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Application of Gauss-Kronrod Quadrature Method for Solving Non-Linear Integro-differential Equations of the second kind

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Summary

In this paper, we deal with the non-linear integro-differential equation in which we propose a quadrature method in order to find an approximate solution. We show the convergence of the method and we give the algorithm. We conclude by providing some numerical similation in order to confirm our theoretical results.

KEYWORDS:
Nonlinear integro-differential equation, Gauss-Kronrod, Quadrature, Backward Differnce.

1 | INTRODUCTION

Nowadays, Integral and Integro-differential equations arises in a lot of fields of real-life problems and many mathematician are working on them. For further information in this respect, see[2,4,5,9,10,12]. In our case we are interested in studying the Integro-differential equations where, we intend to present an iterative method based on Gauss-Kronrod quadrature[6,8,11] to find an approximate solution for the following nonlinear integro-differential equation of the second kind

$$x'(s) = y(s) + \int_{-1}^{1} k(s,t,x(t))dt, \tag{1}$$
$$x(-1) = 0,$$

where $x(s)$ is the unknown function that occurs inside and outside the integral sign, $K(.,.)$ is the kernel and $f(s)$ is the source term. The existence and uniqueness of this kind of equations have already proved[1,7].

The outline of the paper is as follows: In Section 2, we presented the transformation of the equation. In Section 3, we present Gauss-Kronrod Quadrature. Then, we solve the Eq(1). In section 4, we give the algorithm of the method. In the end, We present some numerical examples to illustrate our theoretical results.

2 | SOLVING THE INTEGRO-DIFFERENTIAL EQUATION

In this section, we transform Eq(1) by using a numerical method of integration called Gauss-Kronrod quadrature and we proved that the error vanished when $n \to \infty$. We have the following nonlinear integro-differential equation of the second kind

$$x'(s) = f(s) + \int_{-1}^{1} k(s,t,x(t))dt,$$
$$x(-1) = 0.$$

Let $\Gamma = s_1, s_2, \cdots, s_n$ be the Gauss points.
We have the Backward difference:

$$x'(s) \approx \frac{x(s) - x(s-h)}{h},$$

where $h = s_i - s_{i-1}$.
Now if $x^*(s)$ be an analytical solution of (1), then for Γ on $[-1, 1]$, we have

$$x^*(s_i) - x^*(s_{i-1}) = f(s_i) + \int_{-1}^{1} k(s_i, t, x^*(t))dt, \; i = 1, 2, \cdots, n. \tag{2}$$

In (2), the term integral can be estimated by a numerical method of integration. In our case we use Gauss-Kronrod quadrature. Then (2) becomes

$$x_i^* - x_{i-1}^* = f_i + \sum_{j=1}^{n} w_j k(s_i, t_j, x_j^*) + E(\varepsilon), \quad i = 1, 2, \cdots, n, \tag{3}$$

where $x_i^* = x^*(s_i), \; f_i = f(s_i), \; w_i$ are the wight $i = 1, 2, \cdots, n$, which satisfy

$$\sum_{i=1}^{n} w_i = 2$$

and $E(\varepsilon)$ is the error of Gauss-Kronrod quadrature, which is given as follows

$$R(\varepsilon) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} k^{(2n)}(s, \varepsilon, x^*), \; -1 < \varepsilon < 1.$$

For Γ, we consider a nonlinear system obtained by neglecting the truncation error of (2), as follows,

$$\xi_i - \xi_{i-1} = f_i + \sum_{j=1}^{n} w_j k(s_i, t_j, \xi_j), \quad i = 1, 2, \cdots, n, \tag{4}$$

and suppose that $\xi^* = (\xi_1^*, \xi_2^*, \cdots, \xi_n^*)^T$ is the exact solution of the nonlinear system (4). Now, we look for the conditions of vanishing $\|x^* - \xi^*\|_\infty$, where $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$.

