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RESEARCH ARTICLE

Exact and Approximate Solutions of Fractional Partial Differential Equations Using Aboodh Transform Iterative Method

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Abstract

In this article, we proposed a new iterative method for solving time-fractional partial differential equations. This method is a combination of Aboodh transform method and new iterative method which we termed Aboodh transform iterative method. The proposed method requires that the nonlinear term be decomposed using new iterative method without need for constructing homotopy and Adomian polynomial. Some illustrative examples are presented with there solution profiles to display the effect of the fractional order. The fractional derivative is described in Caputo sense.

KEYWORDS:
Fractional derivative, iterative method, Caputo fractional derivative, Aboodh Transform, Laplace transform

1 | INTRODUCTION

Fractional calculus is a field of mathematics which consist of both ordinary and partial derivatives of positive noninteger order. It has attracted huge interest because it provide realistic models than the integer derivatives,[1,2,3,4,5,6] for comprehensive study on fractional calculus see[7,8,9,10,11].

Integral transform for solving problems in mathematical physics and engineering science is traceable to the work of P.S. Laplace (1749-1827) on probability theory in the 1780's also, to the treatise of J.B Fourier (1768-1830) titled "La Thèorie Analytique de La chaleur" published in 1822,[12] since then the development and establishment of new integral transforms with various modifications have been of great interest to researchers. Sumudu transform was introduced by G.K Watugala in 1993,[13] and was successfully applied to solve differential equations and control problems in engineering science. In 2008, Khan and Khan[14] introduced the natural transform method. T. Elzaki, established the Elzaki transform in 2011 to facilitate the process of solving ordinary and partial differential equations in time domain.[15] Aboodh transform was introduced in 2013,[16] it was derived from the classical fourier integral. Shehu transform was invented in 2019,[17] while the ARA transform was introduced in 2020, the ARA transform unified some variants of the classical laplace transform such as sumudu transform, Elzaki transform, the natural transform and the Shehu transform.[18]

Some researchers combined these integral transform methods with analytical method such as homotopy analysis method, Adomian decomposition method, variational iterative method, homotopy perturbation method, new iterative method and so on. They employed these combinations to solve linear and nonlinear differential equations. Laplace transform method was combined with Adomian decomposition method, new iterative method and homotopy analysis method,[19,20,21,22] Sumudu transform method was also combined with homotopy analysis method, Adomian decomposition method and new iterative method,[23,24,25,26,27,28] similarly, natural transform method was coupled with homotopy analysis method and pertubation method,[29,30,31,32,33] Elzaki transform method was also coupled with homotopy anaylsis method, Adomian decomposition method and new iterative

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

method.[34,35,36] The combinations of Aboodh transform method and variational iterative method, homotopy analysis method, Adomian decomposition method was also presented in,[37,38,39,40] while Shehu transform was combined with homotopy analysis method, new iterative method and Adomian decomposition method.[41,42,43,44] To the best of our knowledge Aboodh transform is yet to be coupled with the new iterative method.

An iterative method called "new iterative method" was proposed by Daftardar-Gejji and Jafari[45] to solve functional equations, the solution was presented in series form. Several authors have used the new iterative method to solve linear and nonlinear fractional partial differential equations.[46,47,48,49,50]

In this article, we propose Aboodh transform iterative method to obtain the exact and approximate series solutions of fractional partial differential equations, the solutions are given in terms of rapid convergent series. This method is easy to implement without requirement for lagrange multiplier, linearization, discretization and perturbation. A major advantage of this method is the prevention of round-off errors. This paper is arrange as follows, we discussed briefly some definitions, properties of Aboodh transform and important notations in section 2, while in section 3, we considered the fundamental concept of Aboodh transform iterative method and in section 4, the accuracy and efficiency were illustrated with the application of the method with examples. Finally, we presented the conclusion in section 5.

2 | PRELIMINARY CONCEPT

In this section, we present important definitions and preliminary ideas which are used further in this article.

Definition 1

Aboodh transform of function u(t) is obtained over the set A,[16]

$$A = \left\{u(t) : \exists B, k_1, k_2 > 0, |u(t)| < Be^{-vt}\right\}, \tag{1}$$

where u(t) is denoted by

$$A\left[u(t)\right] = H(v), \tag{2}$$

and defined as

$$A\left[u(t)\right] = \frac{1}{v}\int_0^\infty u(t)e^{-vt}dt = H(v), \quad t \geq 0, \quad k_1 \leq v \leq k_2. \tag{3}$$

It is important to note that dual relationship exist between Aboodh transform and some integral transform such as Laplace transform, natural transform and Elzaki transform[51].

Definition 2

The inverse Aboodh transform is defined by Alfaqeih and Özis,[52]

$$A^{-1}\left[H(v)\right] = u(t), t \geq 0. \tag{4}$$

Equivalently,

$$u(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ve^{-vt}H(v)dv, \tag{5}$$

where v is the Aboodh transform variable and γ is a real constant.

Lemma 1: Aboodh transform linearity property[53]

If Aboodh transform of $u_1(t)$ and $u_2(t)$ are H(v) and P(v) respectively, then

$$A\left[\xi_1 u_1(t) + \xi_2 u_2(t)\right] = \xi_1 A\left[u_2(t)\right] + \xi_2 A\left[u_2(t)\right] \tag{6}$$

$$= \xi_1 H(v) + \xi_2 P(v),$$

where ξ_1 and ξ_2 are arbitrary constants.

