Application of the Riemann-Hilbert approach to the derivative nonlinear Schrödinger hierarchy*

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Abstract: In this paper, the main work is to study the N-soliton solutions for the derivative nonlinear Schrödinger hierarchy. Then, the matrix Riemann-Hilbert problem is constructed for this integrable hierarchy by analyzing the spectral problem of the Lax pair. Based on the scattering relationship, the N-soliton solutions for this system are given explicitly.

Keywords: derivative nonlinear Schrödinger hierarchy; Riemann-Hilbert problem; Lax pair; the scattering relationship; N-soliton solutions.

MSC codes: 35Q51; 35Q15; 37K10

1 Introduction

As is well known, the well-known (1+1) dimensional derivative nonlinear Schrödinger (DNLS) equation

$$iu_t + u_{xx} + i\alpha(|u|^2u)_x = 0, \tag{1.1}$$

is a very important integrable equation, it has several applications in many branches of physics and applied mathematics, particularly in optics, water wave.

The derivative nonlinear Schrödinger hierarchy is meaningful to find their symmetries and algebraic structure [1]. In fact, the DNLS hierarchy have been studied in many ways, such as: Hamiltonian structure [2], Darboux transformation [3], Tau-Symmetry [4].

In this paper, we consider the DNLS hierarchy via the matrix Riemann-Hilbert problem, namely Fokas method [5]. Learning from the method and experience of predecessors, Fokas method perfectly combined the initial value and boundary value problems. Also, it brings important advantage to the methodology that yields precise information about the long-time asymptotic behavior of the solutions in [6].

Moreover, with the help of nonlinearization in the steepest descent method [7], Fokas method may represent how the solution of large t splits into a collection of solitons traveling at constant speeds. However, when the solution is away from these solitons the asymptotic displays a dispersive character [8]. The global relation which imposes a constraint on the given initial and boundary values provides the perfect solution to this problem at $t = 0$ or $x = 0$ [9–20]. And, the

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RHP also was applied to solve the long-time asymptotic behavior of integrable equations [21,22], the soliton solutions of integrable equations [23–26] and so on.

The present paper is arranged as follows. In **Section 2**, we firstly recall the Lax pair for the DNLS hierarchy. Meanwhile, we consider the matrix spectral analysis of the DNLS hierarchy by introducing the Jost solution, the eigenfunctions and the scattering matrix $S(\lambda)$. In **Section 3**, in order to propose the Riemann-Hilbert problem for the DNLS hierarchy, we introduce two matrix functions and the jump matrix $G(x, t_{2n}; \lambda)$. In **Section 4**, we construct N-soliton solutions for the DNLS hierarchy by the Riemann-Hilbert method and the scattering relationship. In **Section 5**, we give a brief conclusion for this paper.

2 The Lax pair and matrix spectral analysis for the DNLS hierarchy

In order to derive the DNLS hierarchy, we firstly recall the Lax pair as follows,

$$\begin{cases} \Phi_x = U(x, t_{2n}; \lambda)\Phi, \\ \Phi_{t_{2n}} = V(x, t_{2n}; \lambda)\Phi, n \geq 2, \end{cases} \tag{2.1}$$

where $U(x, t_{2n}; \lambda) = (\frac{i}{2}\lambda^2\sigma + \lambda P)$, $V(x, t_{2n}; \lambda) = \sum\limits_{j=0}^{2n} V_{2n-j}\lambda^j$ and $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & r(x, t_{2n}) \\ q(x, t_{2n}) & 0 \end{pmatrix}$, this system above can be given by the zero curvature equation $V_x = [U, V]$. In order to facilitate the later batter research work, under the conditions $V_0 = \sigma$ and $V_1 = P$, the Lax pair (2.1) can be rewritten as

$$\begin{cases} \Phi_x = \frac{i}{2}\lambda^2\sigma\Phi + \hat{P}\Phi, \\ \Phi_{t_{2n}} = \frac{i}{2}\lambda^{2n}\sigma\Phi + \hat{Q}\Phi, n \geq 2, \end{cases} \tag{2.2}$$

where $\hat{P} = \lambda P, \hat{Q} = \lambda^{2n-1}P + \sum\limits_{j=0}^{2n-2} V_{2n-j}\lambda^j$. Then, one can introduce the Jost solution for the equation (2.2) with asymptotic from read

$$\Phi \sim e^{\frac{i}{2}\lambda^2\sigma x + \frac{i}{2}\lambda^{2n}\sigma t_{2n}}, \ |x| \longrightarrow \infty. \tag{2.3}$$