Proposition 1. Suppose that

(i) $k(s, t, x(s)) \in C^{2n}([-1, 1] \times [-1, 1] \times \mathbb{R}, \mathbb{R}_+)$

(ii) $f(s) \in C([-1, 1], \mathbb{R}_+)$

(iii) $k_x(s, t, x(s))$ exists on $[-1, 1] \times [-1, 1] \times \mathbb{R}$ and $\gamma < \frac{1-\alpha}{2}$, where $0 < \alpha < 1$ and

$$\gamma = \sup_{s,t \in [-1,1]} |k_x(s, t, x(t))|.$$

Then

$$\|x^* - \xi^*\|_\infty \leq \frac{1}{1 - (2\gamma + \alpha)} \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n!)]^3} |O(\epsilon^{2n})|, \quad -1 < \varepsilon < 1. \tag{5}$$

Proof. Let

$$|x_p^* - \xi_p^*| = \|x^* - \xi^*\|_\infty,$$

in which $1 \leq p \leq n$.
Firstly, from (i) and (ii) we have

$$x' \geq 0,$$

which means that x is increasing.
Now by substituting (3) and (4), we have

$$x_p^* - \xi_p^* = x_{p-1}^* - \xi_{p-1}^* + \sum_{j=1}^{n} w_j \left[k(s_p, t_j, x_j^*) - k(s_p, t_j, \xi_j^*) \right] + E(\varepsilon).$$

According to (iii) and by using the Mean Value theorem implies the existence of η_j between x_j^* and ξ_j^* such that

$$k(s_p, t_j, x_j^*) - k(s_p, t_j, \xi_j^*) = \frac{\partial k}{\partial x}(s_p, t_j, \eta_j)(x_j^* - \xi_p^*), \quad j = 1, 2, \cdots, n.$$

By using (iii) for the second time and the above equalities, we obtain

$$\begin{aligned}
x_p^* - \xi_p^* &= x_{p-1}^* - \xi_{p-1}^* + \sum_{j=1}^{n} w_j k_x(s_p, t_j, \eta_j)(x_j^* - \xi_j^*) + E(\varepsilon) \\
\Rightarrow |x_p^* - \xi_p^*| &\leq |x_{p-1}^* - \xi_{p-1}^*| + |\sum_{j=1}^{n} w_j k_x(s_p, t_j, \eta_j)(x_j^* - \xi_j^*)| + |E(\varepsilon)| \\
&\leq |x_{p-1}^* - \xi_{p-1}^*| + \sum_{j=1}^{n} w_j \sup_{s,t\in[a,b]} |k_x(s_p, t_j, \eta_j)| \, |(x_j^* - \xi_j^*)| + |E(\varepsilon)| \\
&= |x_{p-1}^* - \xi_{p-1}^*| + \gamma \sum_{j=1}^{n} w_j |x_j^* - \xi_j^*| + |E(\varepsilon)|.
\end{aligned}$$

Since $\sum_{j=1}^{n} w_j = 2$. Then

$$|x_p^* - \xi_p^*| \leq |x_{p-1}^* - \xi_{p-1}^*| + 2\gamma |x_j^* - \xi_j^*| + |E(\varepsilon)|.$$

We divide the previous inequality by $|x_p^* - \xi_p^*|$, we get

$$\frac{|x_p^* - \xi_p^*|}{|x_p^* - \xi_p^*|} \leq 2\gamma \frac{|x_p^* - \xi_p^*|}{|x_p^* - \xi_p^*|} + \frac{|x_{p-1}^* - \xi_{p-1}^*|}{|x_p^* - \xi_p^*|} + \frac{|E(\varepsilon)|}{|x_p^* - \xi_p^*|}.$$

Since x is increasing then,

$$\frac{|x_{p-1}^* - \xi_{p-1}^*|}{|x_p^* - \xi_p^*|} \leq \alpha,$$

where $0 < \alpha < 1$. Thus

$$\|x^* - \xi^*\|_\infty \leq \frac{1}{1 - (2\gamma + \alpha)} \left[\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n!)]^3} |k^{(2n)}(s, \varepsilon, x^*)| \right], \quad -1 < \varepsilon < 1.$$

which completes the proof. □

Now, from Eq(5), we get the following Corollary

Corollary 1. The error estimates $\|x^* - \xi^*\|_\infty \to 0$ (vanishes) when $n \to \infty$.

Up to now, A nonlinear system with a special form obtained given by (4). Now, we could find a numerical approach for obtaining the approximate solution.

