

Stability analysis and optimal control of avian influenza model on complex networks

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Abstract

In this paper, an avian influenza model with saturation and psychological effect on heterogeneous complex networks is proposed. Firstly, the basic reproduction number \mathcal{R}_0 is given through mathematical analysis, which is a threshold to determine whether or not the disease spreads. Secondly, the locally and globally asymptotical stability of the disease-free equilibrium point and the endemic equilibrium point are investigated by using Lyapunov functions and Kirchhoff's matrix tree theorem. If $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable and the disease will die out. If $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable. Thirdly, an optimal control problem is established by taking slaughter rate and cure rate as control variables. Finally, numerical simulations are given to demonstrate the main results.

Keywords: Complex networks; Avian influenza model; Lyapunov functions; Asymptotical stability; Optimal control; Basic reproduction number

1. Introduction

In recent decades, avian influenza viruses have become widespread all over the world, threatening public safety and causing huge enormous economic losses. For example, since the first outbreak of avian influenza H5N1 in Hong Kong in 1997, the virus has infected more than 400 people worldwide, with a mortality rate close to 60% [1]. In 2013, the avian influenza H7N9 crossed the species barrier for the first outbreak in mainland China. More than 400 people have been infected, and the mortality rate is close to 40% [1]. To provide effective control and prevention strategies, mathematical models and methods have been widely adopted to study the epidemiological characteristics of infectious diseases.

Among these important mathematical models, the most famous one is the compartmental model proposed by Kermack and Mckendrick [2–4] in 1927. Since then, there are a large number of different

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mathematical models have been developed and used to analyze the spread mechanism of diseases [5, 6]. In 2007, Iwami et al. [7] proposed an ordinary differential equation model to characterize the dynamical behavior of avian influenza between human and avian populations. Hu [8] constructed an avian influenza model with nonlinear incidence and analyzed the stability of the model. Ma and Wang [9] established a discrete-time model to evaluate the impact of avian influenza transmission in poultry population. Bourouiba et al. [10] established a delayed avian influenza model to investigate the role of migrating birds in the spread of avian influenza. In 2014, Chong et al. [11] proposed a bird-human coupling dynamic model with half-saturated incidence and mutation of virus strains.

However, all the above models are obtained under the assumption that all individuals are uniformly mixed, which means they have the same contact rate with other individuals in the region. That is, the mixture between individuals is homogeneous, but, the contact between poultry-to-poultry, and poultry-to-human are obviously heterogeneous [12] in reality. In order to reflect the heterogeneity of contacting between individuals, some scholars have used complex networks to analyze the epidemic model. For example, Liu et al. [13] considered the spread of epidemic diseases with birth and death on networks, and obtained the epidemic threshold. Fu et al. [14] proposed a network model for differential infectivity and calculated the basic reproduction number. However, these two papers only studied single population. Unfortunately, there are little literatures about avian influenza model on complex networks because of the spread of avian influenza involves two populations, and individual infection rates and contact patterns may be different in different populations. So, it is of great significance to explore the spread of avian influenza on coupled networks.

As we all known, avian influenza viruses that infect usually only birds, sometimes infects humans [15]. The spread of avian influenza in the population seriously threatens the safety of human life. How to formulate optimal control problems to study optimal control strategies is an important issue. Therefore, we introduce the slaughtering for poultry and treatment for humans as control variables, and establish an optimal control problem to decrease the number infected poultry and humans. Inspired by references [8] and [16–18], we propose the avian influenza model on one-way-coupled networks with two subnetworks and prove the locally and globally asymptotical stability of equilibrium points, further prove the uniqueness and existence of optimal control pairs. The main contributions of this paper are as follows:

- One-way-coupled network is applied to the avian influenza model.
- The asymptotically stability of the disease-free and endemic equilibrium points are proved for this model.
- We establish an optimal control problem by introducing slaughtering for poultry and treatment for humans.

The rest of this paper is organized as follows. In [section 2](#), an avian influenza model is constructed, and the positivity and boundedness of its solution is discussed. In [section 3](#), the locally and globally asymptotical stability of the equilibrium points are proved for the avian-only subsystem. The locally and globally asymptotical stability of the equilibrium points are presented for the avian influenza model in [section 4](#). In [section 5](#), the optimal control problem is proposed, and the existence and uniqueness of optimal control is proven. In [section 6](#), several numerical simulations are given to demonstrate the theory results. Finally, we give a brief conclusion and future work in [section 7](#).

2. Model Formulation and the Positivity of Solution

In this section, we formulate an avian influenza model on one-way-coupled network and discuss some simple properties of solution. In [\[8\]](#), the authors proposed the following avian influenza model with nonlinear incidence rate

$$\begin{cases} \frac{dS_a}{dt} = \Lambda_a - \frac{\lambda_a S_a I_a}{1 + \alpha_1 I_a} - \mu_a S_a, \\ \frac{dI_a}{dt} = \frac{\lambda_a S_a I_a}{1 + \alpha_1 I_a} - \delta_a I_a - \mu_a I_a, \\ \frac{dS_h}{dt} = \Lambda_h - \frac{\lambda_h S_h I_a}{1 + \alpha_2 I_h^2} - \mu_h S_h, \\ \frac{dI_h}{dt} = \frac{\lambda_h S_h I_a}{1 + \alpha_2 I_h^2} - \gamma_h I_h - \delta_h I_h - \mu_h I_h, \\ \frac{dR_h}{dt} = \gamma_h I_h - \mu_h R_h. \end{cases} \quad (1)$$