Definition 3

The Aboodh transform of a Caputo fractional derivative is [37]

$$A\left[(D_t^{\alpha}u(t)); v\right] = v^{\alpha} A\left[u(t)\right] - \sum_{s=0}^{n-1} \frac{u^{(s)}(0)}{v^{2-\alpha+s}}, \quad n-1 < \alpha \leq n, \quad n = 1, 2, \ldots \tag{7}$$

Definition 4

Aboodh transform of n-th derivative of u(t) is given as [37]

$$A\left[u^{(n)}t; v\right] = v^{n} A\left[u(t)\right] - \sum_{s=0}^{n-1} \frac{u^{(s)}(0)}{v^{2-n+s}}, \quad n-1 < \alpha \leq n, \quad n = 1, 2, \ldots \tag{8}$$

See table 1 for Aboodh transform of some elementary functions and table 2 for the inverse transform.

Definition 5

Mittage -Leffler function for one parameter is given as [54]

$$E_{\alpha}(z) = \sum_{s=0}^{\infty} \frac{z^{s}}{\Gamma(s\alpha + 1)}, \quad \alpha \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{9}$$

3 | BASIC IDEA OF ABOODH TRANSFORM ITERATIVE METHOD

In this section, we described the idea of Aboodh transform iterative method.

3.1 | Aboodh transform iterative method

Consider the fractional partial derivative equation

$$D_t^{\alpha}u(x,t) = Q(u(x,t)) + N(u(x,t)) + g(x,t), \quad n-1 < \alpha \leq n, \tag{10}$$

with the initial conditions

$$u^{s}(x,0) = k_s(x), \quad s = 0, 1, 2, \ldots, \alpha - 1. \tag{11}$$

D^{α} denote the Caputo fractional derivative operator, g(x,t) is the source term, u(x,t) is the unknown function while Q and N denote the linear and the nonlinear differential operators respectively. Operating the Aboodh transform on both sides of Equation (10), we get

$$A\left[D_t^{\alpha}\right] = A\left[Q(u(x,t)) + N(u(x,t)) + g(x,t)\right]. \tag{12}$$

Simplifying and taking inverse Aboodh transform of Equation (12), we get

$$u(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{s=0}^{n-1} \frac{u^{(s)}(x,0)}{v^{2-\alpha+s}}\right) + \frac{1}{v^{\alpha}}\left(A\left[Q(u(x,t)) + N(u(x,t)) + g(x,t)\right]\right)\right]. \tag{13}$$

The nonlinear operator N can be decomposed as, [45]

$$N(u(x,t)) = N\left(\sum_{s=0}^{\infty} u_s(x,t)\right) \tag{14}$$

$$= N(u_0(x,t)) + \sum_{s=1}^{\infty}\left\{N\left(\sum_{j=0}^{s} u_j(x,t)\right) - N\left(\sum_{j=0}^{s-1} u_j(x,t)\right)\right\}.$$

Define

$$u^{(r)}(x,t) = \sum_{s=0}^{r} u_s(x,t). \tag{15}$$

Assume the solution of Equation (10) as

$$u(x,t) = \lim_{r\to\infty} u^{(r)}(x,t) = \sum_{s=0}^{\infty} u_s(x,t). \tag{16}$$

If N and Q are contractions, then the series in Equation (16) absolutely and uniformly converges to a unique solution for Equation (10).[45] Substitute Equation (14) into Equation (13), we get

$$\sum_{m=0}^{\infty} u_m(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{u^{(s)}(x,0)}{v^{2-\alpha+s}}\right) + A\left[g(x,t)\right]\right] + A^{-1}\left[\frac{1}{v^{\alpha}}A\right.$$
$$\left.\left[Q(u_0(x,t)) + N(u_0(x,t)) + \sum_{s=1}^{\infty}\left(Q(u_s(x,t)) + \left\{N\left(\sum_{j=0}^{s}u_j(x,t)\right) - N\left(\sum_{j=0}^{s-1}u_j(x,t)\right)\right\}\right)\right]\right]. \tag{17}$$

Thus, we obtain the following iterations

$$u_0(x,t) = A\left[\frac{1}{v^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{u^{(s)}(x,0)}{v^{2-\alpha+s}}\right) + A[g(x,t)]\right], \quad n-1 < \alpha \le n, \tag{18}$$

$$u_1(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}A\left[Q(u_0(x,t)) + N(u_0(x,t))\right]\right], \tag{19}$$

$$\vdots$$

$$u_{s+1}(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}A\left[Q(u_s(x,t)) + \left\{N\left(\sum_{j=0}^{s}u_j(x,t)\right) - N\left(\sum_{j=0}^{s-1}u_j(x,t)\right)\right\}\right]\right], \quad s = 1,2,\ldots \tag{20}$$

The series solution converges, for the convergence see[45,55].

4 | NUMERICAL EXAMPLE

In this section, we considered examples to demonstrate the efficiency and effectiveness of the proposed method.

Example 1

Consider the fractional diffusion-wave equation[19]

$$D_t^{\alpha}u + \frac{1}{3}(u_{xx} + u_{yy} + u_{zz}) = 0, \quad 0 \le \alpha \le 1, \tag{21}$$

with the initial condition

$$u(x,y,z,0) = e^{-(x+y+z)}.$$

Applying Aboodh transform to both sides of Equation (21) subject to the initial condition and taking the inverse Aboodh transform, we get

$$u(x,y,z,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{e^{(x+y+z)}}{v^{2-\alpha}}\right) - \frac{1}{3}\left(\frac{1}{v^{\alpha}}A\left[Q\left(\sum_{s=0}^{\infty}(u_{sxx} + u_{syy} + u_{szz})\right)\right]\right)\right]. \tag{22}$$

