GENERAL MASSLESS FIELD EQUATIONS FOR HIGHER SPIN IN DIMENSION 4

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Abstract. Massless field equations are fundamental in particle physics. In Clifford analysis, the Euclidean version of these equations has been dealt with but it is not clear, even in dimension 4, what should be the right analogue of massless field equations for fields with values in a general irreducible $Spin(4)$-module. The main aim of the paper is to explain that a good possibility is to take the so-called generalized Cauchy-Riemann equations proposed a long time ago by E. Stein and G. Weiss. For this choice of the equations, we show that their polynomial solutions form different irreducible $Spin(4)$-modules. This is an important step in developing the corrresponding function theory.

1. Introduction

In particle physics, the field equations in the Minkowski space $\mathbb{R}^{1,3}$ for massless particles of lower spins are fundamental, namely, the wave equation for spin 0, the Dirac equation for spin $\frac{1}{2}$, the Maxwell equations for spin 1, the Rarita-Schwinger equation for spin $\frac{3}{2}$ and the equation of the linearized gravity for spin 2. For any spin, R. Penrose initiated a systematic study of solutions of massless field equations in his twistor program, see [13, 17, 18].

In the framework of Clifford analysis, the Euclidean version of massless field equations has been studied first in dimension 4 and then in a general dimension, see [19, 20]. The first difficulty lies in the fact that it is not clear, even in dimension 4, what should be the set of equations defining the massless fields with values in a general irreducible $Spin(4)$-module.

Classical Clifford analysis is a function theory for spinor valued solutions of the Dirac equation in the Euclidean space $\mathbb{R}^N$. Let $e_1, \dots, e_N$ be the standard basis of $\mathbb{R}^N$ and $\mathbb{C}_N$ be the complex Clifford algebra generated by the vectors $e_1, \dots, e_N$ satisfying the relations $e_ie_j + e_je_i = -2\delta_{ij}$. Then the spinor space $\mathbb{S}_N$ is a unique irreducible $\mathbb{C}_N$-module. As $Spin(N)$-module, in even dimensions N, the spinor space $\mathbb{S}_N$ decomposes as $\mathbb{S}_N = \mathbb{S}_N^+ \oplus \mathbb{S}_N^-$ into two different basic spinor representations $\mathbb{S}_N^\pm$ but, in odd dimensions N, $\mathbb{S}_N$ remains irreducible and we shall write $\mathbb{S}_N = \mathbb{S}_N^+ = \mathbb{S}_N^-$. For a smooth function $f : \mathbb{R}^N \to \mathbb{S}_N^\pm$, we define the Dirac operator as

$$\partial f(x) = \sum_{j=1}^{N} e_j \frac{\partial f}{\partial x_j}(x)$$

where $x = x_1e_1 + \cdots + x_Ne_N \in \mathbb{R}^N$.

The spinor spaces $\mathbb{S}_N^\pm$ are the simplest representations of the group $Spin(N)$. So a natural question arises what the best analogue of Clifford analysis is for higher spin representations. In other words, for a general irreducible $Spin(N)$-module V, we want to find the best analogue of the Dirac equation for V-valued functions defined in $\mathbb{R}^N$. There are more possibilities. First let

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1991 *Mathematics Subject Classification.* 30G35, 33C55, 22E46.

Key words and phrases. Massless field equations, higher spins, Stein-Weiss gradients, Clifford analysis, homogeneous polynomial solutions.

The support of the grant GACR 20-11473S is gratefully acknowledged by R. Lávička and V. Souček. The third author is supported by National Nature Science Foundation in China (No. 11971425).

us remark that the Dirac operator ∂ is elliptic and it is not only rotationally invariant but even conformally invariant. For many such V, there is a unique (up to a multiple) conformally invariant elliptic first order differential operator ∂_V which is called a higher spin Dirac operator, see [5]. So the first possibility is to develop function theory for higher spin Dirac operators, see e.g. [14]. The second possibility is to take the generalized Cauchy-Riemann equations (GCR) suggested by E. Stein and G. Weiss in [21]. In this paper, we consider the second possibility.

Now we recall (GCR) for a given irreducible $Spin(N)$-module V. First consider an irreducible decomposition of $Spin(N)$-module

$$V \otimes \mathbb{C}^N = F_0 \oplus F_1 \oplus \cdots \oplus F_r$$

where $F_0 = V \boxtimes \mathbb{C}^N$ is the Cartan product of V and the defining representation $\mathbb{C}^N$. Let π_j be the projection of $V \otimes \mathbb{C}^N$ onto F_j. For each $j = 0, \ldots, r$, define Stein-Weiss gradient

$$D_j(f) = \pi_j(\nabla(f)), \ f \in \mathcal{C}^\infty(\mathbb{R}^N, V)$$

where $\nabla(f)$ is the gradient of f. Then it is well-known that each Stein-Weiss gradient

$$D_j : \mathcal{C}^\infty(\mathbb{R}^N, V) \to \mathcal{C}^\infty(\mathbb{R}^N, F_j)$$

is rotationally invariant, even conformally invariant for a unique conformal weight w_j. For the given irreducible $Spin(N)$-module V, the generalized Cauchy-Riemann equations (GCR) are defined as

$$D_1(f) = 0, \ldots, D_r(f) = 0 \ \text{ for } f \in \mathcal{C}^\infty(\mathbb{R}^N, V). \tag{1}$$

Example 1. *For $V = \mathbb{S}_N^\pm$, we have $\mathbb{S}_N^\pm \otimes \mathbb{C}^N = (\mathbb{S}_N^\pm \boxtimes \mathbb{C}^N) \oplus \mathbb{S}_N^\mp$, $D_1 = \partial$ is the Dirac operator and (GCR) is the Dirac equation $\partial f = 0$.*

In general, higher spin Dirac operators are special examples of Stein-Weiss gradients.

Example 2. *Let $\Lambda^s := \Lambda^s(\mathbb{C}^N)$ denote the s-th antisymmetric power of $\mathbb{C}^N$. For $V = \Lambda^s$ with $s < N/2$, we have*

$$\Lambda^s \otimes \mathbb{C}^N = (\Lambda^s \boxtimes \mathbb{C}^N) \oplus \Lambda^{s+1} \oplus \Lambda^{s-1},$$

$D_1 = d$ and $D_2 = d^*$ is the de-Rham differential and codifferential, respectively, and (GCR) is the Hodge-de Rham system $df = 0$, $d^* f = 0$.