In model [\(1\)](#), $\frac{1}{1 + \alpha_1 I_a}$ denotes the saturation effect in the poultry population and $\frac{\lambda_h S_h I_a}{1 + \alpha_2 I_h^2}$ describes the psychological effects in the human population. All parameters are assumed non-negative and their meanings are described as follows: Λ_a and Λ_h denotes the recruitment rate of poultry and human population, respectively; λ_a and λ_h represents the infected rate of poultry and human populations, respectively; μ_a and μ_h represents the natural mortality rate of poultry and human populations, respectively; δ_a and δ_h represents the mortality due to disease in poultry and human populations, respectively; γ_h denotes recovery rate.

Considering the heterogeneity of the contact between poultry-to-poultry and poultry-to-human, we introduce one-way-coupled networks into avian influenza model. There are two separate networks, \mathcal{A} and \mathcal{H} . Network \mathcal{H} consists of humanity, where each node represents an individual, and each connection between two individuals represents direct contact between them. Network \mathcal{A} is composed of avian. And there is a connection from subnetwork \mathcal{A} to subnetwork \mathcal{H} . We express in degrees (i, j) that there are i edges connected to subnetwork \mathcal{A} and j edges connected to subnetwork \mathcal{H} . And it is expressed in degrees (i, \cdot) that i edges are connected to subnetwork \mathcal{A} , and any edges are connected to subnetwork \mathcal{H} . The same degree (\cdot, j) indicates that any edge

is connected to the subnetwork \mathcal{A} and j edges are connected to the sub-network \mathcal{H} .

Then, model (1) can be written as

$$\begin{cases} \frac{dS_{i,j}^a(t)}{dt} = \Lambda_a - \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \mu_a S_{i,j}^a(t), \\ \frac{dI_{i,j}^a(t)}{dt} = \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t), \\ \frac{dS_{i,j}^h(t)}{dt} = \Lambda_h - \lambda_{ah}(j)S_{i,j}^h(t)\frac{\Theta_{ah}(t)}{1 + \alpha_2\Theta_{ah}(t)} - \mu_h S_{i,j}^h(t), \\ \frac{dI_{i,j}^h(t)}{dt} = \lambda_{ah}(j)S_{i,j}^h(t)\frac{\Theta_{ah}(t)}{1 + \alpha_2\Theta_{ah}(t)} - \gamma_h I_{i,j}^h(t) - \delta_h I_{i,j}^h(t) - \mu_h I_{i,j}^h(t), \\ \frac{dR_{i,j}^h(t)}{dt} = \gamma_h I_{i,j}^h(t) - \mu_h R_{i,j}^h(t). \end{cases} \quad (2)$$

The parameters of the coupling network are described in Tab. 1.

Table 1: The parameters of the coupling network are described in model (2)

Parameter	($X = \mathcal{A} \text{ or } \mathcal{H}$)
$N_{i,j}^X$	The number of nodes with degree (i, j) on subnet X
$S_{i,j}^X$	The number of susceptible nodes with degree (i, j) on subnet X
$I_{i,j}^X$	The number of infected nodes nodes with degree (i, j) on subnet X
$R_{i,j}^h$	The number of recovered nodes nodes with degree (i, j) on subnet X
$p_X(i, j) = \frac{N_{i,j}^a}{N^a}$	Probability of any node degree (i, j) on subnet X
$p_a(i, \cdot) = \sum_{j=1}^n p_a(i, j)$	The boundary degree distribution of subnet \mathcal{A}
$p_a(\cdot, j) = \sum_{i=1}^n p_a(i, j)$	The boundary degree distribution of subnet \mathcal{A}
$\langle k \rangle_a = \sum_{i=1}^n i p_a(i, \cdot)$	The average of nodes in subject \mathcal{A} connected to subnet \mathcal{A}
$\langle k \rangle_{ah} = \sum_{j=1}^n j p_a(\cdot, j)$	The average of nodes in subject \mathcal{A} connected to subnet \mathcal{H}
$\lambda_a(i) = \lambda_a i$	Poultry to poultry transmission rate of degree i
$\lambda_{ah}(j) = \lambda_{ah} j$	Poultry to human transmission rate of degree j

Θ_a denotes the infection probability of susceptible poultry nodes with the degree i in contact with the infected poultry nodes. Θ_{ah} denotes the infection probability of susceptible human nodes with the degree j in contact with the infected poultry nodes. In the uncorrelated networks, Θ_a , Θ_{ah} can be written as

$$\Theta_a(t) = \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) I_{i,j}^a(t),$$

$$\Theta_{ah}(t) = \frac{1}{\langle k \rangle_{ah}} \sum_{j=1}^n j p_a(\cdot, j) I_{i,j}^a(t).$$