3 | THE NUMERICAL APPROACH

The aim of this section is to find approximate solution of nonlinear equations systems, we use iterative methods which are so famous[11]; As we mentioned before, the nonlinear system (4) has a special form that allows to approximate its solution by an iterative method. In linear systems Gauss-Seidel method is an efficient method to solve this type of equations systems. Thus, we apply a successive substitution similar to it and we define an iterative process that leads to the sequence of vectors $\{\xi^{(k)}\}$, where $\xi^{(k)} = \left(\xi_1^{(k)}, \xi_2^{(k)}, \cdots, \xi_n^{(k)}\right)$ and each element of $\xi^{(k)}$ satisfy the following iteration formula,

$$\xi_i^{(k+1)} - \xi_{i-1}^{(k)} = y_i + \sum_{j=1}^{n} w_j k(s_i, t_j, \xi_j^{(k)}), \; i = 1, 2, \cdots, n, \; k = 0, 1, \cdots. \tag{6}$$

Now, we seek for the conditions that guarantee the convergence of the sequence $\{\xi^{(k)}\}$.

Thus, we give the following theorem.

Theorem 1. Under the same assumption as Proposition 1 the sequence $\{\xi^{(k)}\}$ comes from (6) tends to the exact solution ξ^* of (4), for any arbitrary $\xi^{(0)}$.

Proof. By substituting (4) and (6), we get

$$\left(\xi_i^{(k+1)} - \xi_{i-1}^{(k+1)}\right) - \left(\xi_i^* - \xi_{i-1}^*\right) = \sum_{j=1}^{n} w_j \left[k(s_i, t_j, \xi_j^{(k)}) - k(s_i, t_j, \xi_j^*) \right], \; i = 1, 2, \cdots, n, \; k = 0, 1, \cdots.$$

From (i) and (ii) we get the decreasing property and from (iii) we get

$$\sum_{j=1}^{n} w_j \left[k(s_i, t_j, \xi_j^{(k)}) - k(s_i, t_j, \xi_j^{*})\right] = \sum_{j=1}^{n} w_j \frac{\partial k}{\partial x}(s_i, t_j, \eta_j^{(k)}) \left[\xi_j^{(k)} - \xi_j^{*}\right].$$

Thus,

$$\left(\xi_i^{(k+1)} - \xi_i^{*}\right) = \left(\xi_{i-1}^{(k+1)} - \xi_{i-1}^{*}\right) + \sum_{j=1}^{n} w_j \frac{\partial k}{\partial x}(s_i, t_j, \eta_j^{(k)}) \left[\xi_j^{(k)} - \xi_j^{*}\right].$$

Which implies,

$$\begin{aligned}
|\xi_i^{(k+1)} - \xi_i^{*}| &\le |\xi_{i-1}^{(k+1)} - \xi_{i-1}^{*}| + \sum_{j=1}^{n} w_j \left|\frac{\partial k}{\partial x}(s_i, t_j, \eta_j^{(k)})\right| \|\xi^{(k)} - \xi^{*}\|_\infty, \\
|\xi_i^{(k+1)} - \xi_i^{*}| &\le |\xi_{i-1}^{(k+1)} - \xi_{i-1}^{*}| + 2\gamma \|\xi^{(k)} - \xi^{*}\|_\infty, \\
(1-\alpha)\, |\xi_i^{(k+1)} - \xi_i^{*}| &\le 2\gamma \|\xi^{(k)} - \xi^{*}\|_\infty, \\
|\xi_i^{(k+1)} - \xi_i^{*}| &\le \frac{2\gamma}{1-\alpha} \|\xi^{(k)} - \xi^{*}\|_\infty.
\end{aligned}$$

Now, let $\lambda = \frac{2\gamma}{1-\alpha}$, we have

$$\|\xi^{(k+1)} - \xi^{*}\|_\infty \le \lambda \|\xi^{(k)} - \xi^{*}\|_\infty \le \lambda^2 \|\xi^{(k-1)} - \xi^{*}\|_\infty \le \cdots \le \lambda^k \|\xi^{(0)} - \xi^{*}\|_\infty, \; k = 0, 1, \cdots.$$

Since, we have $0 < 2\gamma < 1$ and $0 < 1 - \gamma < 1$, then $0 < \lambda < 1$.