Example 3. *In dimension 4, we give an explicit description of all Stein-Weiss gradients for a general irreducible $Spin(4)$-module V in* (15) *below. We suggest that (GCR) is the best analogue of massless field equations in $\mathbb{R}^4$ for V-valued fields, see* (16).

In [21], it is shown that each $f \in \mathcal{C}^\infty(\mathbb{R}^N, V)$ satisfying (GCR) have the following properties: (i) f is harmonic and (ii) $|f|^p$ is subharmonic for $p \geq \frac{N-2}{N-1}$. The property (ii) enables to develop a theory of Hardy spaces for solutions of (GCR). By (i), in particular, each such a function f can be uniquely decomposed into homogeneous polynomial solutions of (GCR). Indeed, let us denote by $\mathcal{M}_m(V)$ the space of polynomials $P : \mathbb{R}^N \to V$ of degree m satisfying (GCR). Then we have

$$f = \sum_{m=0}^{\infty} f_m \tag{2}$$

for some uniquely determined $f_m \in \mathcal{M}_m(V)$ where the sum converges locally uniformly on $\mathbb{R}^N$. For $V = \mathbb{S}_N^\pm$ or $V = \Lambda^s(\mathbb{C}^N)$, it is well-known that the spaces $\mathcal{M}_m(V)$ form different irreducible $Spin(N)$-modules. Therefore, in these cases, the decomposition (2) is the best possible with respect to the underlying symmetry given by $Spin(N)$. For a general irreducible $Spin(N)$-module V, we prove this result in dimension $N = 4$, see Theorem 2 below, and we believe that this is true in any dimension N. Actually, an analogous result does not hold for homogeneous solutions of higher spin Dirac operators, see [14].

To show irreducibility of the modules $\mathcal{M}_m(V)$ is a first step in developing the corresponding function theory. Next steps are, for example, to describe the Howe duality, the Fischer decomposition and to construct Gelfand-Tsetlin bases of homogeneous solutions. This is well-known, in any dimension N, for the Dirac operator (see [1, 7, 16]) and the Hodge-de-Rham systems (see [10, 11, 12]). In dimension 4, the Fischer decomposition for massless fields of spin 1 is given in [2, 3]. In other cases, these results are unknown and should be investigated.

The content of the paper is the following. In Section 2, we recall, in dimension 4, the decomposition of scalar valued polynomials into spherical harmonics. In Theorem 1, we give explicit bases for homogeneous spherical harmonics. In Section 3, for a general irreducible $Spin(4)$-module V, we derive the generalized Cauchy-Riemann equations (GCR). In Section 4, we find explicit formulæ of highest weight vectors for homogeneous V-valued spherical harmonics. Using these highest weight vectors, we prove in Section 5 the main result of the paper. Namely, we show in Theorem 2 that homogeneous solutions of (GCR) form different irreducible $Spin(4)$-modules. Finally, in Theorem 3, we give explicit bases for the spaces of homogeneous solutions.

2. Harmonic case

First we consider the space $\mathcal{P}(\mathbb{R}^4)$ of polynomials $F : \mathbb{R}^4 \to \mathbb{C}$. On $\mathcal{P}(\mathbb{R}^4)$, we have a natural action of $Spin(4) \simeq SU(2) \otimes SU(2)$ and also its complexification $Spin(4, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. As usual, we identify the Euclidean space $\mathbb{R}^4$ first with the algebra $\mathbb{H}$ of quaternions and then with a real subspace of complex 2×2 matrices $\mathbb{C}^{2\times 2} \simeq \mathbb{C}^4$ in the following way

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H} \mapsto z = \begin{pmatrix} z_{00'} & z_{01'} \\ z_{10'} & z_{11'} \end{pmatrix} = \begin{pmatrix} x_0 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_0 - ix_3 \end{pmatrix} \tag{3}$$

Thus we can view each polynomial $F \in \mathcal{P}(\mathbb{R}^4)$ as a polynomial in the variables $z_{AA'}$ and extend it uniquely to $z = (z_{AA'}) \in \mathbb{C}^4$. Then, on $\mathcal{P}(\mathbb{R}^4)$ we consider the action of $G := SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ given by

$$[\rho(g, g')P](z) = P(g^t z g'), \quad P \in \mathcal{P}(\mathbb{R}^4), z \in \mathbb{C}^4 \text{ and } g, g' \in SL(2, \mathbb{C}).$$

The derived action of $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ on the polynomials $\mathcal{P}(\mathbb{R}^4)$ is then given by

$$X = z_{0A'}\nabla^{1A'}, \qquad Y = z_{1A'}\nabla^{0A'}, \qquad H = z_{0A'}\nabla^{0A'} - z_{1A'}\nabla^{1A'}, \tag{4}$$

$$X' = z_{A0'}\nabla^{A1'}, \qquad Y' = z_{A1'}\nabla^{A0'}, \qquad H' = z_{A0'}\nabla^{A0'} - z_{A1'}\nabla^{A1'}. \tag{5}$$

Here $A \in \{0, 1\}$ and $A' \in \{0', 1'\}$, we denote $\nabla^{AA'} = \frac{\partial}{\partial z_{AA'}}$ and use the Einstein summation convention. Then we have

$$[X, Y] = 2H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y$$

and the same commutation relations are satisfied for X', Y', H'. It is easy to see that the Laplace operator

$$\Delta := \det \begin{pmatrix} \nabla^{00'} & \nabla^{01'} \\ \nabla^{10'} & \nabla^{11'} \end{pmatrix} = \nabla^{00'}\nabla^{11'} - \nabla^{10'}\nabla^{01'} \tag{6}$$

and the multiplication with

$$r^2 := \det \begin{pmatrix} z_{00'} & z_{01'} \\ z_{10'} & z_{11'} \end{pmatrix} = z_{00'}z_{11'} - z_{10'}z_{01'}$$

are invariant operators acting on the polynomials $\mathcal{P}(\mathbb{R}^4)$. By (3), in the real coordinates, we have

$$r^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2, \qquad \Delta = \frac{1}{4}(\partial_{x_0}^2 + \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2).$$

It is well-known that, under the action of G, the module $\mathcal{P}(\mathbb{R}^4)$ has an irreducible decomposition

$$\mathcal{P}(\mathbb{R}^4) = \bigoplus_{m,j=0}^{+\infty} r^{2j}\mathcal{H}_m$$

where $\mathcal{H}_m = Ker(\Delta) \cap \mathcal{P}_m(\mathbb{R}^4)$ is the space of m-homogeneous spherical harmonics. In particular, the space $\mathcal{H}_m$ forms an irreducible G-modul. Actually, it is easy to find an explicit basis of $\mathcal{H}_m$.