Because the 5th equation does not affect the dynamic behavior of the other four equations in model (2), thus model (2) can be decoupled to the following equations

$$\begin{cases} \frac{dS_{i,j}^a(t)}{dt} = \Lambda_a - \lambda_a(i) S_{i,j}^a(t) \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \mu_a S_{i,j}^a(t), \\ \frac{dI_{i,j}^a(t)}{dt} = \lambda_a(i) S_{i,j}^a(t) \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t), \\ \frac{dS_{i,j}^h(t)}{dt} = \Lambda_h - \lambda_{ah}(j) S_{i,j}^h(t) \frac{\Theta_{ah}(t)}{1 + \alpha_2 \Theta_{ah}(t)} - \mu_h S_{i,j}^h(t), \\ \frac{dI_{i,j}^h(t)}{dt} = \lambda_{ah}(j) S_{i,j}^h(t) \frac{\Theta_{ah}(t)}{1 + \alpha_2 \Theta_{ah}(t)} - \gamma_h I_{i,j}^h(t) - \delta_h I_{i,j}^h(t) - \mu_h I_{i,j}^h(t). \end{cases} \quad (3)$$

The initial conditions of model (3) are given as follows

$$S_{i,j}^a(0) \geq 0, I_{i,j}^a(0) \geq 0, S_{i,j}^h(0) \geq 0, I_{i,j}^h(0) \geq 0.$$

The stability of equilibrium points is often governed by a threshold called the basic reproduction number \mathcal{R}_0 . The basic reproduction number \mathcal{R}_0 of model (3) is obtained by using the method in the reference [19].

$$\begin{aligned} \mathcal{R}_0 &= \frac{\Lambda_a}{\mu_a(\delta_a + \mu_a)} \frac{1}{\langle k \rangle_a} \sum_{i=1}^n \lambda_a(i) i p_a(i, \cdot) \\ &= \frac{\Lambda_a \lambda_a}{\mu_a(\delta_a + \mu_a)} \frac{\langle i^2 \rangle}{\langle i \rangle}. \end{aligned}$$

Meanwhile, we can define the epidemic transmission threshold as follows

$$\lambda_c = \frac{\mu_a(\delta_a + \mu_a) \langle i \rangle}{\Lambda_a \langle i^2 \rangle},$$

where $\langle i^2 \rangle = \sum_{i=1}^n i^2 p_a(i, \cdot)$. Model (3) has two equilibrium points, which are the disease-free equilibrium point E^0 and the endemic equilibrium point E^* ,

$$\begin{aligned} E^0 &= (S_{1,j}^{a0}, S_{2,j}^{a0}, \dots, S_{n,j}^{a0}, I_{1,j}^{a0}, I_{2,j}^{a0}, \dots, I_{n,j}^{a0}, S_{i,1}^{h0}, S_{i,2}^{h0}, \dots, S_{i,n}^{h0}, I_{i,1}^{h0}, I_{i,2}^{h0}, \dots, I_{i,n}^{h0}) \\ &= \left(\frac{\Lambda_a}{\mu_a}, \frac{\Lambda_a}{\mu_a}, \dots, \frac{\Lambda_a}{\mu_a}, 0, 0, \dots, 0, \frac{\Lambda_h}{\mu_h}, \frac{\Lambda_h}{\mu_h}, \dots, \frac{\Lambda_h}{\mu_h}, 0, 0, \dots, 0 \right), \\ E^* &= (S_{1,j}^{a*}, S_{2,j}^{a*}, \dots, S_{n,j}^{a*}, I_{1,j}^{a*}, I_{2,j}^{a*}, \dots, I_{n,j}^{a*}, S_{i,1}^{h*}, S_{i,2}^{h*}, \dots, S_{i,n}^{h*}, I_{i,1}^{h*}, I_{i,2}^{h*}, \dots, I_{i,n}^{h*}). \end{aligned}$$

Obviously, the disease-free equilibrium point E^0 of model (3) always exists. It is easy to know that

the endemic equilibrium point E^* of model (3) satisfies

$$\begin{cases} S_{i,j}^{a*} = \frac{\Lambda_a(1 + \alpha_1 \Theta_a^*(t))}{\lambda_a(i) \Theta_a^*(t) + \mu_a(1 + \alpha_1 \Theta_a^*(t))}, \\ I_{i,j}^{a*} = \frac{\lambda_a(i) \Lambda_a \Theta_a^*(t)}{[\lambda_a(i) \Theta_a^*(t) + \mu_a(1 + \alpha_1 \Theta_a^*(t))](\delta_a + \mu_a)}, \\ S_{i,j}^{h*} = \frac{\Lambda_h(1 + \alpha_2 \Theta_{ah}^*(t))}{\lambda_h(j) \Theta_{ah}^*(t) + \mu_h(1 + \alpha_2 \Theta_{ah}^*(t))}, \\ I_{i,j}^{h*} = \frac{\lambda_h(j) \Lambda_h \Theta_{ah}^*(t)}{[\lambda_h(j) \Theta_{ah}^*(t) + \mu_h(1 + \alpha_2 \Theta_{ah}^*(t))](\delta_h + \gamma_h + \mu_h)}. \end{cases} \quad (4)$$

Substitute the second equation of (4) into Θ_a , we obtain

$$1 = \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\lambda_a(i) \Lambda_a}{[\lambda_a(i) \Theta_a^*(t) + \mu_a(1 + \alpha_1 \Theta_a^*(t))](\delta_a + \mu_a)}.$$