Hence, $\|\xi^{(k+1)} - \xi^{*}\|_\infty$ goes to zero, when k goes to ∞.

The proof is completed. □

4 | ALGORITHM OF THE METHOD

The purpose of this section is to define an algorithm for the non linear integro-differential equation (1) and by taking in consideration the assumptions of Proposition 1.

The algorithm is given as follows[3]:

The initial data:

- Choose any $\varepsilon > 0$.
- Let $\Gamma = \{-1 = s_0 = t_0, s_1 = t_1, \cdots, s_{n-1} = t_{n-1}, s_n = t_n = 1\}$ on $[-1, 1]$.
- Let $\epsilon = s_i - s_{i-1}, \; i = 1, \cdots, n$, be the step size.
- The initial vector $\xi^{(0)} = (\xi_0^{(0)}, \xi_1^{(0)}, \cdots, \xi_n^{(0)})^T$ and $\xi_0^{(k)}$ is given.
- Set $k = 0$ and go to the main steps
- Eq(6) is given as follows

$$\xi_i^{(k+1)} - \xi_{i-1}^{(k)} = y_i + \sum_{j=1}^{n} w_j k(s_i, t_j, \xi_j^{(k)}), \; i = 1, 2, \cdots, n, \; k = 0, 1, \cdots. \tag{6}$$

  where $y_i = y(s_i), \; w_j$, and $k(s, t, \xi(t))$ are given functions.

Main steps:

(1) Compute $\xi^{(k+1)}$ using Eq(6) and go to the next step.

(2) Compute the error $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty$ and go the next step.

(3) If $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty \le \varepsilon$, stop; Otherwise, set $k = k + 1$ and go to step (1).

Remark 1. From Gauss-Kronrod Quadrature method we get

$$\int_{-1}^{1} k(s, t, \xi(t))dt = \sum_{j=1}^{n} w_j k(s_i, t_j, \xi_j), i = 1, \cdots, n$$

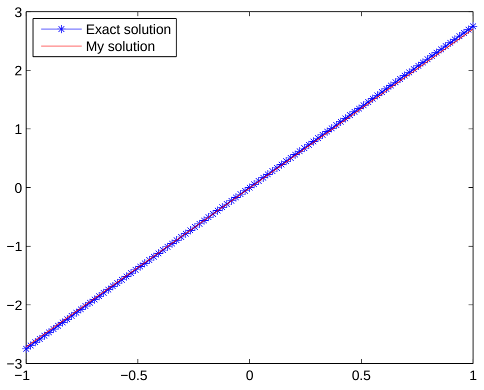

FIGURE 1 The exact and approximate solution of Example 5.1 , with $n = 99$

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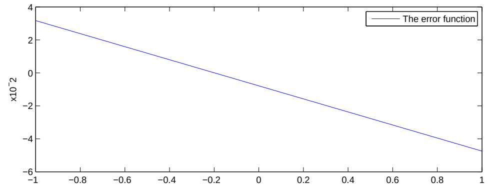

FIGURE 2 The error function for Example 5.1

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5 | NUMERICAL SIMULATION

Let x^* be the exact solution and ξ^* be the approximate solution using Gauss-Kronrod quadrature with the following error function $\delta(s_i) = x^*(s_i) - \xi^*(s_i)$.

5.1 | example 1

Consider the nonlinear integro-differential equations

$$x'(s) = exp(1)s - 1 - \int_{-1}^{1} (t + s)exp(x(t))dt$$
$$x(-1) = 0,$$

with the initial vector $\xi^{(0)} = 0$, $\xi_0^{(k)} = (1, \cdots, 1)^T$, $n = 99$ and $\epsilon = 4.4409e - 16$.
In Figure1 A comparison between exact and approximate solution and in Figure2 the error function $\delta(s)$

