbb{R}^3)$ *and*

$$\|K * \mu\|_\infty \le C\|\mu\|_p^{\left(\frac{2}{3} - \frac{1}{q}\right)/\left(\frac{1}{q} - \frac{1}{p}\right)} \|\mu\|_q^{\left(\frac{1}{p} - \frac{2}{3}\right)/\left(\frac{1}{q} - \frac{1}{p}\right)}.$$

In particular, if $p \in (\frac{3}{2}, 3)$*, we have*

$$\|K * \mu\|_\infty \le \|\mu\|_p^{\frac{2p}{3} - 1} \|\mu\|_{2p}^{2 - \frac{2p}{3}}.$$

Finally, we give the estimates of the heat kernel in Morrey space.

Proposition 2.3. *Let* $S(x,t)$ *be the heat kernel defined as in Definition 2.2. Assume that* $1 \le p_1 \le p_2 \le \infty$*. For any* $t > 0$ *and* $\mu \in \mathcal{M}^{p_1}(\mathbb{R}^3)$*, we define the operators* Λ_1*,* Λ_2 *and* Λ_3 *as following*

$$\Lambda_1 u = S(\cdot,t) * \mu(x), \quad \Lambda_2 u = \nabla S(\cdot,t) * \mu(x), \quad \Lambda_2 u = \partial_t S(\cdot,t) * \mu(x).$$

Then, the operator $\Lambda_i (i = 1,2,3)$ *is a bounded operator from* $\mathcal{M}^{p_1}(\mathbb{R}^3)$ *to* $\mathcal{M}^{p_2}(\mathbb{R}^3)$ *which continuously depends on* t*. Furthermore, we have*

$$\|\Lambda_1 \mu\|_{p_2} \le Ct^{-\frac{3}{2}(\frac{1}{p_1} - \frac{1}{p_2})} \|\mu\|_{p_1},$$

$$\|\Lambda_2 \mu\|_{p_2} \le Ct^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p_1} - \frac{1}{p_2})} \|\mu\|_{p_1},$$

$$\|\Lambda_3 \mu\|_{p_2} \le Ct^{-1 - \frac{3}{2}(\frac{1}{p_1} - \frac{1}{p_2})} \|\mu\|_{p_1}.$$

2.3. Main results

We now state our main results about the local-in-time well-posedness of (1.1) as follows.

Theorem 2.4. (Local-in-time well-posedness). *Let* $p \in (2,3)$ *and*

$$\omega_0, b_0, \nabla b_0 \in \mathcal{M}^p(\mathbb{R}^3).$$

(i) *There exists a time* $T>0$ *and a unique local mild solution* (ω,j) *in* $[0,T]\times\mathbb{R}^3$ *to the problem* (1.2) *such that*

$$\begin{aligned}
&t^{1-\frac{1}{\theta}-\varepsilon}\omega,\ \ t^{1-\frac{1}{\theta}-\varepsilon}j\in L^\infty((0,T);\mathcal{M}^p(\mathbb{R}^3)),\\
&t^{1-\frac{1}{2\theta}}\omega,\ \ t^{1-\frac{1}{2\theta}}j\in L^\infty((0,T);\mathcal{M}^{2p}(\mathbb{R}^3)),\\
&t^{\frac{1}{2}}\nabla\omega,\ \ t^{\frac{1}{2}}\nabla j\in L^\infty((0,T);\mathcal{M}^p(\mathbb{R}^3)),\\
&t^{\frac{1}{2}+\frac{1}{2\theta}}\nabla\omega,\ \ t^{\frac{1}{2}+\frac{1}{2\theta}}\nabla j\in L^\infty((0,T);\mathcal{M}^{2p}(\mathbb{R}^3)),
\end{aligned}$$

where

$$\frac{2p}{3}=\theta\in\left(\frac{4}{3},2\right)\ \text{ and }\ 0<\varepsilon<\min\left(1-\frac{1}{\theta},\frac{3}{2}-\frac{2}{\theta}\right).$$

(ii) *(Regularity) The solution* (ω,j) *solves* (1.2) *in the classical sense for* $t\in(0,T]$.

3. PROOF OF THEOREM 2.4

3.1. Auxiliary equations

This section is devoted to proving Theorem 2.4. To deal with the difficult terms $\nabla\times(u\cdot\nabla)b$ and $\nabla\times(b\cdot\nabla)u$, we introduce the equation of ∇b:

$$\partial_t\nabla b-\eta\triangle\nabla b+\nabla((u\cdot\nabla)b-(b\cdot\nabla)u)=0,$$

with corresponding integral equations:

$$\nabla b(x,t)=S(\cdot,t)*\nabla b_0(x)+\int_0^t\int_{\mathbb{R}^3}S(x-y,t-s)\nabla_y((u\cdot\nabla)b-(b\cdot\nabla)u)(y,s)\mathrm{d}y\mathrm{d}s. \tag{3.1}$$

3.2. Iteration scheme and estimates

Here and after, for convenience we denote

$$\begin{aligned}
W^0_{k,p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{\theta}-\varepsilon}\|\omega^{(k)}(\cdot,t)\|_p, & W^0_{k,2p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{2\theta}}\|\omega^{(k)}(\cdot,t)\|_{2p},\\
W^1_{k,p}&=\sup_{t\in[0,T]}t^{\frac{1}{2}}\|\nabla\omega^{(k)}(\cdot,t)\|_p, & W^1_{k,2p}&=\sup_{t\in[0,T]}t^{\frac{1}{2}+\frac{1}{2\theta}}\|\nabla\omega^{(k)}(\cdot,t)\|_{2p},\\
B^0_{k,p}&=\sup_{t\in[0,T]}\|b^{(k)}(\cdot,t)\|_p, & B^0_{k,2p}&=\sup_{t\in[0,T]}t^{\frac{1}{2\theta}}\|b^{(k)}(\cdot,t)\|_{2p},\\
B^1_{k,p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b^{(k)}(\cdot,t)\|_p, & B^1_{k,2p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{2\theta}}\|\nabla b^{(k)}(\cdot,t)\|_{2p},\\
J^0_{k,p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{\theta}-\varepsilon}\|j^{(k)}(\cdot,t)\|_p, & J^0_{k,2p}&=\sup_{t\in[0,T]}t^{1-\frac{1}{2\theta}}\|j^{(k)}(\cdot,t)\|_{2p},\\
J^1_{k,p}&=\sup_{t\in[0,T]}t^{\frac{1}{2}}\|\nabla j^{(k)}(\cdot,t)\|_p, & J^1_{k,2p}&=\sup_{t\in[0,T]}t^{\frac{1}{2}+\frac{1}{2\theta}}\|\nabla j^{(k)}(\cdot,t)\|_{2p},
\end{aligned}$$

with $\theta \in (\frac{4}{3}, 2)$ and $\varepsilon > 0$.