Using the iterative technique discussed in section 3, we get

$$u(x,y,z,0) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{e^{-(x+y+z)}}{v^{2-\alpha}}\right)\right], n = 1, \tag{23}$$

$$= A^{-1}\left[\frac{e^{-(x+y+z)}}{v^{2}}\right]$$

$$= e^{-(x+y+z)},$$

$$u_1(x,y,z,t) = -\frac{1}{3}\left[\left(\frac{1}{v^\alpha}A\left[Q(u_{0xx}+u_{0yy}+u_{0zz})\right]\right)\right] \tag{24}$$
$$= \frac{-1}{3}A^{-1}\left[\frac{3e^{-(x+y+z)}}{v^{\alpha+2}}\right]$$
$$= -e^{-(x+y+z)}\frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2(x,y,z,t) = -\frac{1}{3}\left[\left(\frac{1}{v^\alpha}A\left[Q(u_{1xx}+u_{1yy}+u_{1zz})\right]\right)\right] \tag{25}$$
$$= \frac{-1}{3}A^{-1}\left[\frac{1}{v^\alpha}\left(\frac{-3e^{-(x+y+z)}}{v^{\alpha+2}}\right)\right]$$
$$= e^{-(x+y+z)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3(x,y,z,t) = -\frac{1}{3}\left[\left(\frac{1}{v^\alpha}A\left[Q(u_{2xx}+u_{2yy}+u_{2zz})\right]\right)\right] \tag{26}$$
$$= \frac{-1}{3}A^{-1}\left[\frac{1}{v^\alpha}\left(\frac{3e^{-(x+y+z)}}{v^{2\alpha+2}}\right)\right]$$
$$= -e^{-(x+y+z)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$\vdots$$
$$u_m(x,y,z,t) = -\frac{1}{3}A^{-1}\left[\left(\frac{1}{v^\alpha}A\left[Q(u_{(m-1)xx}+u_{(m-1)yy}+u_{(m-1)zz})\right]\right)\right] \tag{27}$$
$$= \frac{-1}{3}A^{-1}\left[\frac{1}{v^\alpha}\left(\frac{3e^{-(x+y+z)}}{v^{(m-1)\alpha+2}}\right)\right]$$
$$= (-1)^m e^{-(x+y+z)}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}.$$

The m-th series solution is

$$u^{(m)}(x,y,z,t) = \sum_{s=0}^{m} u_s = u_0+u_1+u_2+u_3+\cdots+u_m, \tag{28}$$
$$= e^{-(x+y+z)}\left(1-\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+\frac{(-1)^m t^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$
$$= e^{-(x+y+z)}\sum_{s=0}^{m}\frac{(-1)^s(t^{s\alpha})}{\Gamma(s\alpha+1)}.$$

Thus,

$$u(x,y,z,t) = \lim_{m\to\infty} u^{(m)}(x,y,z,t) \tag{29}$$
$$= e^{-(x+y+z)}\lim_{m\to\infty}\sum_{s=0}^{m}\frac{(-1)^s(t^{s\alpha})}{\Gamma(s\alpha+1)}$$
$$= e^{-(x+y+z)}E_\alpha(-t^\alpha).$$

The special case when $\alpha = 1$, the exact solution is

$$u(x,y,z,t) = e^{-(x+y+z+t)}. \tag{30}$$

The solution profile with various values of α are given in Figures 1 , 2 , 3 .

Example 2

Consider the time-fractional Schrodinger differential equation[36]

$$iD_t^{\alpha}u + u_{xx} = 0, \;\; i^2 = 1, \;\; 0 < \alpha \leq 1; \tag{31}$$

with initial condition:

$$u(x,0) = ae^{ikx}.$$

Using same procedure and the iterative formula in section 3, we obtain

$$\begin{aligned} u(x,0) &= A^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\sum_{s=0}^{n-1}\frac{ae^{ikx}}{\nu^{2-\alpha+s}}\right)\right], n = 1, \\ &= A^{-1}\left[\frac{ae^{ikx}}{\nu^{\alpha}}\right] \\ &= ae^{ikx}, \end{aligned} \tag{32}$$

$$\begin{aligned} u_1(x,t) &= A^{-1}\left[\frac{1}{\nu^{\alpha}}iA\left[Q(u_{0xx})\right]\right] \\ &= A^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\frac{-ik^2ae^{ikx}}{\nu^2}\right)\right] \\ &= -ak^2e^{ikx}\frac{it^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{33}$$

$$\begin{aligned} u_2(x,t) &= A^{-1}\left[\frac{1}{\nu^{\alpha}}iA\left[Q(u_{1xx})\right]\right] \\ &= A^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\frac{-ak^4e^{ikx}}{\nu^{2+\alpha}}\right)\right] \\ &= ak^4e^{ikx}\frac{(it^{\alpha})^2}{\Gamma(2\alpha+1)}, \end{aligned} \tag{34}$$

$$\begin{aligned} u_3(x,t) &= A^{-1}\left[\frac{1}{\nu^{\alpha}}iA\left[Q(u_{2xx})\right]\right] \\ &= A^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\frac{iak^6e^{ikx}}{\nu^{2\alpha+2}}\right)\right] \\ &= -ak^6e^{ikx}\frac{(it^{\alpha})^3}{\Gamma(3\alpha+1)}, \end{aligned} \tag{35}$$

$$\vdots$$

$$\begin{aligned} u_m(x,t) &= A^{-1}\left[\frac{1}{\nu^{\alpha}}iA\left[Q\left(u_{(m-1)xx}\right)\right]\right] \\ & A^{-1}\left[\frac{1}{\nu^{\alpha}}\left(\frac{iak^{2m}e^{ikx}}{\nu^{(m-1)\alpha+2}}\right)\right] \\ &= (-1)^m ak^{2m}e^{ikx}\frac{(it^{\alpha})^m}{\Gamma(m\alpha+1)}. \end{aligned} \tag{36}$$