Theorem 1. *(i) The G-module $\mathcal{H}_m$ has the highest weight vector $z_{00'}^m/m!$ and it has a basis consisting of the polynomials*

$$f_m^{r,s}(z) = \frac{1}{r!s!} Y^r (Y')^s \left(\frac{z_{00'}^m}{m!}\right), \ r, s = 0, \dots, m.$$

Here Y and Y' are the lowering operators given as in (4) *and* (5).

(ii) Moreover, we have

$$f_m^{r,s}(z) = \frac{z_{00'}^{m-s-r}}{(m-s-r)!} \frac{z_{10'}^r}{r!} \frac{z_{01'}^s}{s!} \, {}_2F_1(-s, -r, m-s-r+1; \alpha) \tag{7}$$

with $\alpha = \frac{z_{11'}z_{00'}}{z_{01'}z_{10'}}$.

(iii) We can express the basis in another form. Indeed, $\mathcal{H}_m$ has a basis consisting of $(m+1)^2$ functions

$$z_{(A_1A_1'}z_{A_2A_2'} \cdot \dots \cdot z_{A_mA_m')}, \tag{8}$$

where $A_1, \dots, A_m$ is a non-decreasing sequence of $\{0, 1\}$, $A_1', \dots, A_m'$ is a non-decreasing sequence of $\{0', 1'\}$ and $(\cdots)$ denotes symmetrization of unprimed indices. The basis (8) *is the same as the basis* (7) *up to a normalization.*

Remark 1. *Obviously, by permuting the primed indices and the unprimed ones in* (8)*, we get the same basis element.*

Proof. (i) Obvious. (ii) Of course, we easily get

$$f_m^{r,0}(z) = \frac{z_{00'}^{m-r}}{(m-r)!} \frac{z_{10'}^r}{r!} \quad \text{and} \quad f_m^{r,s} = \frac{1}{s!}(Y')^s f_m^{r,0}.$$

Moreover, we have

$$\frac{1}{s!}(Y')^s = \sum_{v=0}^{s} \frac{z_{11'}^v}{v!} \frac{z_{01'}^{(s-v)}}{(s-v)!} (\nabla^{00'})^{s-v} (\nabla^{10'})^v$$

and hence we obtain

$$\begin{aligned} f_m^{r,s}(z) &= \sum_{v=0}^{\min(s,r)} \frac{z_{11'}^v}{v!} \frac{z_{01'}^{(s-v)}}{(s-v)!} \frac{z_{10'}^{(r-v)}}{(r-v)!} \frac{z_{00'}^{(m+v-r-s)}}{(m+v-r-s)!} \\ &= \frac{z_{00'}^{m-s-r}}{(m-s-r)!} \frac{z_{10'}^r}{r!} \frac{z_{01'}^s}{s!} \sum_{v=0}^{+\infty} \frac{\alpha^v (-s)_v (-r)_v}{v!(m-s-r+1)_v} \end{aligned}$$

with $\alpha = \frac{z_{11'}z_{00'}}{z_{01'}z_{10'}}$ and $(a)_v$ being the Pochhammer symbol.

(iii) By (i), we know that $Y^r(Y')^s z_{00'}^m$, $r, s = 0, \dots, m$, constitute a basis of $\mathcal{H}_m$. Note that

$$Y'z_{A0'} = z_{A1'}, \qquad Y'z_{A1'} = 0, \qquad Yz_{0A'} = z_{1A'}, \qquad Yz_{1A'} = 0, \tag{9}$$

by expressions of Y and Y' in (4) and (5), and so $(Y')^s z_{00'}^m$ is $z_{00'}^{m-s} z_{01'}^s$ up to a constant. Then $Y^r(Y')^s z_{00'}^m$ is equal to

$$\sum_{A_1+\ldots+A_m=r} z_{A_1 0'} \cdots z_{A_{m-s}0'} \cdot z_{A_{m-s+1}1'} \cdots z_{A_m 1'} \tag{10}$$

up to a constant. This can be proved by induction. Indeed, by (9), we see that Y acting on (10) gives us the similar summation over $A_1+\ldots+A_m = r+1$ multiplied by $m-r$, which is the number of 0 appearing in $A_1, \ldots, A_m$ in (10). Now (8) is just another form of (10) up to a constant. □

3. Higher spin case

Now we study polynomials in dimension 4 which take values in a given (complex and finite dimensional) irreducible G-representation V. So we are interested in the G-module $\mathcal{P}(\mathbb{R}^4, V) := \mathcal{P}(\mathbb{R}^4) \otimes V$.

Spinorial notation. Before working with G-representations recall some simple facts on representations of $SL(2,\mathbb{C})$. Denote by $\mathbb{S} := \mathbb{C}^2$ the defining representation of $SL(2,\mathbb{C})$. Then an irreducible $SL(2,\mathbb{C})$-representation with the highest weight $k \in \mathbb{N}_0$ is equivalent to the symmetric power $\bigodot^k \mathbb{S}$. The dual (contragradient) module $\mathbb{S}^* = \mathbb{C}^2$ is isomorphic to $\mathbb{S}$. The isomorphism $\phi : \mathbb{S} \to \mathbb{S}^*$ is given by $\phi((s^0, s^1)) := (s_0, s_1)$ with $s_0 = -s^1$ and $s_1 = s^0$, which can be written as

$$s^A = \epsilon^{AB} s_B, \qquad s_B = s^A \epsilon_{AB} \qquad \text{with} \qquad \epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11}$$

In what follows, using ϵ, we shall raise and lower indices in general.

We realize $\bigodot^k \mathbb{S}$ as the module $\mathcal{P}_k(\mathbb{S}^*) \subset \mathbb{C}[s_0, s_1]$ of k-homogeneous polynomials with the action of $\mathfrak{sl}(2,\mathbb{C})$ given by

$$X = s_0 \partial^1, \qquad Y = s_1 \partial^0, \qquad H = s_0 \partial^0 - s_1 \partial^1. \tag{12}$$

Here we denote $\partial^A = \frac{\partial}{\partial s_A}$.