Now let us consider the following standard iterative scheme:

$$\begin{cases} \omega^{(k+1)}(x,t) = \omega^{(0)}(x,t) + \int_0^t \int_{\mathbb{R}^3} S(x-y,t-s) \\ \quad \times \partial_{y_i}(u^{i,(k)}\omega^{(k)} - u^{(k)}\omega^{i,(k)} - b^{i,(k)}j^{(k)} + b^{(k)}j^{i,(k)})(y,s)\mathrm{d}y\mathrm{d}s, \\ \nabla b^{(k+1)}(x,t) = \nabla b^{(0)}(x,t) \\ \quad + \int_0^t \int_{\mathbb{R}^3} S(x-y,t-s)\nabla_y((u^{(k)}\cdot\nabla)b^{(k)} - (b^{(k)}\cdot\nabla)u^{(k)})(y,s)\mathrm{d}y\mathrm{d}s, \end{cases} \tag{3.2}$$

with $j^{(0)}$, $\omega^{(0)}$, $b^{(0)}$ and $\nabla b^{(0)}$ given by

$$j^{(0)}(x,t) = S(\cdot,t) * (\nabla \times b_0)(x), \quad \omega^{(0)}(x,t) = S(\cdot,t) * \omega_0(x),$$
$$b^{(0)}(x,t) = S(\cdot,t) * b_0(x), \qquad \nabla b^{(0)}(x,t) = S(\cdot,t) * \nabla b_0(x).$$

We shall establish the following results, which are the main ingredients of proving Theorem 2.4.

Lemma 3.1. *Let $\frac{2p}{3} = \theta \in (\frac{4}{3}, 2)$ and $0 < \varepsilon < \min\{1 - \frac{1}{\theta}, \frac{3}{2} - \frac{2}{\theta}\}$. Then for $k \geq 0$, it holds that*

$$\begin{aligned} W^0_{k+1,p} \leq\, & C(T^{1-\frac{1}{\theta}-\varepsilon}\|\omega_0\|_p \\ & + C(\tfrac{1}{2}, \tfrac{1}{\theta} - \tfrac{1}{2} + \theta\varepsilon)T^{(\theta-1)\varepsilon}\left((W^0_{k,p})^{\theta}(W^0_{k,2p})^{2-\theta} + (B^1_{k,p})^{\theta}(B^1_{k,2p})^{2-\theta})\right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} W^0_{k+1,2p} \leq\, & C(T^{1-\frac{1}{\theta}}\|\omega_0\|_p \\ & + C(\tfrac{1}{2} - \tfrac{1}{2\theta}, \theta\varepsilon + \tfrac{1}{\theta} - \tfrac{1}{2})T^{\theta\varepsilon}\left((W^0_{k,p})^{\theta}(W^0_{k,2p})^{2-\theta} + (B^1_{k,p})^{\theta}(B^1_{k,2p})^{2-\theta})\right), \end{aligned} \tag{3.4}$$

$$\begin{aligned} B^1_{k+1,p} \leq\, & C(T^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b_0\|_p + C\big(\tfrac{1}{2}, \tfrac{1}{\theta} - \tfrac{1}{2} + \theta\varepsilon\big)T^{(\theta-1)\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\ & + C\big(\tfrac{1}{2}, 1 - \tfrac{1}{\theta}\big)T^{\frac{3}{2}-\frac{2}{\theta}-\varepsilon}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}), \end{aligned} \tag{3.5}$$

$$\begin{aligned} B^1_{k+1,2p} \leq\, & C(T^{1-\frac{1}{\theta}}\|\nabla b_0\|_p + C\big(\tfrac{1}{2} - \tfrac{1}{2\theta}, \tfrac{1}{\theta} - \tfrac{1}{2} + \theta\varepsilon\big)T^{\theta\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\ & + C\big(\tfrac{1}{2} - \tfrac{1}{2\theta}, 1 - \tfrac{1}{\theta}\big)T^{\frac{3}{2}-\frac{2}{\theta}}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}). \end{aligned} \tag{3.6}$$

Proof. By Proposition 2.3, it holds that

$$\|S(\cdot,t) * \omega_0\|_p \leq C_1\|\omega_0\|_p, \tag{3.7}$$

$$\|S(\cdot,t) * \omega_0\|_{2p} \leq C_1 t^{-\frac{1}{2\theta}}\|\omega_0\|_p, \tag{3.8}$$

$$\|S(\cdot,t)*\nabla b_0\|_p \le C_1\|\nabla b_0\|_p, \tag{3.9}$$

$$\|S(\cdot,t)*\nabla b_0\|_{2p} \le C_1 t^{-\frac{1}{2\theta}}\|\nabla b_0\|_p. \tag{3.10}$$

What's more, it follows from Proposition 2.2 that

$$\begin{aligned}
&\|(u^i\omega - u\omega^i - b^i j + b j^i)(\cdot,s)\|_p \\
\le\ & 2(\|u(\cdot,s)\|_\infty\|\omega(\cdot,s)\|_p + \|b(\cdot,s)\|_\infty\|j(\cdot,s)\|_p) \\
\le\ & C_2(\|\omega(\cdot,s)\|_p^\theta\|\omega(\cdot,s)\|_{2p}^{2-\theta} + \|j(\cdot,s)\|_p^\theta\|j(\cdot,s)\|_{2p}^{2-\theta}) \\
\le\ & C_2(\|\omega(\cdot,s)\|_p^\theta\|\omega(\cdot,s)\|_{2p}^{2-\theta} + \|\nabla b(\cdot,s)\|_p^\theta\|\nabla b(\cdot,s)\|_{2p}^{2-\theta}),
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\|((u\cdot\nabla)b - (b\cdot\nabla)u)(\cdot,s)\|_p \\
\le\ & \|u(\cdot,s)\|_\infty\|\nabla b(\cdot,s)\|_p + \|\nabla u(\cdot,s)\|_\infty\|b(\cdot,s)\|_p \\
\le\ & C_2(\|\omega(\cdot,s)\|_p^{\theta-1}\|\omega(\cdot,s)\|_{2p}^{2-\theta}\|\nabla b(\cdot,s)\|_p \\
& + \|\nabla\omega(\cdot,s)\|_p^{\theta-1}\|\nabla\omega(\cdot,s)\|_{2p}^{2-\theta}\|b(\cdot,s)\|_p).
\end{aligned} \tag{3.12}$$