Let

$$\mu(x, t_{2n}; \lambda) = \Phi e^{-\frac{i}{2}\lambda^2\sigma x - \frac{i}{2}\lambda^{2n}\sigma t_{2n}} \tag{2.4}$$

being a new matrix spectral function, it shows that through the above transformation.

$$\mu \longrightarrow I, \ |x| \longrightarrow \infty, \ \textit{where } I \textit{ is a } 2\times 2 \textit{ identity matrix.} \tag{2.5}$$

Therefore, the Lax equation (2.2) can be rewritten as

$$\begin{cases} \mu_x - \frac{i}{2}\lambda^2[\sigma, \mu] = \hat{P}\mu, \\ \mu_{t_{2n}} - \frac{i}{2}\lambda^{2n}[\sigma, \mu] = \hat{Q}\mu, \end{cases} \tag{2.6}$$

where $[\sigma, \mu] = \sigma\mu - \mu\sigma$. By introducing the eigenfunctions of the Lax equation (2.2) expressed as

$$\mu_1 = I + \int_{-\infty}^{x} e^{i\lambda^2(x-y)\sigma} \hat{P} \mu_1 e^{-i\lambda^2(x-y)\sigma} dy, \tag{2.7}$$

$$\mu_2 = I - \int_{x}^{+\infty} e^{i\lambda^2(x-y)\sigma} \hat{P} \mu_2 e^{-i\lambda^2(x-y)\sigma} dy. \tag{2.8}$$

Considering the new spectral problem(2.4), we have two fundamental matrix solutions

$$\Phi_1 = \mu_1 e^{\frac{i}{2}\lambda^2\sigma x + \frac{i}{2}\lambda^{2n}\sigma t_{2n}}, \quad \Phi_2 = \mu_2 e^{\frac{i}{2}\lambda^2\sigma x + \frac{i}{2}\lambda^{2n}\sigma t_{2n}}. \tag{2.9}$$

Then, the scattering matrix $S(\lambda) = S_{2\times 2}(\lambda)$, which can be expressed as

$$\mu_1 = \mu_2 e^{\frac{i}{2}\lambda^2\sigma x + \frac{i}{2}\lambda^{2n}\sigma t_{2n}} S(\lambda) e^{-\frac{i}{2}\lambda^2\sigma x - \frac{i}{2}\lambda^{2n}\sigma t_{2n}}. \tag{2.10}$$

The analyticity and symmetry of the eigenfunctions μ_1, μ_2 and scattering matrix $S(\lambda)$ should be paid more attention. According to the integral equations (2.7), (2.8) and integral region $y < x$, we have

$$e^{\frac{i}{2}\lambda^2(x-y)\sigma} \hat{P} e^{-\frac{i}{2}\lambda^2(x-y)\sigma} = \begin{pmatrix} 0 & \lambda r e^{i\lambda^2(x-y)} \\ \lambda q e^{-i\lambda^2(x-y)} & 0 \end{pmatrix}. \tag{2.11}$$

Therefore, note $([\mu_1]_1, [\mu_1]_2) = \mu_1, ([\mu_2]_1, [\mu_2]_2) = \mu_2$, it can be shown that $[\mu_1]_1$ is analytic in $\{Im\lambda^2 < 0\}$, $[\mu_1]_2$ is analytic in $\{Im\lambda^2 > 0\}$. And $[\mu_2]_1$ is analytic in $\{Im\lambda^2 > 0\}$, $[\mu_2]_2$ is analytic in $\{Im\lambda^2 < 0\}$. On the basis of the Able's formula and $tr\hat{P} = 0$, it can be derived that $|\mu_{1(2)}|$ are independent for all x. Then, we have

$$det(\mu_{1(2)}) = 1 \tag{2.12}$$

with the asymptotic conditions $|\mu_{1(2)}| \to I$, as $|x| \to \infty$. It is obvious to obtain

$$det(S(\lambda)) = 1 \tag{2.13}$$

from the equation (2.12). Furthermore, let

$$\mu_1^{-1} = \begin{pmatrix} [\mu_1^{-1}]_1, & [\mu_1^{-1}]_2 \end{pmatrix}^T, \tag{2.14}$$

$$\mu_2^{-1} = \begin{pmatrix} [\mu_2^{-1}]_1, & [\mu_2^{-1}]_2 \end{pmatrix}^T. \tag{2.15}$$

Then, we know that $[\mu_1^{-1}]_1$ is analytic in $\{Im\lambda^2 > 0\}$, $[\mu_1^{-1}]_2$ is analytic in $\{Im\lambda^2 < 0\}$, $[\mu_2^{-1}]_1$ is analytic in $\{Im\lambda^2 < 0\}$ and $[\mu_2^{-1}]_2$ is analytic in $\{Im\lambda^2 > 0\}$. From the equation (2.10), we obtain

$$\mu_1 E = \mu_2 E S(\lambda), \ E = e^{\frac{i}{2}\lambda^2\sigma x}. \tag{2.16}$$