In more details, an element of $\bigodot^k \mathbb{C}^2$ is given by a 2^k-tuple $(F^{A_1 \ldots A_k}) \in \bigotimes^k \mathbb{C}^2$ such that $A_1, \ldots, A_k \in \{0, 1\}$, and $F^{A_1 \ldots A_k}$ is invariant under the permutations of superscripts, i.e.

$$F^{A_1 \ldots A_k} = F^{A_{\sigma(1)} \ldots A_{\sigma(k)}},$$

for any $\sigma \in S_k$, the group of permutations of k letters. The corresponding k-homogeneous polynomial is given by

$$F = F^{A_1 \ldots A_k} s_{A_1} \ldots s_{A_k} = k! \sum_{a=0}^{k} F^{0_a 1_{k-a}} \frac{s_0^a}{a!} \frac{s_1^{k-a}}{(k-a)!},$$

where $0_a = \underbrace{0 \ldots 0}_{a}$. In particular, we have $\mathbb{S} \simeq \{F(s) = a^A s_A | a^0, a^1 \in \mathbb{C}\}$.

It is easy to see and well-known that $\mathbb{S} \otimes \bigodot^k \mathbb{S} \simeq \bigodot^{k+1} \mathbb{S} \oplus \bigodot^{k-1} \mathbb{S}$. To describe the projections $\pi^\pm$ of $\mathbb{S} \otimes \bigodot^k \mathbb{S}$ onto $\bigodot^{k\pm1} \mathbb{S}$ in our polynomial language, note that both the operator of multiplication by $s = a^A s_A$ and the differential operator $a^A \partial_A$ are G-equivariant, hence the projections $\pi^\pm$ are given by

$$\pi^+(s \otimes F) := sF, \qquad \pi^-(s \otimes F) := a^A \partial_A(F). \tag{13}$$

with $\partial_A = \partial^B \epsilon_{BA}$.

Now it is easy to describe irreducible representations of $G = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Indeed, any irreducible G-representation is of the form

$$\mathbb{S}^{k,l} := \bigodot^k \mathbb{S} \otimes \bigodot^l \mathbb{S}'$$

for a unique label $(k,l) \in \mathbb{N}_0^2$ where $\mathbb{S}$ and $\mathbb{S}'$ are the defining representations of the first and second copy of $SL(2,\mathbb{C})$ in G, respectively. For $\mathbb{S}'$ we get the formulas analogous to (11), (12) and (13) if we replace all unprimed indices $A, B, 0, 1$ with the corresponding primed ones $A', B', 0', 1'$.

Remark 2. *Of course, we identify the spinor spaces $\mathbb{S}_4^+$ and $\mathbb{S}_4^-$ with $\mathbb{S}$ and $\mathbb{S}'$, respectively.*

Finally, we realize the G-module $\mathcal{P}(\mathbb{R}^4, \mathbb{S}^{k,l})$ as the module of scalar-valued polynomials in the variables $z_{AA'}$, s_B, $s_{B'}$ which are k-homogeneous in the variables s_B and l-homogeneous in the variables $s_{B'}$.

Stein-Weiss construction. Next we construct four G-invariant first order differential operators on $\mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^{k,l})$ following E. Stein and G. Weiss [21]. To do this, let us first decompose the tensor product $\mathbb{C}^4 \otimes \mathbb{S}^{k,l}$ under the action of the group G. Using the fact that $\mathbb{C}^4 \simeq \mathbb{S} \otimes \mathbb{S}'$, we get

$$\mathbb{C}^4 \otimes \mathbb{S}^{k,l} \simeq (\mathbb{S} \otimes \bigodot^k \mathbb{S}) \otimes (\mathbb{S}' \otimes \bigodot^l \mathbb{S}') \simeq \mathbb{S}^{k+1,l+1} \oplus \mathbb{S}^{k+1,l-1} \oplus \mathbb{S}^{k-1,l+1} \oplus \mathbb{S}^{k-1,l-1}$$

where $\mathbb{S}^{c,d} = 0$ unless $c, d \in \mathbb{N}_0$. We denote by $\pi^{\pm\pm}$ (4 possibilities) the projections of $\mathbb{C}^4 \otimes \mathbb{S}^{k,l}$ onto $\mathbb{S}^{k\pm1,l\pm1}$. For $F \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^{k,l})$, we have that

$$\nabla F := \begin{pmatrix} \nabla^{00'} F & \nabla^{01'} F \\ \nabla^{10'} F & \nabla^{11'} F \end{pmatrix} \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{C}^4 \otimes \mathbb{S}^{k,l}),$$

and we define

$$\mathcal{D}^{\pm\pm} F := \pi^{\pm\pm}(\nabla F). \tag{14}$$

It is well-known that $\mathcal{D}^{\pm\pm}$ are G-invariant first order differential operators on $\mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^{k,l})$ and, using (13) and the corresponding primed analogue, we have that

$$\begin{aligned}
\mathcal{D}^{++} &= s_A s_{A'} \nabla^{AA'}, \\
\mathcal{D}^{+-} &= s_A \partial_{A'} \nabla^{AA'} = s_A \det \begin{pmatrix} \partial^{0'} & \nabla^{A0'} \\ \partial^{1'} & \nabla^{A1'} \end{pmatrix}, \\
\mathcal{D}^{-+} &= s_{A'} \partial_A \nabla^{AA'} = s_{A'} \det \begin{pmatrix} \partial^{0} & \nabla^{0A'} \\ \partial^{1} & \nabla^{1A'} \end{pmatrix}, \\
\mathcal{D}^{--} &= \partial_A \partial_{A'} \nabla^{AA'}.
\end{aligned} \tag{15}$$

Standard notation. It is possible to translate the Stein-Weiss gradients $\mathcal{D}^{\pm\pm}$ into the standard spinorial notation (see [18]) as follows. Let $F \in \mathcal{P}(\mathbb{R}^4, \mathbb{S}^{k,l})$ and

$$F = F^{A_1 \dots A_k B'_1 \dots B'_l} s_{A_1} \dots s_{A_k} s_{B'_1} \dots s_{B'_l}.$$