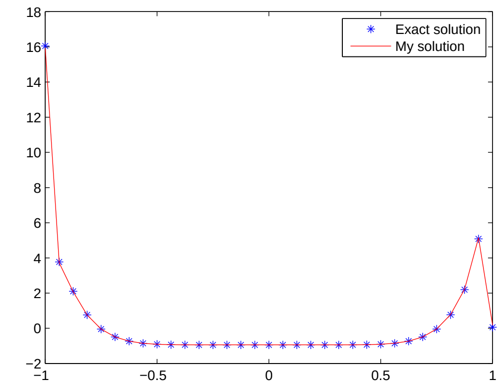

FIGURE 3 The exact and approximate solution of Example 5.2 , with $n = 33$

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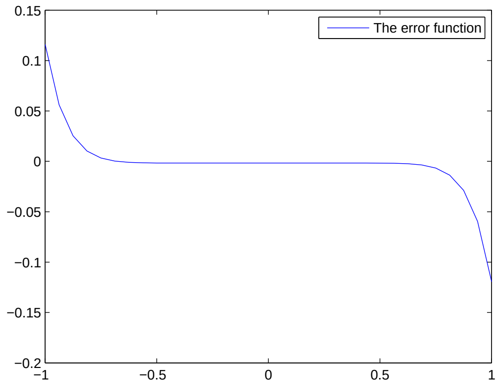

FIGURE 4 The error function for Example 5.2

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5.2 | example 2

Consider the linear Fredholm integral equation of the second kind

$$x(s) = 50s^{49} - 58s^{57} + \frac{16}{3009}s^{11} + 9s^{8} - \frac{257}{273} - \int_{-1}^{1}(s^{11} - t^{11})x(t))dt$$

$$x(-1) = 0,$$

with the initial vector $\xi^{(0)} = 0$, $\xi_0^{(k)} = 0$, $n = 33$ and $\epsilon = 4.4409e - 16$.
In Figure3 A comparison between exact and approximate solution and in Figure4 the error function $\delta(s)$

Remark 2. The most important in this example it's not just that the approximate and the exact solution are to similar but the time of calculation in our method is very fast then the exact.
Where **My time** = 0.0026 and **Time exact** = 0.0732

References

1. V. BABENKO, Theorems of Existence and Uniqueness for Nonlinear Fredholm and Volterra Integral Equations for Functions with Values in L-spaces, Conference Proceeding, (2015).

2. E. BABOLIAN, A. SHAHSAVARAN, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, Journal of Computational and Applied Mathematics **225** (2009) 87-95.

3. A. H. BORZABADI , O. S. FARD, A numerical scheme for a class of nonlinear Fredholm integral equations of the second kind, Journal of Computational and Applied Mathematics **232** (2009) 449-454.

4. A. H. BORZABADI , A. V. KAMYAD and H. H. MEHNE, A different approach for solving the nonlinear Fredholm integral equations of the second kind, Applied Mathematics and Computation **173** (2006) 724–735.

5. H. CHEBBAH,, A. MENNOUNI and N.E. RAMDANI, Numerical Solution of Generalized Logarithmic Integral Equations of the Second Kind by Projections, Malaysian Journal of Mathematical Sciences **12(3)**: 349–367 (2018).

6. J.P. CALVI, Analyse Numérique, UPS Université de Toulouse, (2014).

7. M. DOBRITOIU, A Nonlinear Fredholm Integral Equation. TJMM; **1** (2009), No. 1-2, 25-32.

8. D.P. LAURIE, Calculation of Gauss-Kronrod Quadrature Rules, MATHEMATICS OF COMPUTATION Volume **66**, Number 219, (1997).

9. A. MENNOUNI, N.E. RAMDANI, Collocation method to solve second order Cauchy integro-differential equations, in Differential and Difference Equations with Applications, ICDDEA 2017 proceedings, Editors: Pinelas, S., Caraballo, T., Kloeden, P., Graef, J.R. (Eds.), springer (2018), ISBN 978-3-319-75647-9.

10. A. MENNOUNI, N.E. RAMDANI and Kh. ZENNIR, A New Class of Fredholm Integral Equations of the Second Kind with Non Symmetric Kernel: Solving by Wavelets Method, Boletim da Sociedade Paranaense de Matemática, (2018).

11. J. STOER and R. BULIRSCH, Introduction to Numerical Analysis, Springer-Verlag, New York, (1993).

12. A. M WAZWAZ, Nonlinear Fredholm Integro-Differential Equations. Linear and Nonlinear Integral Equations,(2011) 517–546.

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