In order to prove the existence and uniqueness of the endemic equilibrium point E^* , we define a function

$$F(\Theta_a) := 1 - \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\lambda_a(i) \Lambda_a}{(\lambda_a(i) \Theta_a + \mu_a(1 + \alpha_1 \Theta_a))(\delta_a + \mu_a)}.$$

It is easy to see that, $F(0) = 1 - \mathcal{R}_0$, $\lim_{\Theta_a \rightarrow \infty} F(\Theta_a) = 1$,

$$\frac{dF(\Theta_a)}{d\Theta_a} = \frac{\Lambda_a}{(\delta_a + \mu_a)} \frac{1}{\langle k \rangle_a} \sum_{i=1}^n \frac{\lambda_a(i) i p_a(i, \cdot) (\lambda_a(i) + \mu_a \alpha_1)}{[\lambda_a(i) \Theta_a^*(t) + \mu_a(1 + \alpha_1 \Theta_a^*(t))]^2} > 0.$$

Obviously, $F(\Theta_a)$ has a unique positive solution when $\mathcal{R}_0 > 1$, and consequently model (3) has a unique endemic equilibrium point E^* .

Based on model (3), we get

$$\begin{cases} \frac{dN_{i,j}^a(t)}{dt} = \Lambda_a - \mu_a N_{i,j}^a(t) - \delta_a I_{i,j}^a(t) \leq \Lambda_a - \mu_a N_{i,j}^a(t), \\ \frac{dN_{i,j}^h(t)}{dt} = \Lambda_h - \mu_h N_{i,j}^h(t) - \delta_h I_{i,j}^h(t) \leq \Lambda_h - \mu_h N_{i,j}^h(t), \end{cases}$$

furthermore, we get

$$\begin{cases} 0 < N_{i,j}^a(t) \leq \frac{\Lambda_a}{\mu_a}, t \rightarrow \infty, \\ 0 < N_{i,j}^h(t) \leq \frac{\Lambda_h}{\mu_h}, t \rightarrow \infty. \end{cases}$$

Thus, the closed set $\Omega = \{(S_{i,j}^a, I_{i,j}^a, S_{i,j}^h, I_{i,j}^h) \in R_+^{4n} : 0 < N_{i,j}^a \leq \frac{\Lambda_a}{\mu_a}, 0 < N_{i,j}^h \leq \frac{\Lambda_h}{\mu_h}\}$ is a bounded and positively invariant.

3. Stability Analysis of Avian-only Subsystem

In this section, we will study the locally and globally asymptotical stability of equilibrium points for the avian-only subsystem. The avian-only subsystem is

$$\begin{cases} \frac{dS_{i,j}^a(t)}{dt} = \Lambda_a - \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \mu_a S_{i,j}^a(t), \\ \frac{dI_{i,j}^a(t)}{dt} = \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t). \end{cases} \quad (5)$$

Model (5) has two equilibrium points, one is the disease-free equilibrium point

$$E_1^0 = \left(\frac{\Lambda_a}{\mu_a}, \frac{\Lambda_a}{\mu_a}, \dots, \frac{\Lambda_a}{\mu_a}, 0, 0, \dots, 0 \right), \quad (6)$$

which always exists. Another one is the endemic equilibrium point E_1^* , which satisfies

$$\begin{cases} S_{i,j}^{a*} = \frac{\Lambda_a(1 + \alpha_1\Theta_a^*(t))}{\lambda_a(i)\Theta_a^*(t) + \mu_a(1 + \alpha_1\Theta_a^*(t))}, \\ I_{i,j}^{a*} = \frac{\lambda_a(i)\Lambda_a\Theta_a^*(t)}{[\lambda_a(i)\Theta_a^*(t) + \mu_a(1 + \alpha_1\Theta_a^*(t))](\delta_a + \mu_a)}. \end{cases} \quad (7)$$

3.1. Locally asymptotical stability of avian-only subsystem

Theorem 3.1. (i) If $\mathcal{R}_0 < 1$, then the disease-free equilibrium point E_1^0 of model (5) is locally asymptotically stable; (ii) If $\mathcal{R}_0 > 1$, then the endemic equilibrium point E_1^* of model (5) is locally asymptotically stable.

In order to analyze locally asymptotical stability of the disease-free equilibrium point and the endemic equilibrium point of model (5).

Proof. (i) The Jacobian matrix $J(E_1^0)$ of disease-free equilibrium point can be calculated as follows

$$J(E_1^0) = \begin{bmatrix} -\mu_a & \cdots & 0 & -\lambda_a(1)S_{1,j}^{a0}f(1) & \cdots & -\lambda_a(1)S_{1,j}^{a0}f(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_a & -\lambda_a(n)S_{n,j}^{a0}f(1) & \cdots & -\lambda_a(n)S_{n,j}^{a0}f(n) \\ 0 & \cdots & 0 & -(\delta_a + \mu_a) + \lambda_a(1)S_{1,j}^{a0}f(1) & \cdots & \lambda_a(1)S_{1,j}^{a0}f(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_a(n)S_{n,j}^{a0} & \cdots & -(\delta_a + \mu_a) + \lambda_a(n)S_{n,j}^{a0}f(n) \end{bmatrix},$$

where $f(i) = \frac{ip_a(i,\cdot)}{\langle k \rangle_a}$. The characteristic polynomial of linear model (5) is