Then, by (3.2), (3.7), (3.11) and Proposition 2.3, we can get

$$\begin{aligned}
W_{k+1,p}^0 \le\ & \sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\Big(\|S(\cdot,t)*\omega_0\|_p \\
& + C_3\int_0^t (t-s)^{-\frac{1}{2}}\|(u^{i,(k)}\omega^{(k)} - u^{(k)}\omega^{i,(k)} - b^{i,(k)}j^{(k)} + b^{(k)}j^{i,(k)})(\cdot,s)\|_p \mathrm{d}s\Big) \\
\le\ & C_1 T^{1-\frac{1}{\theta}-\varepsilon}\|\omega_0\|_p + C_2C_3 \sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\int_0^t (t-s)^{-\frac{1}{2}} \\
& \times(\|\omega^{(k)}(\cdot,s)\|_p^\theta\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta} + \|\nabla b^{(k)}(\cdot,s)\|_p^\theta\|\nabla b^{(k)}(\cdot,s)\|_{2p}^{2-\theta})\mathrm{d}s \\
\le\ & C_1 T^{1-\frac{1}{\theta}-\varepsilon}\|\omega_0\|_p + C_2C_3((W_{k,p}^0)^\theta(W_{k,2p}^0)^{2-\theta} + (B_{k,p}^1)^\theta(B_{k,2p}^1)^{2-\theta}) \\
& \times \sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\int_0^t (t-s)^{-\frac{1}{2}} s^{\theta\varepsilon+\frac{1}{\theta}-\frac{3}{2}}\mathrm{d}s \\
\le\ & C_1 T^{1-\frac{1}{\theta}-\varepsilon}\|\omega_0\|_p \\
& + C_2C_3C(\tfrac{1}{2}, \theta\varepsilon+\tfrac{1}{\theta}-\tfrac{1}{2})T^{(\theta-1)\varepsilon}((W_{k,p}^0)^\theta(W_{k,2p}^0)^{2-\theta} + (B_{k,p}^1)^\theta(B_{k,2p}^1)^{2-\theta}),
\end{aligned}$$

which proves (3.3).

By (3.2), (3.8), (3.11) and Proposition 2.3, one has

$$\begin{aligned}
W^0_{k+1,2p} \le & \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\Big(\|S(\cdot,t)*\omega_0\|_{2p} \\
& +C_4\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}}\|(u^{i,(k)}\omega^{(k)}-u^{(k)}\omega^{i,(k)}-b^{i,(k)}j^{(k)}+b^{(k)}j^{i,(k)})(\cdot,s)\|_p \mathrm{d}s\Big) \\
\le & C_1T^{1-\frac{1}{\theta}}\|\omega_0\|_p + C_2C_4 \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} \\
& \times(\|\omega^{(k)}(\cdot,s)\|_p^{\theta}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}+\|\nabla b^{(k)}(\cdot,s)\|_p^{\theta}\|\nabla b^{(k)}(\cdot,s)\|_{2p}^{2-\theta})\mathrm{d}s \\
\le & C_1T^{1-\frac{1}{\theta}}\|\omega_0\|_p + C_2C_4((W^0_{k,p})^{\theta}(W^0_{k,2p})^{2-\theta}+(B^1_{k,p})^{\theta}(B^1_{k,2p})^{2-\theta}) \\
& \times \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} s^{\theta\varepsilon+\frac{1}{\theta}-\frac{3}{2}}\mathrm{d}s \\
\le & C_1T^{1-\frac{1}{\theta}}\|\omega_0\|_p \\
& +C_2C_4C(\tfrac{1}{2}-\tfrac{1}{2\theta},\theta\varepsilon+\tfrac{1}{\theta}-\tfrac{1}{2})T^{\theta\varepsilon}((W^0_{k,p})^{\theta}(W^0_{k,2p})^{2-\theta}+(B^1_{k,p})^{\theta}(B^1_{k,2p})^{2-\theta}).
\end{aligned}$$

And that gives (3.4).

By (3.2), (3.9), (3.12) and Proposition 2.3, it holds that

$$\begin{aligned}
B^1_{k+1,p} \le & \sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\Big(\|S(\cdot,t)*\nabla b_0\|_p \\
& +C_5\int_0^t (t-s)^{-\frac{1}{2}}\|((u^{(k)}\cdot\nabla)b^{(k)}-(b^{(k)}\cdot\nabla)u^{(k)})(\cdot,s)\|_p\mathrm{d}s\Big) \\
\le & C_1T^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b_0\|_p + C_2C_5\sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\int_0^t (t-s)^{-\frac{1}{2}} \\
& \times(\|\omega(\cdot,s)\|_p^{\theta-1}\|\omega(\cdot,s)\|_{2p}^{2-\theta}\|\nabla b(\cdot,s)\|_p \\
& +\|\nabla\omega(\cdot,s)\|_p^{\theta-1}\|\nabla\omega(\cdot,s)\|_{2p}^{2-\theta}\|b(\cdot,s)\|_p)\mathrm{d}s.
\end{aligned} \tag{3.13}$$

Note that

$$
\begin{aligned}
&\int_0^t (t-s)^{-\frac{1}{2}} (\|\omega(\cdot,s)\|_p^{\theta-1} \|\omega(\cdot,s)\|_{2p}^{2-\theta} \|\nabla b(\cdot,s)\|_p \\
&\quad + \|\nabla\omega(\cdot,s)\|_p^{\theta-1} \|\nabla\omega(\cdot,s)\|_{2p}^{2-\theta} \|b(\cdot,s)\|_p) \mathrm{d}s \\
&\le (W_{k,p}^0)^{\theta-1} (W_{k,2p}^0)^{2-\theta} B_{k,p}^1 \int_0^t (t-s)^{-\frac{1}{2}} s^{\theta\varepsilon+\frac{1}{\theta}-\frac{3}{2}} \mathrm{d}s \\
&\quad + (W_{k,p}^1)^{\theta-1} (W_{k,2p}^1)^{2-\theta} B_{k,p}^0) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{\theta}} \mathrm{d}s \\
&\le t^{\theta\varepsilon+\frac{1}{\theta}-1} C(\tfrac{1}{2}, \theta\varepsilon+\tfrac{1}{\theta}-\tfrac{1}{2}) (W_{k,p}^0)^{\theta-1} (W_{k,2p}^0)^{2-\theta} B_{k,p}^1 \\
&\quad + t^{\frac{1}{2}-\frac{1}{\theta}} C(\tfrac{1}{2}, 1-\tfrac{1}{\theta}) (W_{k,p}^1)^{\theta-1} (W_{k,2p}^1)^{2-\theta} B_{k,p}^0,
\end{aligned}
\tag{3.14}
$$

which together with (3.13) yields (3.5).