The m-th series solution is

$$\begin{aligned} u^m(x,t) &= \sum_{s=0}^{m} u_s = u_0 + u_1 + u_2 + u_3 + \cdots + u_m \\ &= ae^{ikx}\left(1 - \frac{ik^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(ikt)^2}{\Gamma(3\alpha+1)} - \frac{(ik^2t^{\alpha})^3}{3\alpha+1} + \cdots + \frac{(-1)^m(ik^2t^{\alpha})^m}{\Gamma(m\alpha+1)}\right) \\ &= ae^{ikx}\sum_{s=0}^{m}(-1)^m\frac{(ik^2t^{\alpha})^m}{\Gamma(m\alpha+1)}. \end{aligned} \tag{37}$$

Hence,

$$u(x,t) = \lim_{m\to\infty} u^m(x,t) \tag{38}$$

$$= ae^{ikx} \lim_{m\to\infty} \sum_{s=0}^{m} (-1)^s \frac{(ik^2 t^\alpha)^s}{\Gamma(s\alpha+1)}.$$

The case where $\alpha = 1$, the exact solution is

$$u(x,t) = ae^{ik(x-kt)}. \tag{39}$$

The solution profile with various values of α are given in Figures 4 ,5 ,6 .

Example 3

Consider the nonhomogeneous time fractional backward Klomogorov equation[41]

$$D_t^\alpha u = -x^2 e^t u_{xx} + (x+1)u_x + xt, \quad 0 < \alpha < 1, \tag{40}$$

with initial condition

$$u(x,0) = x+1.$$

Using same procedure and the iterative formula in section 3, We obtain

$$u(x,0) = A^{-1}\left[\frac{1}{v^\alpha}\left(\sum_{s=0}^{n-1} \frac{x+1}{v^{2-\alpha+s}}\right) + [xt]\right], n = 1, \tag{41}$$

$$= A^{-1}\left[\frac{x+1}{v^2} + \frac{x}{v^{\alpha+3}}\right]$$

$$= (x+1) + \frac{xt^{\alpha+1}}{\Gamma(2+\alpha)},$$

$$u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(-x^2 e^t u_{0xx} + (x+1)u_{0x}\right)\right]\right)\right] \tag{42}$$

$$= (x+1)A^{-1}\left[\frac{1}{v^{2+\alpha}} + \frac{1}{v^{2\alpha+3}}\right]$$

$$= (x+1)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right),$$

$$u_2(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(-x^2 e^t u_{1xx} + (x+1)u_{1x}\right)\right]\right)\right] \tag{43}$$

$$= (x+1)A^{-1}\left[\frac{1}{v^{2\alpha+2}} + \frac{1}{v^{3\alpha+3}}\right]$$

$$= (x+1)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right),$$

$$u_3(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(-x^2 e^t u_{2xx} + (x+1)u_{2x}\right)\right]\right)\right] \tag{44}$$

$$= (x+1)A^{-1}\left[\frac{1}{v^{3\alpha+3}} + \frac{1}{v^{4\alpha+4}}\right]$$

$$= (x+1)\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right),$$

$$\vdots$$

$$u_m(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(-x^2 e^t u_{(m-1)xx} + (x+1)u_{(m-1)x}\right)\right]\right)\right] \tag{45}$$

$$= (x+1)A^{-1}\left[\frac{1}{v^{m\alpha+2}} + \frac{1}{v^{(m+1)\alpha+3}}\right]$$

$$= (x+1)\left(\frac{t^{m\alpha}}{\Gamma(m\alpha+1)} + \frac{t^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right).$$

The m-th series solution is

$$u^m(x,t) = \sum_{s=0}^{m} u_s = u_0 + u_1 + u_2 + u_3 + \cdots + u_m \tag{46}$$

$$= (x+1)\left(\left\{\sum_{s=0}^{m} \frac{t^{s\alpha}}{\Gamma(s\alpha+1)} + \sum_{s=0}^{m} \frac{t^{(s+1)\alpha+1}}{\Gamma((s+1)\alpha+2)}\right\}\right) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Thus,

$$u(x,t) = \lim_{m\to\infty} u^{(m)}(x,t) \tag{47}$$

$$= (x+1)\left(\lim_{m\to\infty} \sum_{s=0}^{m} \frac{t^{s\alpha}}{\Gamma(s\alpha+1)} + \lim_{m\to\infty} \sum_{s=0}^{m} \frac{t^{(s+1)\alpha+1}}{\Gamma((s+1)\alpha+2)}\right) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$= (x+1)\left(E_\alpha(t^\alpha) + \lim_{m\to\infty} \sum_{s=0}^{m} \frac{t^{(s+1)\alpha+1}}{\Gamma((s+1)\alpha+2)}\right) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

The special case when $\alpha = 1$, the exact solution is

$$u(x,t) = (x+1)\left(e^t + \lim_{m\to\infty} \sum_{s=0}^{m} \frac{t^{s+2}}{\Gamma(s+3)}\right) - \frac{t^2}{2}.$$

The solution profile with various values of α are given in Figures 7 , 8 , 9 .