$$(z + \mu_a)^n |zE - F| = 0,$$

where

$$F = \begin{bmatrix} \omega_a + \lambda_a(1)S_{1,j}^{a0}f(1) & \lambda_a(1)S_{1,j}^{a0}f(2) & \cdots & \lambda_a(1)S_{1,j}^{a0}f(n) \\ \lambda_a(2)S_{2,j}^{a0}f(1) & \omega_a + \lambda_a(2)S_{2,j}^{a0}f(2) & \cdots & \lambda_a(2)S_{2,j}^{a0}f(n) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_a(n)S_{n,j}^{a0}f(1) & \lambda_a(n)S_{n,j}^{a0}f(2) & \cdots & \omega_a + \lambda_a(n)S_{n,j}^{a0}f(n) \end{bmatrix},$$

where $\omega_a = -(\mu_a + \delta_a)$. It is easy to see that the Jacobian matrix $J(E_1^0)$ has n eigenvalues equal to $-\mu_a$, and the rest n eigenvalues of matrix $J(E_1^0)$ are the eigenvalues of matrix F . The characteristic polynomial of matrix F is given by

$$\begin{aligned} |zE - F| &= \begin{vmatrix} z - \omega_a & 0 & \cdots & -\lambda_a(1)S_{1,j}^{a0}f(n) \\ 0 & z - \omega_a & \cdots & -\lambda_a(2)S_{2,j}^{a0}f(n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z - \omega_a - \sum_{i=1}^n \lambda_a(i)S_{i,j}^{a0}f(i) \end{vmatrix} \\ &= (z - \omega_a)^{n-1} \left(z - \omega_a - \sum_{i=1}^n \lambda_a(i)S_{i,j}^{a0}f(i) \right), i = 1, 2, \dots, n. \end{aligned}$$

By performing similar transformation to the n -dimensional matrix, the $n - 1$ eigenvalues equal to $\omega_a < 0$, and the n th eigenvalue is

$$\begin{aligned} \rho &= \omega_a + \sum_{i=1}^n \lambda_a(i)S_{i,j}^{a0}f(i) \\ &= \omega_a + \frac{\Lambda_a}{\mu_a} \frac{1}{\langle k \rangle_a} \sum_{i=1}^n \lambda_a(i)ip_a(i, \cdot) \\ &= -\omega_a \left(\frac{\Lambda_a}{\mu_a(-\omega_a)} \frac{1}{\langle k \rangle_a} \sum_{i=1}^n \lambda_a(i)ip_a(i, \cdot) - 1 \right) \\ &= -\omega_a(\mathcal{R}_0 - 1) < 0. \end{aligned}$$

Hence, the disease-free equilibrium point E_1^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.

(ii) The Jacobian matrix $J(E_1^*)$ of endemic equilibrium point can be calculated as follows

$$J(E_1^*) = \begin{bmatrix} -\mu_a - p_1 & \cdots & 0 & -m_1f(1) & \cdots & -m_1f(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_a - p_n & -m_nf(1) & \cdots & -m_nf(n) \\ p_1 & \cdots & 0 & \omega_a + m_1f(1) & \cdots & m_1f(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & p_n & m_nf(1) & \cdots & \omega_a + m_nf(n) \end{bmatrix},$$

where $p_i = \lambda_a(i) \frac{\Theta_a^*(t)}{1+\alpha_1 \Theta_a^*(t)}$, $m_i = \lambda_a(i) \frac{S_{i,j}^{a*}}{[1+\alpha_1 \Theta_a^*(t)]^2}$. Based on the principle of matrix similar transform, regard invertible matrix M , N as similarity transformation matrix. By using similarity transformation to the matrix $J(E_1^*)$, we get $MJ(E_1^*)N = B$, where E is a n -dimensional unit normal matrix.

$$M = \begin{bmatrix} E & E \\ 0 & E \end{bmatrix}, N = \begin{bmatrix} E & -\frac{\mu_a + \delta_a}{\mu_a} \\ 0 & E \end{bmatrix},$$

and

$$B = \begin{bmatrix} -\mu_a & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_a & 0 & \cdots & 0 \\ p_1 & \cdots & 0 & \omega_a + m_1 f(1) - \frac{\mu_a + \delta_a}{\mu_a} p_1 & \cdots & m_1 f(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & p_n & m_n f(1) & \cdots & \omega_a + m_1 f(1) - \frac{\mu_a + \delta_a}{\mu_a} p_1 \end{bmatrix}.$$

The characteristic equation of Jacobian matrix $J(E_1^*)$ is

$$(z + \mu_a)^n |zE - H| = 0,$$

where

$$H = \begin{bmatrix} \omega_a - f_1 + m_1 f(1) & m_1 f(2) & \cdots & m_1 f(n) \\ m_2 f(1) & \omega_a - f_2 + m_2 f(2) & \cdots & m_2 f(n) \\ \vdots & \vdots & \ddots & \vdots \\ m_n f(1) & m_n f(2) & \cdots & \omega_a - f_n + m_n f(n) \end{bmatrix},$$