Thus, the analyticity of $S(\lambda)$ can be expressed by

$$E S(\lambda) E^{-1} = \mu_2^{-1}\mu_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.17}$$

where $a_{11} = [\mu_2^{-1}]_1[\mu_1]_1, a_{12} = [\mu_2^{-1}]_1[\mu_1]_2, a_{21} = [\mu_2^{-1}]_2[\mu_1]_1$ and $a_{22} = [\mu_2^{-1}]_2[\mu_1]_2$, and $s_{11}(\lambda)$ is analytic in $\{Im\lambda^2 < 0\}$, $s_{22}(\lambda)$ is analytic in $\{Im\lambda^2 > 0\}$.

3 The construction of the Riemann-Hilbert problem for the DNLS hierarchy

We introduce two matrix functions for constructing the Riemann-Hilbert problem,

$$P^{+}(x,t_{2n};\lambda)=([\mu_1]_1,[\mu_2]_2)(x,t_{2n};\lambda)=\mu_1A_1+\mu_2A_2, \tag{3.1}$$

$$P^{-}(x,t_{2n};\lambda)=([\mu_1^{-1}]_1,[\mu_2^{-1}]_2)^T(x,t_{2n};\lambda)=A_1\mu_1+A_2\mu_2, \tag{3.2}$$

where $A_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $A_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, it is easy to see that $\begin{cases}P^{+}(x,t_{2n};\lambda)\to I,\\ P^{-}(x,t_{2n};\lambda)\to I,\end{cases}$ as $\lambda\to\infty$.

On the basis of the preparatory knowledge above, we can propose the Riemann-Hilbert problem:

1. P^+ is analytic in $\{Im\lambda^2<0\}$, P^- is analytic in $\{Im\lambda^2>0\}$,

2. $P^{-}(x,t_{2n};\lambda)P^{+}(x,t_{2n};\lambda)=G(x,t_{2n};\lambda)$,

3. $P^{\pm}(x,t_{2n};\lambda)\to\ I$, as $\lambda\to\ \infty$,

where the jump matrix $G(x,t_{2n};\lambda)=e^{\frac{i}{2}\lambda^2\sigma x+\frac{i}{2}\lambda^{2n}\sigma t_{2n}}\begin{pmatrix}1&h_{12}\\s_{21}&1\end{pmatrix}e^{-\frac{i}{2}\lambda^2\sigma x-\frac{i}{2}\lambda^{2n}\sigma t_{2n}}$, and $H(\lambda)=\begin{pmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{pmatrix}=S^{-1}(\lambda)$, $h_{11}s_{11}+h_{12}s_{21}$=1. This is the associated matrix RHp, which is built for the derivative nonlinear Schrödinger hierarchy.

4 N-soliton solutions for the DNLS hierarchy

In this section, we construct N-soliton solutions for the derivative nonlinear Schrödinger hierarchy by the Riemann-Hilbert method. According to the definition of $P^{\pm}$ and the scattering relationship (2.10), we have

$$\begin{cases}det(P^{+})=det(\mu_1A_1+\mu_2A_2)=s_{11}(\lambda),\\ det(P^{-})=det(A_1\mu_1^{-1}+A_2\mu_2^{-1})=h_{11}(\lambda),\end{cases} \tag{4.1}$$

which shows that the zeros of $det(P^+),det(P^-)$ are the same as s_{11},h_{11} respectively. Because of the symmetry of anti-Hermitian matrix U and the relation $\mu_{1(2)}^{H}(\lambda^*)=\mu_{1(2)}^{-1}$, where the superscript H represents conjugate transpose. One can obtain $(P^+)^H(\lambda^*)=P^-(\lambda)$, $S^H(\lambda^*)=S^{-1}(\lambda)$ from the equation (2.12), it implies that

$$h_{11}^{H}(\lambda^*)=s_{11}(\lambda) \tag{4.2}$$

and each zero $\pm\lambda_k^*$ of h_{11} corresponds to each zero $\pm\lambda_k$ of s_{11}.