Example 4

Consider the nonlinear time fractional gas dynamics equation[56]

$$D_t^\alpha u + \frac{1}{2}u_x^2 - u + u^2 = 0, \;\; 0 < \alpha < 1, \tag{48}$$

with initial condition

$$u(x,0) = e^{-x}.$$

Using same procedure and the iterative formula in section 3, we obtain

$$u(x,0) = A^{-1}\left[\frac{1}{v^\alpha}\left(\sum_{s=0}^{n-1} \frac{e^{-x}}{v^{2-\alpha+s}}\right)\right], \;\; n = 1, \tag{49}$$

$$A^{-1}\left[\frac{e^{-x}}{v^2}\right]$$

$$= e^{-x},$$

$$u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(u_0\right) - N\left(\frac{1}{2}u_{0x}^2 + u_0^2\right)\right]\right)\right] \tag{50}$$

$$= A^{-1}\left[\frac{e^{-x}}{v^{\alpha+2}}\right]$$

$$= \frac{e^{-x}t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q(u_1) + \left\{N\left(\frac{-1}{2}(u_0+u_1)_x^2 - (u_0+u_1)^2\right) - N\left(\frac{-1}{2}u_{0x}^2 - u_0^2\right)\right\}\right]\right)\right] \tag{51}$$

$$= A^{-1}\left[\frac{e^{-x}}{v^{2\alpha+2}}\right]$$

$$= \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q(u_2)+\left\{N\left(\frac{-1}{2}(u_0+u_1+u_2)_x^2-(u_0+u_1+u_2)^2\right)-N\left(\frac{-1}{2}(u_0+u_1)_x^2-(u_0+u_1)^2\right)\right\}\right]\right)\right] \quad (52)$$

$$= A^{-1}\left[\frac{e^{-x}}{v^{3\alpha+2}}\right]$$

$$= \frac{e^{-x}t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

$$u_m(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(u_{(m-1)}\right)+\left\{N\left(\sum_{j=0}^{m-1}\left(\frac{1}{2}u_{jx}^2-u^{2_j}\right)-N\left(\sum_{j=0}^{m-2}\left(\frac{1}{2}u_{jx}^2-u_j^2\right)\right)\right)\right\}\right]\right)\right] \quad (53)$$

$$= A^{-1}\left[\frac{e^{-x}}{v^{m\alpha+2}}\right]$$

$$= \frac{e^{-x}}{\Gamma(m\alpha+1)}.$$

The m-th series solution is

$$u^{(m)}(x,t) = \sum_{s=0}^{m} u_s = u_0+u_1+u_2+u_3+\cdots+u_m \quad (54)$$

$$= e^{-x}\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$= e^{-x}\sum_{s=0}^{m}\frac{t^{s\alpha}}{\Gamma(s\alpha+1)}.$$

Hence,

$$u(x,t) = \lim_{m\to\infty} u^m(x,t) \quad (55)$$

$$= e^{-x}\lim_{m\to\infty}\sum_{s=0}^{m}\frac{t^{s\alpha}}{\Gamma(s\alpha+1)}$$

$$= e^{-x}E_\alpha(t^\alpha). \quad (56)$$

The special case when $\alpha = 1$, the exact solution is

$$u(x,t) = e^{t-x}. \quad (57)$$

The solution profile with various values of α are given in Figures 10 , 11 , 12 .

Example 5

Consider the nonlinear time fractional Fokker-Planck equation[57]

$$D_t^\alpha u = \left(\frac{1}{3}xu-\frac{4}{x}u^2\right)_x + u_{xx}^2, \quad 0<\alpha<1, \quad x\in R, \quad (58)$$

with initial condition

$$u(x,0) = x^2.$$

Using same procedure and the iterative formula in section 3, we obtain

$$u(x,0) = A^{-1}\left[\frac{1}{v^\alpha}\left(\frac{x^2}{v^{2-\alpha+s}}\right)\right], n=1, \quad (59)$$

$$= A^{-1}\left[\frac{x^2}{v^2}\right]$$

$$= x^2,$$

$$u_1(x,t) = A^{-1}\left[\frac{1}{v^\alpha}\left(A\left[Q\left(\frac{x}{3}u_{0x}\right)-N\left(\frac{4}{x}u_{0x}^2-u_{0xx}^2\right)\right]\right)\right] \quad (60)$$

$$= A^{-1}\left[\frac{x^2}{v^{\alpha+2}}\right]$$

$$= \frac{x^2t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(A\left[Q\left(\frac{x}{3}u_{1x}\right)+\left\{N\left(\frac{-4}{x}\left(u_0+u_1\right)_x^2+\left(u_0+u_1\right)_{xx}^2\right)-N\left(\frac{-4}{x}u_{0x}^2+u_{0xx}^2\right)\right\}\right]\right)\right] \tag{61}$$

$$= A^{-1}\left[\frac{x^2}{v^{2+2\alpha}}\right]$$

$$= \frac{x^2t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(A\left[Q\left(\frac{x}{3}u_{2x}\right)+N\left(\frac{-4}{x}\left(u_0+u_1+u_2\right)_x^2+\left(u_0+u_1+u_2\right)_{xx}^2\right)-N\left(\frac{-4}{x}\left(u_0+u_1\right)_x^2+\left(u_0+u_1\right)_{xx}^2\right)\right]\right)\right] \tag{62}$$

$$= A^{-1}\left[\frac{x^2}{v^{2+3\alpha}}\right]$$

$$= \frac{x^2t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

$$u_m(x,t) = A^{-1}\left[\frac{1}{v^{\alpha}}\left(A\left[Q\left(u_{(m-1)x}^2\right)-\left\{N\left(\sum_{j=0}^{m-1}\left(\frac{4}{x}u_{jx}^2-u_{jxx}^2\right)-N\left(\sum_{j=0}^{m-2}\left(\frac{4}{x}u_{jx}^2-u_{jxx}^2\right)\right)\right\}\right]\right)\right] \tag{63}$$