$f_i = \frac{\delta_a + \mu_a}{\mu_a} p_i = \frac{\delta_a + \mu_a}{\mu_a} \lambda_a(i) \frac{\Theta_a^*(t)}{1+\alpha_1 \Theta_a^*(t)}$. Clearly, matrix $J(E_1^*)$ has n negative eigenvalues. In the following, we calculate the rest n eigenvalues of matrix $J(E_1^*)$. For $|zE - H| = 0$, we consider the following two cases:

Case I: If $z - \omega_a + f_i = 0$, namely, $z = \omega_a - f_i (i = 1, 2, \dots, n)$, then

$$|zE - H| = \begin{vmatrix} -m_1 f(1) & -m_1 f(2) & \cdots & -m_1 f(n) \\ -m_2 f(1) & -m_2 f(2) & \cdots & -m_2 f(n) \\ \vdots & \vdots & \ddots & \vdots \\ -m_n f(1) & -m_n f(2) & \cdots & -m_n f(n) \end{vmatrix} \equiv 0.$$

Therefore, we obtain n eigenvalues $z_i = \omega_a - f_i < 0 (i = 1, 2, \dots, n)$.

Case II: If $z - \omega_a + f_i \neq 0$, then

$$|zE - H| = \prod_{i=1}^n (z - \omega_a + f_i) \left(1 - \sum_{i=1}^n \frac{m_i f(i)}{z - \omega_a + f_i}\right).$$

Let

$$\begin{aligned} \psi(x) &= \prod_{i=1}^n (x - \omega_a + f_i) \left(1 - \sum_{i=1}^n \frac{m_i f(i)}{x - \omega_a + f_i}\right) \\ &= (x - \omega_a + f_1)(x - \omega_a + f_2) \cdots (x - \omega_a + f_n) \\ &\quad - m_1 f(1)(x - \omega_a + f_2)(x - \omega_a + f_3) \cdots (x - \omega_a + f_n) \\ &\quad - m_2 f(2)(x - \omega_a + f_1)(x - \omega_a + f_3) \cdots (x - \omega_a + f_n) \\ &\quad - \cdots - m_n f(n)(x - \omega_a + f_1)(x - \omega_a + f_2) \cdots (x - \omega_a + f_{n-1}). \end{aligned}$$

Since $\psi(x)$ is continuous, f_k is increasing and note that

$$\psi[-(-\omega_a + f_n)]\psi[-(-\omega_a + f_{n+1})] < 0, i = 1, 2, \dots, n.$$

Thus, there exists at least one root in $[-(-\omega_a + f_i), -(-\omega_a + f_{i+1})]$. In other words, there are $n - 1$ negative roots in $[-(-\omega_a + f_n), -(-\omega_a + f_1)]$. On the other hand, $\psi[-(-\omega_a + f_1)] < 0$, and

$$\begin{aligned} \psi(0) &= \prod_{i=1}^n (-\omega_a + f_i) \left(1 - \sum_{i=1}^n \frac{m_i f(i)}{-\omega_a + f_i}\right) \\ &= \prod_{i=1}^n (\mu_a + \delta_a + f_i) \left[1 - \sum_{i=1}^n \frac{\lambda_a(i) \frac{S_{i,j}^{a*}}{[1 + \alpha_1 \Theta_a^*(t)]^2} i p_a(i, \cdot)}{\langle k \rangle_a (\mu_a + \delta_a + \frac{\delta_a + \mu_a}{\mu_a} \lambda_a(i) \frac{\Theta_a^*(t)}{1 + \alpha_1 \Theta_a^*(t)})}\right] \\ &= \prod_{i=1}^n (\mu_a + \delta_a + f_i) \left[1 - \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\frac{\lambda_a(i) \Lambda_a (1 + \alpha_1 \Theta_a^*(t))}{[\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))](1 + \alpha_1 \Theta_a^*(t))^2}}{\mu_a + \delta_a + \frac{\delta_a + \mu_a}{\mu_a} \lambda_a(i) \frac{\Theta_a^*(t)}{1 + \alpha_1 \Theta_a^*(t)}}\right] \\ &= \prod_{i=1}^n (\mu_a + \delta_a + f_i) \left[1 - \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\mu_a \lambda_a(i) \Lambda_a}{(\mu_a + \delta_a) [\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))]^2}\right] > 0. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &\frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\mu_a \lambda_a(i) \Lambda_a}{(\mu_a + \delta_a) [\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))]^2} \\ &= \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\lambda_a(i) \Lambda_a \Theta_a^*(t)}{(\mu_a + \delta_a) [\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))]} \frac{\mu_a}{\Theta_a^*(t) [\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))]} \\ &< \frac{1}{\langle k \rangle_a} \sum_{i=1}^n i p_a(i, \cdot) \frac{\lambda_a(i) \Lambda_a \Theta_a^*(t)}{(\mu_a + \delta_a) [\lambda_a(i) \Theta_a^*(t) + \mu_a (1 + \alpha_1 \Theta_a^*(t))]} \frac{\mu_a}{\Theta_a^*(t) [\mu_a (1 + \alpha_1 \Theta_a^*(t))]} \\ &= \Theta_a^*(t) \frac{1}{\Theta_a^*(t) (1 + \alpha_1 \Theta_a^*(t))} < 1. \end{aligned}$$

Hence, the matrix N has n negative roots in $[-(-\omega_a + f_n), 0]$. It is proven that all the eigenvalues of the Jacobian matrix $J(E_1^*)$ are negative. That is to say, the endemic equilibrium point E_1^* is locally asymptotically stable. \square

3.2. Globally asymptotical stability of avian-only subsystem

The aim of this subsection is to investigate the globally asymptotical stability of avian-only subsystem (5). We have the following results:

Theorem 3.2. (i) If $\mathcal{R}_0 < 1$, the disease-free equilibrium point E_1^0 of model (5) is globally asymptotically stable; (ii) If $\mathcal{R}_0 > 1$, the endemic equilibrium point E_1^* of model (5) is globally asymptotically stable.