It remains to show (3.6). By (3.2), (3.10), (3.12) and Proposition 2.3, one may get that

$$
\begin{aligned}
B_{k+1,2p}^1 &\le \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}} \Big(\|S(\cdot,t) * \nabla b_0\|_{2p} \\
&\quad + C_6 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} \|((u^{(k)}\cdot\nabla) b^{(k)} - (b^{(k)}\cdot\nabla) u^{(k)})(\cdot,s)\|_p \mathrm{d}s \Big) \\
&\le C_1 T^{1-\frac{1}{\theta}} \|\nabla b_0\|_p + C_2 C_6 \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} \\
&\quad \times (\|\omega(\cdot,s)\|_p^{\theta-1} \|\omega(\cdot,s)\|_{2p}^{2-\theta} \|\nabla b(\cdot,s)\|_p \\
&\quad + \|\nabla\omega(\cdot,s)\|_p^{\theta-1} \|\nabla\omega(\cdot,s)\|_{2p}^{2-\theta} \|b(\cdot,s)\|_p) \mathrm{d}s.
\end{aligned}
\tag{3.15}
$$

We can write

$$
\begin{aligned}
&\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} (\|\omega(\cdot,s)\|_p^{\theta-1} \|\omega(\cdot,s)\|_{2p}^{2-\theta} \|\nabla b(\cdot,s)\|_p \\
&\quad + \|\nabla\omega(\cdot,s)\|_p^{\theta-1} \|\nabla\omega(\cdot,s)\|_{2p}^{2-\theta} \|b(\cdot,s)\|_p) \mathrm{d}s \\
&\le (W_{k,p}^0)^{\theta-1} (W_{k,2p}^0)^{2-\theta} B_{k,p}^1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} s^{\theta\varepsilon+\frac{1}{\theta}-\frac{3}{2}} \mathrm{d}s \\
&\quad + (W_{k,p}^1)^{\theta-1} (W_{k,2p}^1)^{2-\theta} B_{k,p}^0) \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} s^{-\frac{1}{\theta}} \mathrm{d}s \\
&\le t^{\theta\varepsilon+\frac{1}{2\theta}-1} C(\tfrac{1}{2}-\tfrac{1}{2\theta}, \theta\varepsilon+\tfrac{1}{\theta}-\tfrac{1}{2}) (W_{k,p}^0)^{\theta-1} (W_{k,2p}^0)^{2-\theta} B_{k,p}^1 \\
&\quad + t^{\frac{1}{2}-\frac{3}{2\theta}} C(\tfrac{1}{2}-\tfrac{1}{2\theta}, 1-\tfrac{1}{\theta}) (W_{k,p}^1)^{\theta-1} (W_{k,2p}^1)^{2-\theta} B_{k,p}^0,
\end{aligned}
$$

which combining (3.15) leads to (3.6). $\square$

In order to estimate $B^0_{k+1,p}$, $B^0_{k+1,2p}$, $J^0_{k+1,p}$ and $J^0_{k+1,2p}$, we shall use the following standard successive approximations:

$$b^{(k+1)}(x,t) = S(\cdot,t) * b_0(x) + \int_0^t \int_{\mathbb{R}^3} S(x-y,t-s)\nabla_y \times (u^{(k)} b^{(k)})(y,s)\mathrm{d}y\mathrm{d}s, \tag{3.16}$$

$$\begin{aligned} j^{(k+1)}(x,t) = & S(\cdot,t) * (\nabla \times b_0)(x) \\ & + \int_0^t \int_{\mathbb{R}^3} S(x-y,t-s)\nabla_y \times ((u^{(k)}\cdot\nabla)b^{(k)} - (b^{(k)}\cdot\nabla)u^{(k)})(y,s)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.17}$$

Lemma 3.2. *Let $\frac{2p}{3} = \theta \in (\frac{4}{3},2)$ and $0<\varepsilon<\min\{1-\frac{1}{\theta}, \frac{3}{2}-\frac{2}{\theta}\}$. Then, for all $k\geq 0$, it holds that*

$$B^0_{k+1,p} \leq C(\|b_0\|_p + C(\frac{1}{2},\frac{1}{2}+(\theta-1)\varepsilon)T^{(\theta-1)\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^0_{k,p}), \tag{3.18}$$

$$B^0_{k+1,2p} \leq C(\|b_0\|_p + C(\frac{1}{2}-\frac{1}{2\theta},\frac{1}{2}+(\theta-1)\varepsilon)T^{(\theta-1)\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^0_{k,p}), \tag{3.19}$$

$$\begin{aligned} J^0_{k+1,p} \leq & C(T^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b_0\|_p + C(\tfrac{1}{2},\tfrac{1}{\theta}-\tfrac{1}{2}+\theta\varepsilon)T^{(\theta-1)\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\ & + C\big(\tfrac{1}{2},1-\tfrac{1}{\theta}\big)T^{\frac{3}{2}-\frac{2}{\theta}-\varepsilon}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}), \end{aligned} \tag{3.20}$$

$$\begin{aligned} J^0_{k+1,2p} \leq & C(T^{1-\frac{1}{\theta}}\|\nabla b_0\|_p + C\big(\tfrac{1}{2}-\tfrac{1}{2\theta},\tfrac{1}{\theta}-\tfrac{1}{2}+\theta\varepsilon\big)T^{\theta\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\ & + C(\tfrac{1}{2}-\tfrac{1}{2\theta},1-\tfrac{1}{\theta})T^{\frac{3}{2}-\frac{2}{\theta}}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}). \end{aligned} \tag{3.21}$$

Proof. By Proposition 2.2, it holds that

$$\|(ub)(\cdot,s)\|_p \leq \|u(\cdot,s)\|_\infty \|b(\cdot,s)\|_p \leq C_2\|\omega(\cdot,s)\|_p^{\theta-1}\|\omega(\cdot,s)\|_{2p}^{2-\theta}\|b(\cdot,s)\|_p. \tag{3.22}$$

(3.22) together with (3.7), (3.16) and Proposition 2.3 implies that

$$
\begin{aligned}
B^0_{k+1,p} &\le C_1\|b_0\|_p + C_3 \sup_{t\in[0,T]} \int_0^t (t-s)^{-\frac{1}{2}} \|(u^{(k)}b^{(k)})(\cdot,s)\|_p \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 \sup_{t\in[0,T]} \int_0^t (t-s)^{-\frac{1}{2}} \|\omega^{(k)}(\cdot,s)\|_p^{\theta-1} \|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta} \|b^{(k)}(\cdot,s)\|_p \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 (W^0_{k,p})^{\theta-1} (W^0_{k,2p})^{2-\theta} \\
&\quad \times \sup_{t\in[0,T]} \int_0^t (t-s)^{-\frac{1}{2}} s^{(1-\frac{1}{\theta}-\varepsilon)(1-\theta)} s^{(1-\frac{1}{2\theta})(\theta-2)} \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 C(\frac{1}{2}, (\theta-1)\varepsilon + \frac{1}{2}) T^{(\theta-1)\varepsilon} (W^0_{k,p})^{\theta-1} (W^0_{k,2p})^{2-\theta},
\end{aligned}
$$

which proves (3.18).