$$= A^{-1}\left[\frac{x^2}{v^{2+m\alpha}}\right]$$

$$= \frac{x^2t^{m\alpha}}{\Gamma(m\alpha+1)}.$$

The m-th series solution is

$$u^m(x,t) = \sum_{s=0}^{m} u_s = u_0+u_1+u_2+u_3+\cdots+u_m \tag{64}$$

$$= x^2\left(1+\frac{t^{\alpha}}{\Gamma(s\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma 3\alpha+1}+\cdots+\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\right)$$

$$= x^2\sum_{s=0}^{m}\frac{t^{s\alpha}}{\Gamma(s\alpha+1)}.$$

Hence,

$$u(x,t) = \lim_{m\to\infty} u^{(m)}(x,t) \tag{65}$$

$$= x^2\lim_{m\to\infty}\sum_{s=0}^{m}\frac{t^{s\alpha}}{\Gamma(s\alpha+1)}$$

$$= x^2E_{\alpha}(t^{\alpha}).$$

The special case when $\alpha = 1$, the exact solution is

$$u(x,t) = x^2e^t. \tag{66}$$

The solution profile with various values of α is given in Figure13 .

5 | CONCLUSION

In this article, we have presented both approximate and exact solutions of time-fractional diffusion-wave equation, linear time-fractional Schrodinger equation, nonhomogeneous time-fractional backward Klomogorov equation, nonlinear time-fractional gas dynamics equation and the nonlinear time-fractional Fokker-Planck equation. The outcomes reveals that Aboodh transform iterative method is efficient and easy to use. In order to study the effect of the fractional order, we presented the solution profiles for various values of alpha, however the interpretation is left to the readers. We conclude that the method is reliable and effective in obtaining both exact and approximate solutions without the need for prescribed assumption, discretization, linearization and free of round-off error.

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Conflict of interest

The authors declare no potential conflict of interests.

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□

TABLE 1 Aboodh transform for some elementary functions.[37]

$u(t)$	$\mathrm{A}[u(t)] = \mathrm{H}(v)$
1	$\frac{1}{v^2}$
t	$\frac{1}{v^3}$
t^n	$\frac{n!}{v^{n+2}}$, n = 0,1,2,...
t^α	$\frac{\Gamma(\alpha+1)}{v^{\alpha+2}},\ \alpha \geq 0$

TABLE 2 Inverse Aboodh transform for some elementary functions.[37]

$H(v)$	$u(t) = A^{-1}[H(v)]$
$\frac{1}{v^2}$	1
$\frac{1}{v^3}$	t
$\frac{n!}{v^{n+2}}$, n = 0,1,2,...	t^n
$\frac{\Gamma(\alpha+1)}{v^{\alpha+2}},\ \alpha \geq 0$	t^α