Then we have (up to a non-zero multiple)

$$(\mathcal{D}^{+-} F)^{A_1 \dots A_{k+1} B'_1 \dots B'_{l-1}} = \nabla^{(A_1}_{B'_l} F^{A_2 \dots A_{k+1}) B'_1 \dots B'_l}$$

where $(\cdots)$ denotes the symmetrization in the unprimed indices. Indeed, it is easy to see that

$$\begin{aligned}
(\mathcal{D}^{+-} F) &= -s_{A_1} \partial^{A'} \nabla^{A_1}_{A'} \left(F^{A_2 \dots A_{k+1} B'_1 \dots B'_l} s_{A_2} \dots s_{A_{k+1}} s_{B'_1} \dots s_{B'_l} \right) = \\
&= -(k+1) \sum_{i=1}^{l} \delta^{A'}_{B'_i} \nabla^{(A_1}_{A'} F^{A_2 \dots A_{k+1}) B'_1 \dots B'_l} s_{A_1} \dots s_{A_{k+1}} s_{B'_1} \dots \widehat{s_{B'_i}} \dots s_{B'_l} \\
&= -(k+1) l \, \nabla^{(A_1}_{B'_l} F^{A_2 \dots A_{k+1}) B'_1 \dots B'_l} s_{A_1} \dots s_{A_{k+1}} s_{B'_1} \dots s_{B'_{l-1}}.
\end{aligned}$$

where $\widehat{s_{B'_i}}$ denotes the omission of the factor $s_{B'_i}$. Similarly, we have (up to non-zero multiples)

$$(\mathcal{D}^{-+} F)^{A_1 \dots A_{k-1} B'_1 \dots B'_{l+1}} = \nabla^{(B'_1}_{A_k} F^{B'_2 \dots B'_{l+1}) A_1 \dots A_k}$$

where $(\cdots)$ denotes the symmetrization in the primed indices, and

$$(\mathcal{D}^{--}F)^{A_1\ldots A_{k-1}B'_1\ldots B'_{l-1}} = \nabla_{A_kB'_l}F^{A_1\ldots A_kB'_1\ldots B'_l}.$$

Generalized C-R equations. For $\mathbb{S}^{k,l}$, the generalized Cauchy-Riemann equations (GCR) are defined as

$$\mathcal{D}^{+-}(f) = 0, \quad \mathcal{D}^{-+}(f) = 0, \quad \mathcal{D}^{--}(f) = 0 \quad \text{for } f \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^{k,l}). \tag{16}$$

Denote by $\mathcal{M}_m(\mathbb{S}^{k,l})$ the space of polynomials $F \in \mathcal{P}_m(\mathbb{R}^4, \mathbb{S}^{k,l})$ satisfying (GCR). The main result of the paper is that $\mathcal{M}_m(\mathbb{S}^{k,l})$ forms an irreducible G-module with the label $(m+k, m+l)$, see Theorem 2 below.

Remark 3. *For $l = 0$ in* (16)*, we get (GCR) $D^{-+}(f) = 0$. This is the usual form of massless field equation for spin $k/2$ particles, see* [19]*. Moreover, by the identification (3), the spin $k/2$ massless field operator*

$$D^{-+} : \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^k) \to \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^{k-1} \otimes \mathbb{S}')$$

can be viewed as the k-Cauchy-Fueter operator studied in [15, 22, 23]*, since the spin* 1/2 *massless field operator is exactly Cauchy-Fueter operator. The k-Cauchy-Fueter operator can be defined over the quaternionic space $\mathbb{H}^n$ and a function annihilated by it is called k-regular. See also* [4, 6, 8, 9] *and references therein for Cauchy-Fueter operator of several quaternionic variables. By using twistor method, all k-regular polynomials are found in* [15]*. But formulae of these polynomials are complicated. Let us remark that Theorem 3 below gives us general form of all k-regular polynomials on $\mathbb{H}$.*

4. Highest weight vectors

In this section, we find explicit formulæ of highest weight vectors for homogeneous $\mathbb{S}^{k,l}$-valued spherical harmonics. By (4), (5) and (12), the action of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ on the polynomials $\mathcal{P}(\mathbb{R}^4, \mathbb{S}^{k,l})$ is obviously given by

$$X = z_{0A'}\nabla^{1A'} + s_0\partial^1,\ Y = z_{1A'}\nabla^{0A'} + s_1\partial^0,\ H = z_{0A'}\nabla^{0A'} - z_{1A'}\nabla^{1A'} + s_0\partial^0 - s_1\partial^1, \tag{17}$$

$$X' = z_{A0'}\nabla^{A1'} + s_{0'}\partial^{1'},\ Y' = z_{A1'}\nabla^{A0'} + s_{1'}\partial^{0'},\ H' = z_{A0'}\nabla^{A0'} - z_{A1'}\nabla^{A1'} + s_{0'}\partial^{0'} - s_{1'}\partial^{1'}. \tag{18}$$

Of course, the module $\mathcal{P}(\mathbb{R}^4, \mathbb{S}^{k,l})$ has a decomposition

$$\mathcal{P}(\mathbb{R}^4, \mathbb{S}^{k,l}) = \bigoplus_{m,j=0}^{+\infty} r^{2j}\mathcal{H}_m \otimes \mathbb{S}^{k,l}.$$

Because $\mathcal{M}_m(\mathbb{S}^{k,l}) \subset \mathcal{H}_m \otimes \mathbb{S}^{k,l}$ we are interested in the G-module $\mathcal{H}_m \otimes \mathbb{S}^{k,l}$, which has an irreducible decomposition:

$$\begin{aligned}\mathcal{H}_m \otimes \mathbb{S}^{k,l} \cong\ & (m+k, m+l) \oplus (m+k-2, m+l) \oplus \ldots \oplus (|m-k|, m+l)\\ & \oplus (m+k, m+l-2) \oplus (m+k-2, m+l-2) \oplus \ldots \oplus (|m-k|, m+l-2)\\ & \vdots\\ & \oplus (m+k, |m-l|) \oplus (m+k-2, |m-l|) \oplus \ldots \oplus (|m-k|, |m-l|).\end{aligned} \tag{19}$$

Here we write its label instead of an irreducible G-module. Denote

$$D := \det\begin{pmatrix} z_{00'} & s_0 \\ z_{10'} & s_1 \end{pmatrix}, \qquad D' := \det\begin{pmatrix} z_{00'} & s_{0'} \\ z_{01'} & s_{1'} \end{pmatrix},$$

and

$$\mathbb{D} := z^{BB'}s_Bs_{B'} = z_{00'}s_1s_{1'} + z_{11'}s_0s_{0'} - z_{01'}s_1s_{0'} - z_{10'}s_0s_{1'}.$$

Proposition 1. *The highest weight vector of $(m+k-2j, m+l-2i)$ for $j \geq i$ in the decomposition (19) can be written as*

$$\varphi = \sum_{a=0}^{\min\{i,m-j\}} C_a z_{00'}^{m-j-a} (D')^a D^{j-i+a} \mathbb{D}^{i-a} s_0^{k-j} s_{0'}^{l-i} \tag{20}$$

for some positive constants $C_a > 0$. The case $j < i$ is similar by interchanging primed and unprimed indices.