Concerning (3.19), by (3.8), (3.16), (3.22) and Proposition 2.3, it holds that

$$
\begin{aligned}
B^0_{k+1,2p} &\le C_1\|b_0\|_p + C_3 \sup_{t\in[0,T]} t^{\frac{1}{2\theta}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} \|(u^{(k)}b^{(k)})(\cdot,s)\|_p \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 \sup_{t\in[0,T]} t^{\frac{1}{2\theta}} \int_0^t (t-s)^{-\frac{1}{2}} \|\omega(\cdot,s)\|_p^{\theta-1} \|\omega(\cdot,s)\|_{2p}^{2-\theta} \|b(\cdot,s)\|_p \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 (W^0_{k,p})^{\theta-1} (W^0_{k,2p})^{2-\theta} \\
&\quad \times \sup_{t\in[0,T]} t^{\frac{1}{2\theta}} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}} s^{(\theta-1)\varepsilon-\frac{1}{2}} \mathrm{d}s \\
&\le C_1\|b_0\|_p + C_3C_2 C(\frac{1}{2}-\frac{1}{2\theta}, (\theta-1)\varepsilon + \frac{1}{2}) T^{(\theta-1)\varepsilon} (W^0_{k,p})^{\theta-1} (W^0_{k,2p})^{2-\theta}.
\end{aligned}
$$

Thus we obtain (3.19).

For the estimate of $J^0_{k+1,p}$, according to (3.14), (3.17) and (3.22), one has

$$
\begin{aligned}
J^0_{k+1,p} \leq & \sup_{t\in[0,T]} t^{\frac{1}{2}-\frac{1}{\theta}-\varepsilon}\Big(C_1\|\nabla b_0\|_p + C_4\int_0^t (t-s)^{-\frac{1}{2}} \\
& \qquad \times \|((u^{(k)}\cdot\nabla)b^{(k)} - (b^{(k)}\cdot\nabla)u^{(k)})(\cdot,s)\|_p \mathrm{d}s\Big) \\
\leq & C_1 T^{\frac{1}{2}-\frac{1}{\theta}-\varepsilon}\|b_0\|_p \\
& + C_5 \sup_{t\in[0,T]} t^{\frac{1}{2}-\frac{1}{\theta}-\varepsilon}\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla b^{(k)}(\cdot,s)\|_p \\
& \quad + \|\nabla\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|b^{(k)}(\cdot,s)\|_p\Big)\mathrm{d}s \\
\leq & C(T^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b_0\|_p + C(\frac{1}{2},\frac{1}{\theta}-\frac{1}{2}+\theta\varepsilon)T^{(\theta-1)\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\
& + C(\tfrac{1}{2},1-\tfrac{1}{\theta})T^{\frac{3}{2}-\frac{2}{\theta}-\varepsilon}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}).
\end{aligned}
$$

Finally, for $J^0_{k+1,2p}$, following the analogue analysis of (3.15), we can get

$$
\begin{aligned}
J^0_{k+1,2p} \leq & \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\Big(\|S(\cdot,t)*\nabla b_0\|_{2p} \\
& + C_6\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2\theta}}\|((u^{(k)}\cdot\nabla)b^{(k)} - (b^{(k)}\cdot\nabla)u^{(k)})(\cdot,s)\|_p \mathrm{d}s\Big) \\
\leq & C(T^{1-\frac{1}{\theta}}\|\nabla b_0\|_p + C(\frac{1}{2}-\frac{1}{2\theta},\frac{1}{\theta}-\frac{1}{2}+\theta\varepsilon)T^{\theta\varepsilon}(W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}B^1_{k,p} \\
& + C(\tfrac{1}{2}-\tfrac{1}{2\theta},1-\tfrac{1}{\theta})T^{\frac{3}{2}-\frac{2}{\theta}}(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}B^0_{k,p}).
\end{aligned}
$$

□

In order to estimate $W^1_{k,p}$, $W^1_{k,2p}$, $J^1_{k,p}$ and $J^1_{k,2p}$, we will need the following successive approximation scheme:

$$
\begin{cases}
\nabla\omega^{(k+1)}(x,t) = S(\cdot,t)*\nabla\omega_0(x) + \displaystyle\int_0^t\int_{\mathbb{R}^3} S(x-y,t-s)\nabla_y((u^{(k)}\cdot\nabla)\omega^{(k)} \\
\qquad\qquad - (\omega^{(k)}\cdot\nabla)u^{(k)} - (b^{(k)}\cdot\nabla)j^{(k)} + (j^{(k)}\cdot\nabla)b^{(k)})(y,s)\mathrm{d}y\mathrm{d}s, \\
\nabla j^{(k+1)}(x,t) = S(\cdot,t)*\nabla j_0(x) + \displaystyle\int_0^t\int_{\mathbb{R}^3} S(x-y,t-s)\nabla_y((u^{(k)}\cdot\nabla)j^{(k)} \\
\qquad\qquad + \nabla u^{i,(k)}\times b^{(k)}_{x_i} - (b^{(k)}\cdot\nabla)\omega^{(k)} - \nabla b^{i,(k)}\times u^{(k)}_{x_i})(y,s)\mathrm{d}y\mathrm{d}s.
\end{cases}
\tag{3.23}
$$