Assuming there exist n simple zeros $\{\lambda_k\}_{1\le k\le n}$ of $det(P^+)$ in $\{Im\lambda^2<0\}$ and n simple zeros $\{\lambda_k\}_{1\le k\le n}$ of $det(P^-)$ in $\{Im\lambda^2>0\}$, then,

$$\lambda_k^*=\lambda_k,\ 1\le k\le n. \tag{4.3}$$

Let ϕ_k, ϕ_k^* are nonzero column vector, row vector and they satisfy the following liner equations respectively:

$$\begin{cases} P^+(\lambda_k)\phi_k(\lambda_k) = 0, \\ \phi_k^*(\lambda_k^*)P^-(\lambda_k^*) = 0. \end{cases} \tag{4.4}$$

Comparing with the formulas above, we may get

$$\phi_k^* = \phi_k^H, \ 1 \leq k \leq n. \tag{4.5}$$

From the Lax equation (2.2) and (4.4), ϕ_k, ϕ_k^* can be expressed as

$$\begin{cases} \phi_k = e^{\frac{i}{2}\lambda_k^2\sigma x + \frac{i}{2}\lambda_k^{2n}\sigma t_{2n}}\phi_{k,0}, 1 \leq k \leq n, \\ \phi_k^* = \phi_{k,0}^H e^{-\frac{i}{2}(\lambda_k^*)^2\sigma x - \frac{i}{2}(\lambda_k^*)^{2n}\sigma t_{2n}}, 1 \leq k \leq n, \end{cases} \tag{4.6}$$

where $\phi_{k,0}$ is two-dimension constant column vector. In order to present soliton solutions explicitly, the scatting date must be satisfied the condition $s_{21} = 0$. Therefore, the solutions to the Riemann-Hilbert problem are explicitly given by

$$\begin{cases} P^+(\lambda) = I - \sum\limits_{k,j=1}^{N} \frac{\phi_k\phi_j^*(M^{-1})_{kj}}{\lambda - \lambda_j^*}, \\ P^-(\lambda) = I + \sum\limits_{k,j=1}^{N} \frac{\phi_k\phi_j^*(M^{-1})_{kj}}{\lambda - \lambda_k^*}, \end{cases} \tag{4.7}$$

where the $n \times n$ matrix M is defined by

$$M_{kj} = \frac{\phi_k^*\phi_j}{\lambda_j - \lambda_k^*}, \ 1 \leq k, j \leq n. \tag{4.8}$$

The asymptotic expansion of $P^+(\lambda)$ can expressed as

$$P^+(\lambda) = I + \lambda^{-1}P_1^+ + \lambda^{-2}P_2^+ + O(\lambda^{-3}), \ \lambda \to \infty. \tag{4.9}$$

And substitute asymptotic expansion (4.9) into the first equation of (2.6), we get

$$-\frac{i}{2}[\sigma, P_1^+] = P, \tag{4.10}$$

then, $q(x, t_{2n}), r(x, t_{2n})$ can be expressed as

$$\begin{cases} q(x, t_{2n}) = -i(P_1^+)_{12}, \\ r(x, t_{2n}) = -i(P_1^+)_{21}, \end{cases} \tag{4.11}$$

where $(P_1^+)_{12}$ is the $(1,2)-$element of the matrix P_1^+, $(P_1^+)_{21}$ is the $(2,1)-$element of the matrix P_1^+. From the equation (4.7), thus, we obtain

$$P_1^+ = -\sum_{k=1}^{N}\sum_{j=1}^{N} \phi_k\phi_j^*(M^{-1})_{kj}. \tag{4.12}$$

As a result, the $N-$soliton solutions to the derivative nonlinear Schrödinger hierarchy (2.2) are read by

$$q(x,t_{2n}) = -i\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\beta_j^* e^{\xi_j^*-\xi_k}(M^{-1})_{kj}, \tag{4.13}$$

where $m_{kj} = \frac{\alpha_k\alpha_k^* e^{-\xi_j-\xi_k^*}+\beta_k\beta_k^* e^{\xi_j+\xi_k^*}}{\lambda_j-\lambda_k^*}$ and $\xi_k = \frac{i}{2}\lambda_k^2 x + \frac{i}{2}\lambda_k^{2n}t_{2n}$, $\phi_{k,0} = (\alpha_k,\beta_k)^T, 1\leq k\leq N$. In particular, assuming $\lambda_1 = 1+\frac{i}{2}, \alpha_1 = \beta_1 = 1$, then,

$$q(x,t_{2n}) = 2\beta_1^* e^{\xi_1^*-\xi_1} sech(\xi_1^*+\xi_1). \tag{4.14}$$

5 Conclusion and Discussion

In this paper, the method of constructing N-soliton solutions to the derivative nonlinear Schrödinger hierarchy is Riemann-Hilbert method. In fact, scholars have done a lot of research on the DNLS equation via other approaches to the soliton solutions, such as the Hirota direct method, the Wronskian technique, the Darboux transformation. But we focus on the soliton solutions for the DNLS hierarchy in this paper, and provide a new way to solve the integrable hierarchy in integrable systems. At the same time, the mathematical structure and physical properties of the DNLS hierarchy are also need to study.

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