Proof. First let us remark that, by (iii) of Theorem 1, $\mathcal{H}_m \otimes \mathbb{S}^{k,l}$ is spanned by polynomials of the form

$$z_{(A_1A_1'} z_{A_2A_2'} \cdot \ldots \cdot z_{A_mA_m')} s_{B_1} \cdot \ldots \cdot s_{B_k} s_{B_1'} \cdot \ldots \cdot s_{B_l'}. \tag{21}$$

Note that $i \leq \min\{m, l\}$, $j \leq \min\{m, k\}$, and

$$\begin{aligned} \varphi =& z_{(00'} \cdot \ldots \cdot z_{00'} \cdot z_{A_{m-j+1}0'} \cdots z_{A_{m-i}0'} z_{A_{m-i+1}A'_{m-i+1}} \cdots z_{A_mA'_m)} \\ & \cdot s_0^{k-j} \prod_{t=1}^{j} s_{B_{k-j+t}} s_{0'}^{l-i} \prod_{t=1}^{i} s_{B'_{l-i+t}} \cdot \prod_{t=1}^{j} \epsilon^{B_{k-j+t}A_{m-j+t}} \prod_{t=1}^{i} \epsilon^{B'_{l-i+t}A'_{m-i+t}} \end{aligned} \tag{22}$$

is an element of $\mathcal{H}_m \otimes \mathbb{S}^{k,l}$, since it is a linear combination of terms in (21) with $A_1 = \cdots = A_{m-j} = 0$, $A_1' = \cdots = A'_{m-i} = 0'$. By raising indices and then taking symmetrization, we see that it can be written as

$$\varphi = z^{(11'} \cdot \ldots \cdot z^{B_{k-j+1}1'} \cdots z^{B_{k-i}1'} z^{B_{k-i+1}B'_{l-i+1}} \cdots z^{B_kB_l')} s_0^{k-j} \prod_{t=1}^{j} s_{B_{k-j+t}} s_{0'}^{l-i} \prod_{t=1}^{i} s_{B'_{l-i+t}}. \tag{23}$$

up to a constant $(-1)^{i+j}$.

Denote $E_t = B_{k-t+1}$ for $t = 1, \ldots, j$, and $E_t = 1$ for $t = j+1, \ldots, m$. Then φ can be written as

(24)

$$\begin{aligned} \varphi =& z^{(E_m1'} \cdots z^{E_j1'} \cdots z^{E_{i+1}1'} z^{E_iB'_{l-i+1}} \cdots z^{E_1B_l')} s_0^{k-j} \prod_{t=1}^{j} s_{E_t} s_{0'}^{l-i} \prod_{t=1}^{i} s_{B'_{l-i+t}} \\ =& \frac{1}{m!} \sum_{\sigma \in S_m} z^{E_{\sigma(m)}1'} s_{E_{\sigma(m)}} \cdots z^{E_{\sigma(j)}1'} s_{E_{\sigma(j)}} \cdots z^{E_{\sigma(i)}B'_{l-i+1}} s_{E_{\sigma(i)}} \cdots z^{E_{\sigma(1)}B'_l} s_{E_{\sigma(1)}} s_{0'}^{l-i} \prod_{t=1}^{i} s_{B'_{l-t+1}}, \end{aligned}$$

since

$$s_0^{k-j} \prod_{t=1}^{j} s_{E_t} = \prod_{t=1}^{m} s_{E_{\sigma(t)}}.$$

For a fixed $\sigma \in S_m$, if we denote by a the number of $t \in \{1, \ldots, i\}$ such that $\sigma(t) > j$, then there are exactly a indices in $E_{\sigma(i)}, \ldots, E_{\sigma(1)}$ fixed to be 1, $a = 0, \ldots, \min\{i, m-j\}$. So there remains $m-j-a$ indices in $E_{\sigma(m)}, \ldots, E_{\sigma(i+1)}$ fixed to be 1. If $E_{\sigma(t)}$ for $t \in \{1, \ldots, i\}$ is fixed to be 1, i.e. $\sigma(t) \in \{m, \ldots, i+1\}$, then

$$z^{E_{\sigma(t)}B'_{l-t+1}} s_{B'_{l-t+1}} = z^{1B'_{l-t+1}} s_{B'_{l-t+1}} = D',$$

by $D' = z^{1B'} s_{B'} = z_{00'}s_{1'} - z_{01'}s_{0'}$, whose number is a. If $E_{\sigma(t)}$ for $t \in \{1, \ldots, i\}$ is not fixed to be 1, then

$$z^{E_{\sigma(t)}B'_{l-t+1}} s_{E_{\sigma(t)}} s_{B'_{l-t+1}} = \mathbb{D},$$

whose number is $i - a$.

If $E_{\sigma(t)}$ for $t \in \{m, \ldots, i+1\}$ is not fixed to be 1, i.e. $\sigma(t) \in \{1, \ldots, j\}$, then

$$z^{E_{\sigma(t)}1'} s_{E_{\sigma(t)}} = D,$$

by $D = z^{B1'}s_B = z_{00'}s_1 - z_{10'}s_0$, whose number is $m - i - (m - j - a) = j - i + a$. If $E_{\sigma(t)}$ for $t \in \{m, \dots, i+1\}$ is fixed to be 1, then $z^{E_{\sigma(t)}1'} = z^{11'}$, whose number is $m - j - a$. So the term for fixed σ in the summation (24) is the product

$$z_{00'}^{m-j-a}(D')^a D^{j-i+a}\mathbb{D}^{i-a}s_0^{k-j}s_{0'}^{l-i}. \tag{25}$$

Thus φ can be written in the form (20) with C_a to be the number of this term (25) appearing in the sum (24) divided by $m!$.

To check that the vector φ in (20) is a highest weight vector of a correct weight, note that the weight of $z_{00'}$ is $(1,1)$, the weight of $z_{10'}$ is $(-1,1)$, the weight of $z_{01'}$ is $(1,-1)$, the weight of $z_{11'}$ is $(-1,-1)$, the weight of s_0 is $(1,0)$, the weight of s_1 is $(-1,0)$ the weight of $s_{0'}$ is $(0,1)$, the weight of $s_{1'}$ is $(0,-1)$, the weight of D is $(0,1)$, the weight of D' is $(1,0)$ and the weight of $\mathbb{D}$ is $(0,0)$. It shows that the weight of the vector (25) is indeed $(m+k-2j, m+l-2i)$.

It is immediately visible that $z_{00'}, D, D', \mathbb{D}, s_0, s_0'$ are killed both by X and X' given in (17) and (18), hence it is true also for any product of such factors. So is for φ and the result follows. □

5. Homogeneous solutions

Now we are ready to prove the main result of the paper.