Lemma 3.3. *Let* $\frac{2p}{3} = \theta \in (\frac{4}{3}, 2)$ *and* $0 < \varepsilon < \min\{1 - \frac{1}{\theta}, \frac{3}{2} - \frac{2}{\theta}\}$. *Then, for all* $k \geq 0$*, we have*

$$\begin{aligned} W^1_{k+1,p} \leq C(\|\omega_0\|_p + \mathcal{C}(\tfrac{1}{2}, \varepsilon(\theta-1))T^{\varepsilon(\theta-1)} \\ \times((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}W^1_{k,p} + (B^1_{k,p})^{\theta-1}(B^1_{k,2p})^{2-\theta}J^1_{k,p}) \\ +\mathcal{C}(\tfrac{1}{2}, \varepsilon)T^{\varepsilon}((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}W^0_{k,p} + (J^1_{k,p})^{\theta-1}(J^1_{k,2p})^{2-\theta}B^1_{k,p})), \end{aligned} \tag{3.24}$$

$$\begin{aligned} W^1_{k+1,2p} \leq C(\|\omega_0\|_p + \mathcal{C}(\tfrac{1}{2} - \tfrac{1}{2\theta}, \varepsilon(\theta-1))T^{\varepsilon(\theta-1)} \\ \times((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}W^1_{k,p} + (B^1_{k,p})^{\theta-1}(B^1_{k,2p})^{2-\theta}J^1_{k,p}) \\ +\mathcal{C}(\tfrac{1}{2} - \tfrac{1}{2\theta}, \varepsilon)T^{\varepsilon}((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}W^0_{k,p} + (J^1_{k,p})^{\theta-1}(J^1_{k,2p})^{2-\theta}B^1_{k,p})), \end{aligned} \tag{3.25}$$

$$\begin{aligned} J^1_{k+1,p} \leq C(\|\nabla b_0\|_p + \mathcal{C}(\tfrac{1}{2}, \varepsilon(\theta-1))T^{\varepsilon(\theta-1)} \\ \times((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}J^1_{k,p} + (J^0_{k,p})^{\theta-1}(J^0_{k,2p})^{2-\theta}W^1_{k,p}) \\ +\mathcal{C}(\tfrac{1}{2}, \varepsilon)T^{\varepsilon}((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}J^0_{k,p}), \end{aligned} \tag{3.26}$$

$$\begin{aligned} J^1_{k+1,2p} \leq C(\|\nabla b_0\|_p + \mathcal{C}(\tfrac{1}{2} - \tfrac{1}{2\theta}, \varepsilon(\theta-1))T^{\varepsilon(\theta-1)} \\ \times((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}J^1_{k,p} + (B^1_{k,p})^{\theta-1}(B^1_{k,2p})^{2-\theta}W^1_{k,p}) \\ +\mathcal{C}(\tfrac{1}{2} - \tfrac{1}{2\theta}, \varepsilon)T^{\varepsilon}((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}J^0_{k,p}). \end{aligned} \tag{3.27}$$

Proof. By Proposition 2.3, it holds that

$$\|S(\cdot,t) * \nabla\omega_0\|_p = \|\nabla S(\cdot,t) * \omega_0\|_p \leq C_1 t^{-\frac{1}{2}}\|\omega_0\|_p, \tag{3.28}$$

and

$$\|S(\cdot,t) * \nabla\omega_0\|_{2p} = \|\nabla S(\cdot,t) * \omega_0\|_{2p} \leq C_1 t^{-\frac{1}{2}-\frac{1}{2\theta}}\|\omega_0\|_p. \tag{3.29}$$

Using (3.12), (3.22), (3.28) and Proposition 2.3, we obtain
(3.30)

$$
\begin{aligned}
W^1_{k+1,p} \leq & \sup_{t\in[0,T]} t^{\frac{1}{2}}\Big(\|S(\cdot,t)*\nabla\omega_0\|_p + C_4\int_0^t (t-s)^{-\frac{1}{2}}\|((u^{(k)}\cdot\nabla)\omega^{(k)} - (\omega^{(k)}\cdot\nabla)u^{(k)} \\
& -(b^{(k)}\cdot\nabla)j^{(k)} + (j^{(k)}\cdot\nabla)b^{(k)})(\cdot,s)\|_p \mathrm{d}s\Big) \\
\leq & C_1\|\omega_0\|_p \\
& + C_5 \sup_{t\in[0,T]} t^{\frac{1}{2}}\Big(\int_0^t (t-s)^{-\frac{1}{2}}(\|\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla\omega^{(k)}(\cdot,s)\|_p \\
& + \|\nabla\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\omega^{(k)}(\cdot,s)\|_p)\mathrm{d}s \\
& + \int_0^t (t-s)^{-\frac{1}{2}}(\|\nabla b^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla b^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla j^{(k)}(\cdot,s)\|_p \\
& + \|\nabla j^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla j^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla b^{(k)}(\cdot,s)\|_p)\mathrm{d}s\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \int_0^t (t-s)^{-\frac{1}{2}}(\|\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla\omega^{(k)}(\cdot,s)\|_p \\
& + \|\nabla\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\omega^{(k)}(\cdot,s)\|_p)\mathrm{d}s \\
& + \int_0^t (t-s)^{-\frac{1}{2}}(\|\nabla b^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla b^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla j^{(k)}(\cdot,s)\|_p \\
& + \|\nabla j^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla j^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla b^{(k)}(\cdot,s)\|_p)\mathrm{d}s \\
\leq & (W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}W^1_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon(\theta-1)-1}\mathrm{d}s \\
& + (W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}W^0_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon-1}\mathrm{d}s \\
& + (B^1_{k,p})^{\theta-1}(B^1_{k,2p})^{2-\theta}J^1_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon(\theta-1)-1}\mathrm{d}s \\
& + (J^1_{k,p})^{\theta-1}(J^1_{k,2p})^{2-\theta}B^1_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon-1}\mathrm{d}s \\
\leq & t^{\varepsilon(\theta-1)-\frac{1}{2}}C(\frac{1}{2},\varepsilon(\theta-1))((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}W^1_{k,p} + (B^1_{k,p})^{\theta-1}(B^1_{k,2p})^{2-\theta}J^1_{k,p}) \\
& + t^{\varepsilon-\frac{1}{2}}C(\frac{1}{2},\varepsilon)((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}W^0_{k,p} + (J^1_{k,p})^{\theta-1}(J^1_{k,2p})^{2-\theta}B^1_{k,p}).
\end{aligned}
$$

This together with (3.30) yields (3.24). Then (3.25) follows from (3.23), (3.29), (3.12), Proposition 2.3 and the arguments similar to those in getting (3.24).