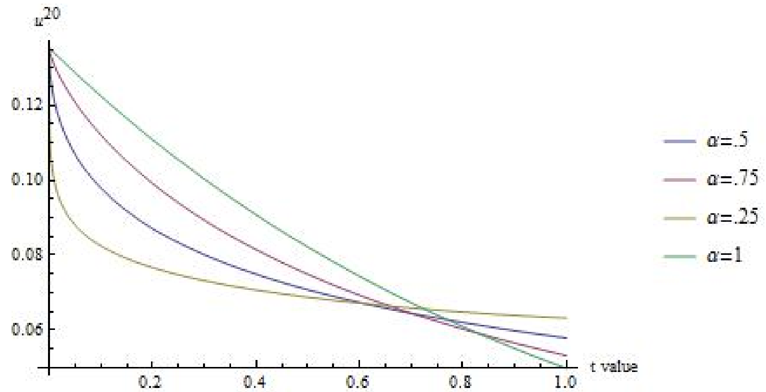

FIGURE 1 Solution profile of example 1 when $x = 1$

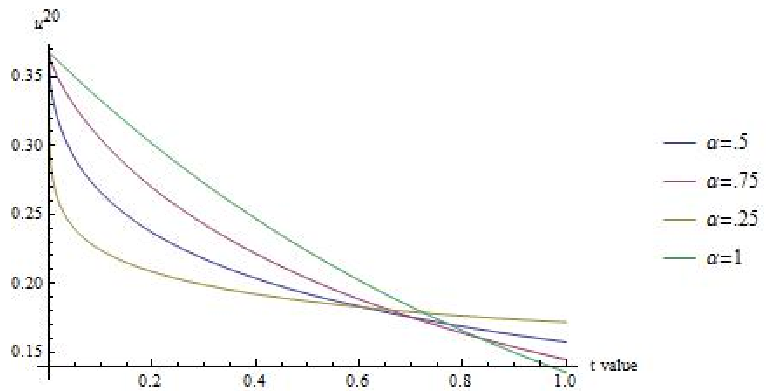

FIGURE 2 Solution profile of example 1 when $x = 0$

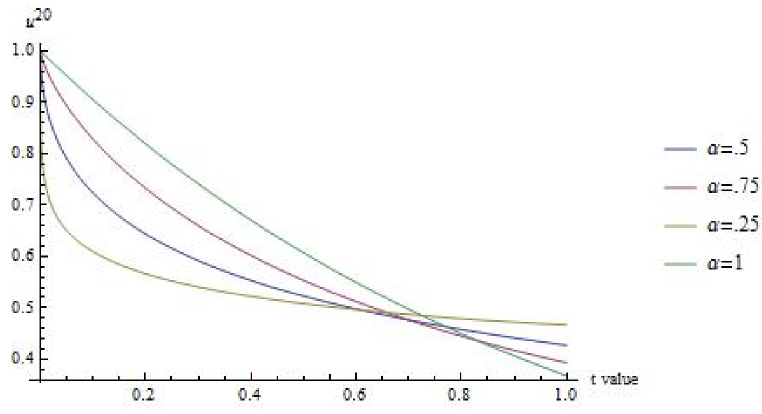

FIGURE 3 Solution profile of example 1 when $x = -1$

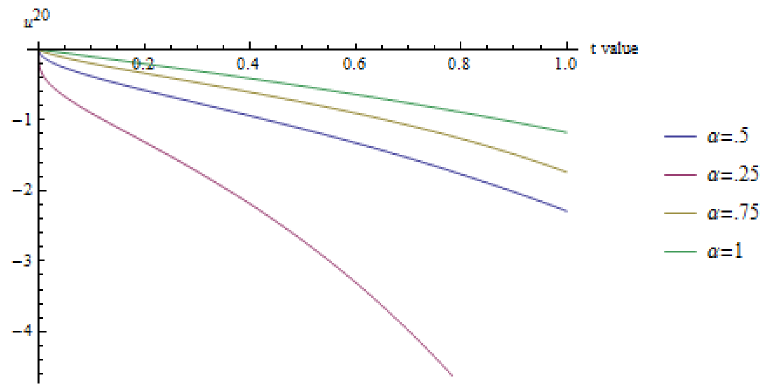

FIGURE 4 Solution profile for example 2 when $x = 0$

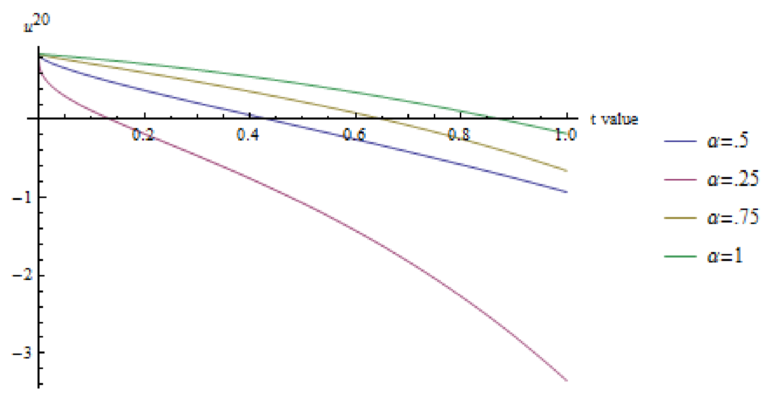

FIGURE 5 Solution profile for example 2 when $x = 1$

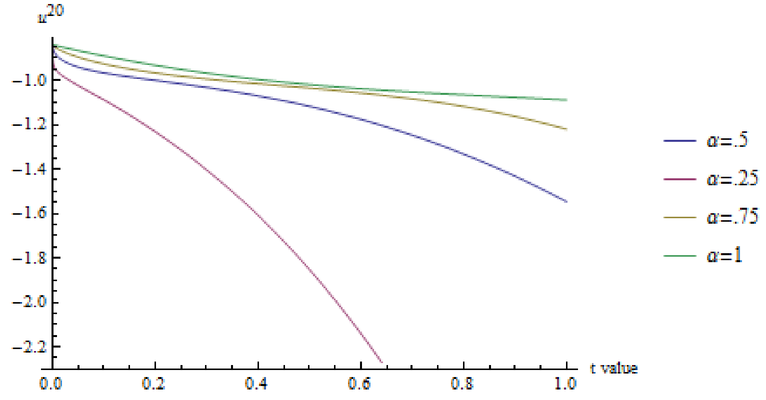

FIGURE 6 Solution profile for example 2 when $x = -1$

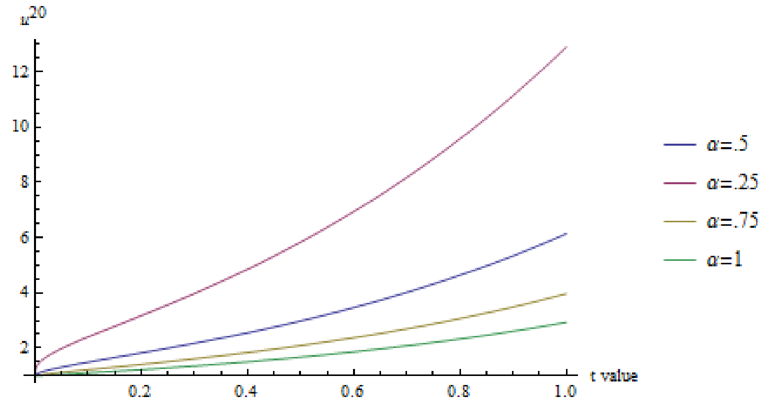