Theorem 2. *Let $\mathcal{M}_m(\mathbb{S}^{k,l})$ denote the space of polynomials $F \in \mathcal{P}_m(\mathbb{R}^4, \mathbb{S}^{k,l})$ satisfying (GCR), see* (16)*. Then $\mathcal{M}_m(\mathbb{S}^{k,l})$ forms an irreducible $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$-module with the label $(m+k, m+l)$,*

Proof. The tensor product of the space $\mathcal{H}_m$ of harmonic polynomials of order m and the space $\mathbb{S}^{k,l}$ decomposes into irreducible components as described in (19). We are going to prove that the highest weight vector of any component in the decomposition is killed by all three operators $\mathcal{D}^{+-}, \mathcal{D}^{-+}, \mathcal{D}^{--}$ if and only if it is the Cartan component $(m+k, m+l)$.

We want to show that if either i or j is positive, then the highest weight vector of the summand $(m+k-2j, m+l-2i)$ is not in $\mathcal{M}_m(\mathbb{S}^{k,l})$. In the following, we suppose that $j \geq i$. The case $j < i$ follows simply by interchanging primed and unprimed indices.

It follows from the expression (24) that φ, the highest weight vector of the summand $(m+k-2j, m+l-2i)$, is homogeneous of degree k in variables s_0, s_1, and homogeneous of degree l in variables $s_{0'}, s_{1'}$.

Case i: Let $\min\{i, m-j\} \geq 1$. Recall that

$$D' = z^{1B'}s_{B'} = z_{00'}s_{1'} - z_{01'}s_{0'}, \qquad D = z^{B1'}s_B = z_{00'}s_1 - z_{10'}s_0.$$

Now consider terms containing the factor $s_0^k s_{0'}^l$ of φ in (20) of Proposition 20. Such a term must come from products of $-z_{01'}s_{0'}$ in D', $-z_{10'}s_0$ in D and $z_{11'}s_0s_{0'}$ in $\mathbb{D}$, while a term containing the factor $s_0^k s_{0'}^{l-1}s_{1'}$ in (20) must come from such a product with only one $-z_{01'}s_{0'}$ in D' replaced by $z_{00'}s_{1'}$, or $z_{11'}s_0s_{0'}$ in $\mathbb{D}$ replaced by $-z_{10'}s_0s_{1'}$. So we have

$$\begin{aligned}\varphi = &\sum_{a=0}^{\min\{i,m-j\}} C_a z_{00'}^{m-j-a}(-z_{01'})^a(-z_{10'})^{j-i+a}z_{11'}^{i-a}s_0^k s_{0'}^l \\ &+ \left\{ \sum_{a=0}^{\min\{i,m-j\}} C_a z_{00'}^{m-j-a+1} a(-z_{01'})^{a-1}(-z_{10'})^{j-i+a}z_{11'}^{i-a} \right. \\ &\left. + \sum_{a=0}^{\min\{i-1,m-j\}} C_a z_{00'}^{m-j-a}(-z_{01'})^a(-z_{10'})^{j-i+a+1}(i-a)z_{11'}^{i-a-1} \right\} s_0^k s_{0'}^{l-1}s_{1'} + \cdots\end{aligned}$$

where $\cdots$ is the summation of terms involving

$$s_0^A s_1^B s_{0'}^C s_{1'}^D \quad (A+B=k, C+D=l) \quad \text{with} \quad A<k \quad \text{or} \quad D \geq 2 \quad \text{if} \quad A=k, \tag{26}$$

and the term involving $(z_{11'})^{i-2}$ does not appear if $i=1$. By mod $z^2_{01'}$, it can be simplified to be

$$\begin{aligned}\varphi =&\left\{C_0(z_{00'})^{m-j}(-z_{10'})^{j-i}(z_{11'})^{i}+C_1(z_{00'})^{m-j-1}(-z_{01'})(-z_{10'})^{j-i+1}(z_{11'})^{i-1}\right\} s_0^k s_{0'}^l\\ &+\left\{(C_1+iC_0)(z_{00'})^{m-j}(-z_{10'})^{j-i+1}(z_{11'})^{i-1}\right.\\ &\quad\left.+[2C_2+(i-1)C_1](z_{00'})^{m-j-1}(-z_{01'})(-z_{10'})^{j-i+2}(z_{11'})^{i-2}\right\} s_0^k s_{0'}^{l-1}s_{1'}+\cdots, \quad mod \quad z^2_{01'}.\end{aligned}$$

It is obvious that $s_0^{k+1}s_{0'}^{l-1}$ does not appear after the action of

$$-\mathcal{D}^{+-}=s_0\nabla^{00'}\partial^{1'}+s_1\nabla^{10'}\partial^{1'}-s_0\nabla^{01'}\partial^{0'}-s_1\nabla^{11'}\partial^{0'} \tag{27}$$

on terms in (26). Moreover, only the derivative $-s_0\nabla^{01'}\nabla^{0'}$ in $\mathcal{D}^{+-}$ acting on the $s_0^k s_{0'}^l$ term of φ produces a $s_0^{k+1}s_{0'}^{l-1}$ term, and only the derivative $s_0\nabla^{00'}\nabla^{1'}$ in $\mathcal{D}^{+-}$ acting on the $s_0^k s_{0'}^{l-1}s_{1'}$ term produces a $s_0^{k+1}s_{0'}^{l-1}$ term. So we get

$$-\mathcal{D}^{+-}\varphi=[lC_1+(C_1+iC_0)(m-j)](z_{00'})^{m-j-1}(-z_{10'})^{j-i+1}(z_{11'})^{i-1}s_0^{k+1}s_{0'}^{l-1}+\cdots\neq 0, \quad mod \quad z_{01'}$$

Thus $(m+k-2j,m+l-2i)\cap\mathcal{M}_m(\mathbb{S}^{k,l})=\{0\}$ in this case.

Case ii$_1$: Let $m-j=0$ and $m-i>0$. In this case we must have $j\neq 0$ and $k\geq m$. Then $\varphi=D^{m-i}\mathbb{D}^i s_0^{k-m}s_{0'}^{l-i}$, and

$$\begin{aligned}\varphi=&(-z_{10'})^{m-i}(z_{11'})^{i}s_0^k s_{0'}^l\\ &+\left[(m-i)z_{00'}(-z_{10'})^{m-i-1}(z_{11'})^{i}+i(-z_{10'})^{m-i}(-z_{10'})(z_{11'})^{i-1}\right]s_0^{k-1}s_1 s_{0'}^l+\cdots,\end{aligned}$$

where $\cdots$ is the summation of terms as in (26) with $C<l$ or $B\geq 2$ if $C=l$. Then apply

$$-\mathcal{D}^{-+}=s_{0'}\nabla^{00'}\partial^{1}+s_{1'}\nabla^{01'}\partial^{1}-s_{0'}\nabla^{10'}\partial^{0}-s_{1'}\nabla^{11'}\partial^{0}. \tag{28}$$

to φ to get

$$-\mathcal{D}^{-+}\varphi=(k+1)(m-i)(-z_{10'})^{m-i-1}(z_{11'})^{i-1}s_0^{k-1}s_{0'}^{l+1}+\cdots\neq 0.$$

Thus $(m+k-2j,m+l-2i)\cap\mathcal{M}_m(\mathbb{S}^{k,l})=\{0\}$ in this case.