Considering the estimate of $J^1_{k+1,p}$, by (3.23) and Proposition 2.3, we get

(3.31)
$$\begin{aligned}
J^1_{k+1,p} \leq & \sup_{t\in[0,T]} t^{\frac{1}{2}}\Big(\|S(\cdot,t)*\nabla\omega_0\|_p + C_4\int_0^t (t-s)^{-\frac{1}{2}}\|((u\cdot\nabla)j+\nabla u^i\times b_{x_i}\\
&\qquad -(b\cdot\nabla)\omega-\nabla b^i\times u_{x_i})(\cdot,s)\|_p\mathrm{d}s\Big)\\
\leq & C_1\|\omega_0\|_p + C_5\sup_{t\in[0,T]} t^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{1}{2}}\Big(\|u^{(k)}(\cdot,s)\|_\infty\|\nabla j^{(k)}(\cdot,s)\|_p\\
&\qquad +\|\nabla u^{(k)}(\cdot,s)\|_\infty\|\nabla b^{(k)}(\cdot,s)\|_p+\|b^{(k)}(\cdot,s)\|_\infty\|\nabla\omega^{(k)}(\cdot,s)\|_p\Big)\mathrm{d}s\\
\leq & C_1\|\omega_0\|_p + C_5\sup_{t\in[0,T]} t^{\frac{1}{2}}\Big(\int_0^t (t-s)^{-\frac{1}{2}}(\|\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla j^{(k)}(\cdot,s)\|_p\mathrm{d}s\\
&+\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|j^{(k)}(\cdot,s)\|_p)\mathrm{d}s\\
&+\int_0^t (t-s)^{-\frac{1}{2}}(\|j^{(k)}(\cdot,s)\|_p^{\theta-1}\|j^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla\omega^{(k)}(\cdot,s)\|_p\mathrm{d}s\Big).
\end{aligned}$$

Also note that

$$\begin{aligned}
&\int_0^t (t-s)^{-\frac{1}{2}}(\|\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla j^{(k)}(\cdot,s)\|_p\mathrm{d}s\\
&+\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla\omega^{(k)}(\cdot,s)\|_p^{\theta-1}\|\nabla\omega^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|j^{(k)}(\cdot,s)\|_p\mathrm{d}s\\
&+\int_0^t (t-s)^{-\frac{1}{2}}(\|j^{(k)}(\cdot,s)\|_p^{\theta-1}\|j^{(k)}(\cdot,s)\|_{2p}^{2-\theta}\|\nabla\omega^{(k)}(\cdot,s)\|_p\mathrm{d}s\\
\leq\ & (W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}J^1_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon(\theta-1)-1}\mathrm{d}s\\
&+(W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}J^0_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon-1}\mathrm{d}s \qquad (3.32)\\
&+(J^0_{k,p})^{\theta-1}(J^0_{k,2p})^{2-\theta}W^1_{k,p}\int_0^t (t-s)^{-\frac{1}{2}}s^{\varepsilon(\theta-1)-1}\mathrm{d}s\\
\leq\ & t^{\varepsilon(\theta-1)-\frac{1}{2}}C(\frac{1}{2},\varepsilon(\theta-1))((W^0_{k,p})^{\theta-1}(W^0_{k,2p})^{2-\theta}J^1_{k,p}+(J^0_{k,p})^{\theta-1}(J^0_{k,2p})^{2-\theta}W^1_{k,p})\\
&+t^{\varepsilon-\frac{1}{2}}C(\frac{1}{2},\varepsilon)((W^1_{k,p})^{\theta-1}(W^1_{k,2p})^{2-\theta}J^0_{k,p}).
\end{aligned}$$

Combine (3.31) and (3.32) together, we arrive at (3.26).

Finally, for the estimate of $J^1_{k+1,2p}$, by (3.23), Proposition 2.3 and the similar arguments used to derive (3.26), we can get (3.27). The details are omitted. □

3.3. The proof for Theorem 2.4

Proof of Theorem 2.4. For the convenience of notations, we set

$$\alpha=\min\Big\{1-\frac{1}{\theta}-\varepsilon,1-\frac{1}{2\theta},0\Big\},\ \ \beta=\min\Big\{(\theta-1)\varepsilon,\theta\varepsilon,\frac{3}{2}-\frac{2}{\theta}\Big\},$$

$$L_k=\max\big\{W^0_{k,p},W^0_{k,2p},W^1_{k,p},W^1_{k,2p},B^0_{k,p},B^0_{k,2p},B^1_{k,p},B^1_{k,2p},J^1_{k,p},J^1_{k,2p}\big\},$$

and

$$\begin{aligned}M=\max\Big\{&\mathcal{C}\big(\tfrac12,\tfrac1\theta-\tfrac12+\theta\varepsilon\big),\mathcal{C}\big(\tfrac12-\tfrac{1}{2\theta},\tfrac1\theta-\tfrac12+\theta\varepsilon\big),\mathcal{C}\big(\tfrac12,1-\tfrac1\theta\big),\mathcal{C}\big(\tfrac12-\tfrac{1}{2\theta},1-\tfrac1\theta\big),\\&\mathcal{C}\big(\tfrac12,\tfrac12+(\theta-1)\varepsilon\big),\mathcal{C}\big(\tfrac12-\tfrac{1}{2\theta},\tfrac12+(\theta-1)\varepsilon\big),\mathcal{C}\big(\tfrac12,(\theta-1)\varepsilon\big),\mathcal{C}\big(\tfrac12,\varepsilon\big),\\&\mathcal{C}\big(\tfrac12-\tfrac{1}{2\theta},(\theta-1)\varepsilon\big),\mathcal{C}\big(\tfrac12-\tfrac{1}{2\theta},\varepsilon\big)\Big\}.\end{aligned}$$

Applying the above notations, invoking Lemma 3.1-3.3, it holds that

$$L_{k+1}\le\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^\alpha+T^\beta ML_k^2. \tag{3.33}$$

Assuming that the time T satisfies

$$1-4\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^{\alpha+\beta}M>0,$$

and noting that

$$L_0\le\frac{1-\sqrt{1-4\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^{\alpha+\beta}M}}{2T^\beta M},$$

one can get from (3.33) that for all $k\ge0$,

$$\begin{aligned}L_k&\le\frac{1-\sqrt{1-4\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^{\alpha+\beta}M}}{2T^\beta M}\\&\le2\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^\alpha.\end{aligned}$$

In fact, if

$$L_{k-1}\le\frac{1-\sqrt{1-4\max\{\|\omega_0\|_p,\|b_0\|_p,\|\nabla b_0\|_p\}T^{\alpha+\beta}M}}{2T^\beta M},$$

then we have

$$\begin{aligned} L_k &\leq \max\{\|\omega_0\|_p, \|b_0\|_p, \|\nabla b_0\|_p\}T^\alpha + T^\beta M L_{k-1}^2 \\ &\leq \max\{\|\omega_0\|_p, \|b_0\|_p, \|\nabla b_0\|_p\}T^\alpha, \\ &\quad + T^\beta M \left(\frac{1-\sqrt{1-4\max\{\|\omega_0\|_p, \|b_0\|_p, \|\nabla b_0\|_p\}T^{\alpha+\beta}M}}{2T^\beta M} \right)^2 \\ &= \frac{1-\sqrt{1-4\max\{\|\omega_0\|_p, \|b_0\|_p, \|\nabla b_0\|_p\}T^{\alpha+\beta}M}}{2T^\beta M}. \end{aligned}$$