FIGURE 7 Solution profile for example 3 when $x = 0$

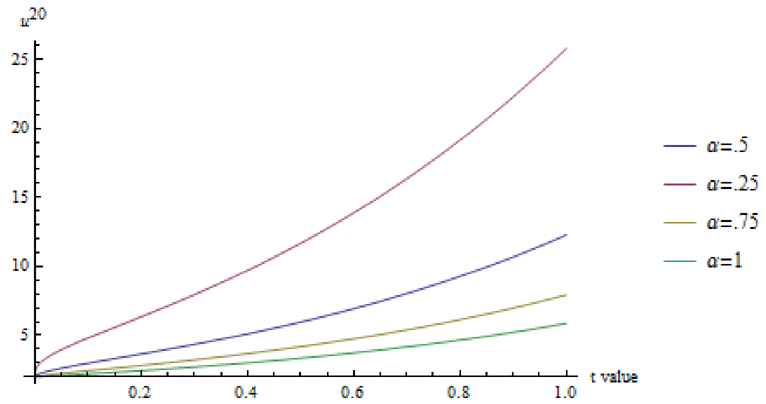

FIGURE 8 Solution profile for example 2 when $x = 1$

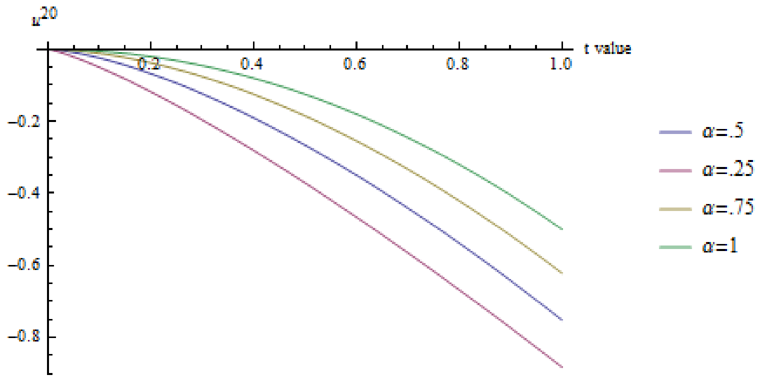

FIGURE 9 Solution profile for example 3 when $x = -1$

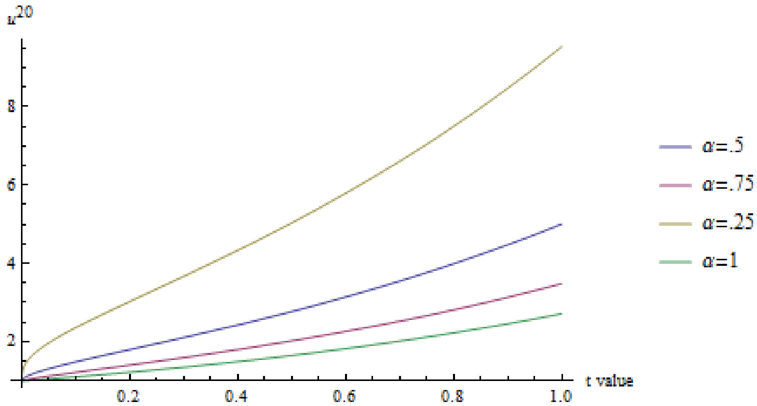

FIGURE 10 Solution profile for example 4 when $x = 0$

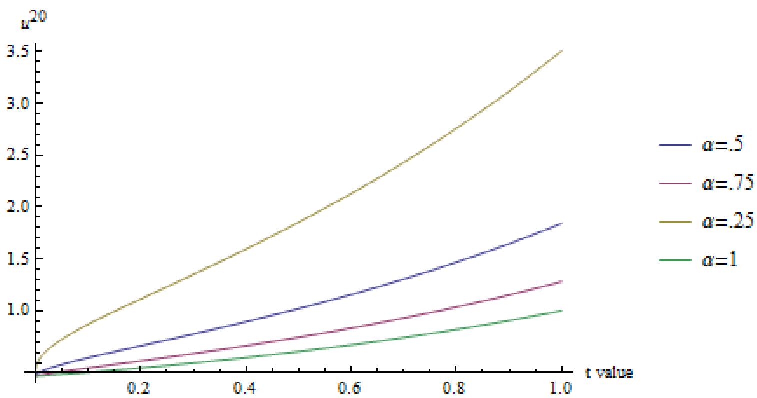

FIGURE 11 Solution profile for example 4 when $x = 1$

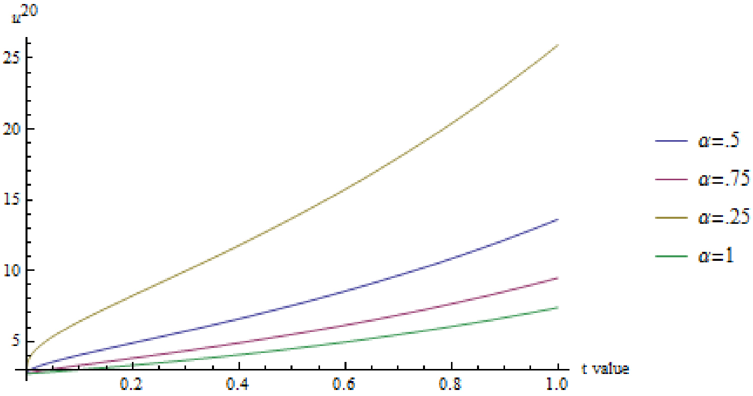

FIGURE 12 Solution profile for example 4 when $x = -1$

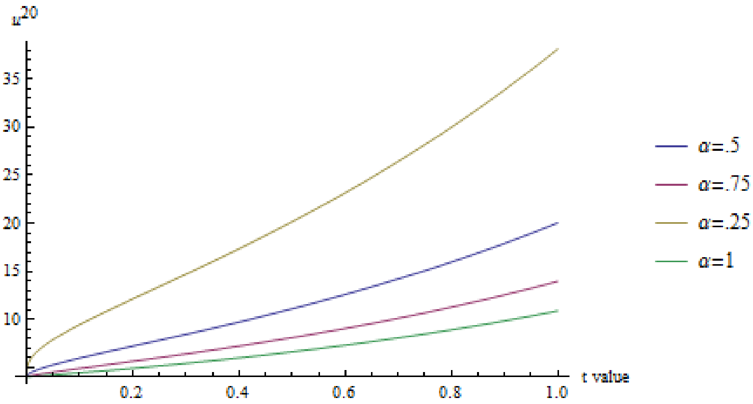

FIGURE 13 Solution profile for example 5 when $x = 2$