Case ii$_2$: Let $m-j=0$ and $m-i=0$. In this case we must have $k,l\geq m$. Then $\varphi=\mathbb{D}^m s_{0'}^{k-m}s_0^{l-m}$, and

$$\begin{aligned}\varphi=&(z_{11'})^m s_0^k s_{0'}^l+m(-z_{10'})(z_{11'})^{m-1}s_0^k s_{0'}^{l-1}s_{1'}+m(-z_{01'})(z_{11'})^{m-1}s_0^{k-1}s_1 s_{0'}^l\\ &+mz_{00'}(z_{11'})^{m-1}s_0^{k-1}s_1 s_{0'}^{l-1}s_{1'}+\cdots,\end{aligned}$$

where $\cdots$ is the summation of terms involving $s_0^A s_1^B s_{0'}^C s_{1'}^D$ $(A+B=k, C+D=l)$ with $A\leq k-2$ or $C\leq l-2$. Then apply

$$\mathcal{D}^{--}=\nabla^{00'}\partial^1\partial^{1'}-\nabla^{01'}\partial^1\partial^{0'}-\nabla^{10'}\partial^0\partial^{1'}+\nabla^{11'}\partial^0\partial^{0'} \tag{29}$$

to φ to get

$$\mathcal{D}^{--}\varphi=[kl+k+l+1]m(z_{11'})^{m-1}s_0^{k-1}s_{0'}^{l-1}+\cdots\neq 0.$$

Thus $(m+k-2j,m+l-2i)\cap\mathcal{M}_m(\mathbb{S}^{k,l})=\{0\}$ in this case.

Case iii: Let $m-j=i=0$. In this case, the module is $(k-m,m+l)$ with the highest weight vector $\varphi=D^m s_0^{k-m}s_{0'}^l$, and

$$\varphi=(-z_{10'})^m s_0^k s_{0'}^l+mz_{00'}(-z_{10'})^{m-1}s_0^{k-1}s_1 s_{0'}^l+\cdots$$

where $\cdots$ is the summation of terms involving $s_0^A s_1^B s_{0'}^l$ $(A+B=k)$ with $A\leq k-2$. Then we have

$$-\mathcal{D}^{-+}\varphi=m(k+1)(-z_{10'})^{m-1}s_0^{k-1}s_{0'}^{l+1}+\cdots\neq 0.$$

Thus $(m+k-2j,m+l-2i)\cap\mathcal{M}_m(\mathbb{S}^{k,l})=\{0\}$ in this case.

$(m+k,m+l)$ has the highest weight vector $z^m_{00'}s_0^k s_{0'}^l$, which is obviously killed by all three operators $\mathcal{D}^{+-},\mathcal{D}^{-+},\mathcal{D}^{--}$ in (27)-(29). Theorem is proved. □

Now we give explicit bases for homogeneous solutions of general massless field equations.

Theorem 3. *(i) The irreducible G-module $\mathcal{M}_m(\mathbb{S}^{k,l})$ has a basis consisting of the polynomials*

$$F_m^{r,s}(z) = \frac{1}{r!s!}Y^r(Y')^s\left(\frac{z_{00'}^m}{m!}\frac{s_0^k}{k!}\frac{s_{0'}^l}{l!}\right)$$

where $r = 0,\dots,m+k$, $s = 0,\dots,m+l$ and $Y = z_{1A'}\nabla^{0A'} + s_1\partial^0$, $Y' = z_{A1'}\nabla^{A0'} + s_{1'}\partial^{0'}$ are the lowering operators given in (17) and (18).

(ii) Moreover, we have

$$F_m^{r,s}(z) = \sum_{u=0}^{\min(r,k)}\sum_{v=0}^{\min(s,l)} f_m^{r-u,s-v}(z)\frac{s_0^{k-u}}{(k-u)!}\frac{s_1^u}{u!}\frac{s_{0'}^{l-v}}{(l-v)!}\frac{s_{1'}^v}{v!}$$

Here $f_m^{r,s}$ are the harmonic polynomials given in Theorem 1.

Proof. (i) This is obvious. In particular,

$$\frac{z_{00'}^m}{m!}\frac{s_0^k}{k!}\frac{s_{0'}^l}{l!}$$

is the highest weight vector of $\mathcal{M}_m(\mathbb{S}^{k,l})$.

(ii) Denote $\tilde{Y} = z_{1A'}\nabla^{0A'}$ and $\tilde{Y}' = z_{A1'}\nabla^{A0'}$. Recall that $\tilde{Y}$ and $\tilde{Y}'$ are the lowering operators from the harmonic case given in (4) and (5). Obviously, we have that

$$\frac{1}{r!}Y^r = \sum_{u=0}^{r}\frac{1}{u!(r-u)!}\tilde{Y}^{r-u}(s_1\partial^0)^u = \sum_{u=0}^{r}\frac{\tilde{Y}^{r-u}}{(r-u)!}\frac{s_1^u}{u!}(\partial^0)^u,$$

$$\frac{1}{s!}(Y')^s = \sum_{v=0}^{s}\frac{(\tilde{Y'})^{s-v}}{(s-v)!}\frac{s_{1'}^v}{v!}(\partial^{0'})^v.$$

Applying these expressions, we get easily

$$F_m^{r,s}(z) = \sum_{u=0}^{\min(r,k)}\sum_{v=0}^{\min(s,l)}\frac{\tilde{Y}^{r-u}}{(r-u)!}\frac{(\tilde{Y'})^{s-v}}{(s-v)!}\left(\frac{z_{00'}^m}{m!}\right)\frac{s_0^{k-u}}{(k-u)!}\frac{s_1^u}{u!}\frac{s_{0'}^{l-v}}{(l-v)!}\frac{s_{1'}^v}{v!}.$$

To complete the proof we use (i) of Theorem 1. □

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