We also set

$$\begin{aligned} N_{k+1} := \max\Big\{ &\sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\|\omega^{(k+1)} - \omega^{(k)}\|_p, \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\|\omega^{(k+1)} - \omega^{(k)}\|_{2p}, \\ &\sup_{t\in[0,T]} t^{\frac{1}{2}}\|\nabla\omega^{(k+1)} - \nabla\omega^{(k)}\|_p, \sup_{t\in[0,T]} t^{\frac{1}{2}+\frac{1}{2\theta}}\|\nabla\omega^{(k+1)} - \nabla\omega^{(k)}\|_{2p}, \\ &\sup_{t\in[0,T]} \|b^{(k+1)} - b^{(k)}\|_p, \sup_{t\in[0,T]} t^{\frac{1}{2\theta}}\|b^{(k+1)} - b^{(k)}\|_{2p} \\ &\sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\|\nabla b^{(k+1)} - \nabla b^{(k)}\|_p, \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\|\nabla b^{(k+1)} - \nabla b^{(k)}\|_{2p}, \\ &\sup_{t\in[0,T]} t^{1-\frac{1}{\theta}-\varepsilon}\|j^{(k+1)} - j^{(k)}\|_p, \sup_{t\in[0,T]} t^{1-\frac{1}{2\theta}}\|j^{(k+1)} - j^{(k)}\|_{2p}, \\ &\sup_{t\in[0,T]} t^{\frac{1}{2}}\|\nabla j^{(k+1)} - \nabla j^{(k)}\|_p, \sup_{t\in[0,T]} t^{\frac{1}{2}+\frac{1}{2\theta}}\|\nabla j^{(k+1)} - \nabla j^{(k)}\|_{2p} \Big\}, \end{aligned}$$

by a similar approach as above, we can obtain

$$N_{k+1} \leq T^\beta M(L_{k+1} + L_k)N_k,$$

where

$$T^\beta M(L_{k+1} + L_k) < 1 - \sqrt{1-4\max\{\|\omega_0\|_p, \|b_0\|_p, \|\nabla b_0\|_p\}T^{\alpha+\beta}M} < 1.$$

With the above estimate in mind, applying Banach's fixed point Theorem, we conclude that IVP (1.2) has a mild solution in $[0,T]$. In order to complete the proof of existence, we also require the weak convergence assumption as in Definition 2.2, which can be ensured by the fact that the heat semigroup is weak-continuous in $\mathcal{M}^p(\mathbb{R}^3)$ as $t \to 0$. The proof of uniqueness and regularity can be obtained via the same method as above, we refer to [3, 8, 21] for technical details. □

Acknowledgements. The research of Zirong Zeng was supported partly by the China Scholarship Council (grant No. 201906230069).

Conflict of Interest: The author would like to declare that no conflict of interest exits in this paper.

REFERENCES

[1] Rubén Agapito and Maria Schonbek. Non-uniform decay of MHD equations with and without magnetic diffusion. *Comm. Partial Differential Equations*, 32(10-12):1791–1812, 2007.

[2] Marco Cannone, A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoam. **13**: 515–541, 1997.

[3] José A. Carrillo and Juan Soler, On the Vlasov-Poisson-Fokker-Planck equations with measures in Morrey spaces as initial data, J. Math. Anal. Appl. **207**: 475–495, 1997.

[4] Georges-Henri Cottet, Équations de Navier-Stokes dans le plan avec tourbillon initial mesure, C. R. Acad. Sci. Paris Sér. I Math. **303**: 105–108, 1986.

[5] Georges Duvaut and Jacques-Louis Lions, Inéquations en thermoélasticité et magnétohydrodynamique, Arch. Rational Mech. Anal. **46**: 241–279, 1972.

[6] Hiroshi Fujita and Tosio Kato, On the Navier-Stokes initial value problem, I, Arch. Rational Mech. Anal. **16**: 269–315, 1964.

[7] Isabelle Gallagher and Thierry Gallay, Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity, Math. Ann. **332**: 287–327, 2005.

[8] Yoshikazu Giga and Tetsuro Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, Commun. Partial Differ. Equ. **14**: 577–618, 1989.

[9] Yoshikazu Giga, Tetsuro Miyakawa and Hirofumi Osada, Two-dimensional Navier-Stokes flow with measures as initial vorticity, Arch. Ration. Mech. Anal. **104**: 223–250, 1988.

[10] Cheng He, Xiangdi Huang, and Yun Wang. On some new global existence results for 3D magnetohydrodynamic equations. *Nonlinearity*, 27(2):343–352, 2014.

[11] Tosio Kato, Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. **187**: 471–480, 1984.

[12] Tosio Kato, Strong solutions of the Navier-Stokes equation in Morrey spaces, Bol. Soc. Brasil. Mat. (N.S.) **22**: 127–155, 1992.

[13] Herbert Koch and Daniel Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math., **157**: 22–35, 2001.

[14] Hideo Kozono and Masao Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. Commun. Partial Differ. Equa. **19**: 959—1014, 1994.

[15] Zhen Lei, On axially symmetric incompressible magnetohydrodynamics in three dimensions. *J. Differential Equations*, 259(7):3202–3215, 2015.

[16] Zhen Lei and Fanghua Lin, Global Mild Solutions of Navier-Stokes Equations, Commun. Pure Appl. Math., **LXIV**: 1297–1304, 2011.

[17] Fabrice Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$, Ann. Inst. Henri Poincare, Anal. Non Lineaire **13**: 319–336, 1996.

[18] Michel Sermange and Roger M. Temam, Some mathematical questions related to the MHD equations, Commun. Pure Appl. Math. **36**: 635–664, 1983.

[19] M. E. Schonbek, T. P. Schonbek, and Endre Süli. Large-time behaviour of solutions to the magnetohydrodynamics equations. *Math. Ann.*, 304(4):717–756, 1996.

[20] Michael E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Commun. Partial Differ. Equa. **17**: 1407–1456, 1992.

[21] Jiahong Wu, Quasi-geostrophic-type equations with initial data in Morrey spaces, Nonlinearity **10**: 1409–1420, 1997.

SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, PEOPLE'S REPUBLIC OF CHINA

Email address: beckzzr@sjtu.edu.cn