Proof. (i) In order to show the disease-free equilibrium point E_1^0 is globally asymptotically stable, we first consider the following Lyapunov function

$$V(t) = \sum_{i=1}^n f(i)(S_{i,j}^a - S_{i,j}^{a0} - S_{i,j}^{a0} \ln \frac{S_{i,j}^a}{S_{i,j}^{a0}}) + \sum_{i=1}^n f(i)I_{i,j}^a(t). \quad (8)$$

The derivative of $V(t)$ is

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=1}^n f(i) \left(1 - \frac{S_{i,j}^{a0}}{S_{i,j}^a}\right) \frac{dS_{i,j}^a(t)}{dt} + \sum_{i=1}^n f(i) \frac{dI_{i,j}^a(t)}{dt} \\ &= \sum_{i=1}^n f(i) \left(1 - \frac{S_{i,j}^{a0}}{S_{i,j}^a}\right) (\Lambda_a - \lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \mu_a S_{i,j}^a(t)) \\ &\quad + \sum_{i=1}^n f(i) (\lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t)) \\ &= -\mu_a \sum_{i=1}^n f(i) \frac{(S_{i,j}^a - S_{i,j}^{a0})^2}{S_{i,j}^a} - \sum_{i=1}^n f(i) \lambda_a(i) \Theta_a(t) \frac{S_{i,j}^a - S_{i,j}^{a0}}{1 + \alpha_1 \Theta_a(t)} \\ &\quad + \sum_{i=1}^n f(i) \lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \sum_{i=1}^n f(i) (\delta_a + \mu_a) I_{i,j}^a(t) \\ &= -\mu_a \sum_{i=1}^n f(i) \frac{(S_{i,j}^a - S_{i,j}^{a0})^2}{S_{i,j}^a} + \sum_{i=1}^n f(i) \lambda_a(i) S_{i,j}^{a0} \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \sum_{i=1}^n f(i) (\delta_a + \mu_a) I_{i,j}^a(t) \\ &< -\mu_a \sum_{i=1}^n f(i) \frac{(S_{i,j}^a - S_{i,j}^{a0})^2}{S_{i,j}^a} + \sum_{i=1}^n f(i) \lambda_a(i) S_{i,j}^{a0} \Theta_a(t) - \sum_{i=1}^n f(i) (\delta_a + \mu_a) I_{i,j}^a(t) \\ &= -\mu_a \sum_{i=1}^n f(i) \frac{(S_{i,j}^a - S_{i,j}^{a0})^2}{S_{i,j}^a} + (\delta_a + \mu_a)(\mathcal{R}_0 - 1)\Theta_a(t), \end{aligned}$$

thus, $\frac{dV(t)}{dt} < 0$ when $\mathcal{R}_0 < 1$, and $\frac{dV(t)}{dt} = 0$ if and only if $I_{i,j}^a(t) = 0$. Moreover, $\lim_{t \rightarrow \infty} S_{i,j}^a(t) = 1$, the largest invariant set of $\frac{dV(t)}{dt} = 0$ is a singleton E_1^0 . Hence the disease-free equilibrium point E_1^0 is globally asymptotically stable when $\mathcal{R}_0 < 1$ by the LaSalle invariance principle.

(ii) In order to show the endemic equilibrium point E_1^* is globally asymptotically stable, we define the following Lyapunov function:

$$V(t) = \sum_{i=1}^n c_i (S_{i,j}^a - S_{i,j}^{a*} - S_{i,j}^{a*} \ln \frac{S_{i,j}^a}{S_{i,j}^{a*}}) + \sum_{i=1}^n c_i (I_{i,j}^a - I_{i,j}^{a*} - I_{i,j}^{a*} \ln \frac{I_{i,j}^a}{I_{i,j}^{a*}}). \quad (9)$$

The derivative of $V(t)$ is

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=1}^n c_i \left(1 - \frac{S_{i,j}^{a*}}{S_{i,j}^a}\right) \frac{dS_{i,j}^a(t)}{dt} + \sum_{i=1}^n c_i \left(1 - \frac{I_{i,j}^{a*}}{I_{i,j}^a}\right) \frac{dI_{i,j}^a(t)}{dt} \\ &= \sum_{i=1}^n c_i \left(1 - \frac{S_{i,j}^{a*}}{S_{i,j}^a}\right) \left[\Lambda_a - \lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \mu_a S_{i,j}^a(t) \right] \\ &\quad + \sum_{i=1}^n c_i \left(1 - \frac{I_{i,j}^{a*}}{I_{i,j}^a}\right) \left[\lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t) \right] \\ &= \sum_{i=1}^n c_i \left(1 - \frac{S_{i,j}^{a*}}{S_{i,j}^a}\right) \left[\lambda_a(i) S_{i,j}^{a*} \frac{\Theta_a^*(t)}{1 + \alpha_1 \Theta_a^*(t)} + \mu_a S_{i,j}^{a*}(t) - \lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \mu_a S_{i,j}^a(t) \right] \\ &\quad + \sum_{i=1}^n c_i \left(1 - \frac{I_{i,j}^{a*}}{I_{i,j}^a}\right) \left[\lambda_a(i) S_{i,j}^a \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \lambda_a(i) S_{i,j}^{a*} \frac{\Theta_a^*(t)}{1 + \alpha_1 \Theta_a^*(t)} \frac{I_{i,j}^a}{I_{i,j}^{a*}} \right] \\ &= -\mu_a \sum_{i=1}^n c_i \frac{(S_{i,j}^a - S_{i,j}^{a*})^2}{S_{i,j}^a} + \sum_{i=1}^n c_i \lambda_a(i) S_{i,j}^{a*} \frac{\Theta_a^*}{1 + \alpha_1 \Theta_a^*} \left(2 - \frac{S_{i,j}^{a*}(t)}{S_{i,j}^a(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)}\right) \\ &\quad + \sum_{i=1}^n c_i \lambda_a(i) S_{i,j}^{a*} \frac{\Theta_a}{1 + \alpha_1 \Theta_a} \left(1 - \frac{S_{i,j}^a}{S_{i,j}^{a*}} \frac{I_{i,j}^{a*}}{I_{i,j}^a}\right) \\ &< -\mu_a \sum_{i=1}^n c_i \frac{(S_{i,j}^a - S_{i,j}^{a*})^2}{S_{i,j}^a} + \sum_{i=1}^n c_i \lambda_a(i) S_{i,j}^{a*} \Theta_a^* \left(2 - \frac{S_{i,j}^{a*}(t)}{S_{i,j}^a(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)}\right) \\ &\quad + \sum_{i=1}^n c_i \lambda_a(i) S_{i,j}^{a*} \Theta_a \left(1 - \frac{S_{i,j}^a(t)}{S_{i,j}^{a*}(t)} \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)}\right) \\ &= -\mu_a \sum_{i=1}^n c_i \frac{(S_{i,j}^a - S_{i,j}^{a*})^2}{S_{i,j}^a} \\ &\quad + \sum_{i=1}^n c_i \frac{\lambda_a(i)}{\langle k \rangle_a} S_{i,j}^{a*} \sum_{l=1}^n l p(l, \cdot) I_{l,j}^{a*} \left(2 - \frac{S_{i,j}^{a*}(t)}{S_{i,j}^a(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} + \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{S_{i,j}^a(t)}{S_{i,j}^{a*}(t)} \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)} \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)}\right). \end{aligned}$$

Because the function $g(x) = 1 - x + \ln x < 0$ when $x > 0$, and $g(x) = 0$ if and only if $x = 1$.

Utilizing the property of function $g(x)$ yields

$$\begin{aligned} &2 - \frac{S_{i,j}^{a*}(t)}{S_{i,j}^a(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} + \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{S_{i,j}^a(t)}{S_{i,j}^{a*}(t)} \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)} \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} \\ &= g\left(\frac{S_{i,j}^{a*}(t)}{S_{i,j}^a(t)}\right) + g\left(\frac{S_{i,j}^a(t)}{S_{i,j}^{a*}(t)} \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)} \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)}\right) - \ln \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)} \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} + \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{I_{i,j}^{a*}(t)}{I_{i,j}^a(t)} \\ &\leq \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \ln \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} + \ln \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)}. \end{aligned}$$

Define a non-negative weight matrix $M = (m_{il})_{n \times n}$, where $m_{il} = \frac{\lambda_a(i)}{\langle k \rangle_a} S_{i,j}^{a*}(t) lp(l) I_{l,j}^{a*}$, we have

$$\frac{dV(t)}{dt} \leq \sum_{i=1}^n \sum_{l=1}^n c_i m_{il} \left(\frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \ln \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} + \ln \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} \right),$$

where c_i are constants ($i = 1, 2, \dots, n$). According to Kirchhoffs matrix tree theorem [20], choosing $c_i = \sum_{T \in T_i} M(T)$ and using the tree cycle identity [21], we obtain the following identity

$$\sum_{i=1}^n \sum_{l=1}^n c_i m_{il} \left(\frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \ln \frac{I_{l,j}^a(t)}{I_{l,j}^{a*}(t)} - \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} + \ln \frac{I_{i,j}^a(t)}{I_{i,j}^{a*}(t)} \right) = 0,$$

which ensures that $\frac{dV(t)}{dt} \leq 0$ for all $t \in \Omega$, and the strict equality $V'(t) = 0$ holds only $S_{i,j}^a = S_{i,j}^{a*}, I_{i,j}^a = I_{i,j}^{a*}$, the positive endemic equilibrium point E_1^* is globally asymptotically stable on Ω . \square

4. Stability analysis of the avian-human influenza model (3)

The aim of this section is to investigate the locally and globally asymptotical stability of equilibrium points for model (3).

4.1. Locally asymptotical stability of model (3)

Theorem 4.1. (i) If $\mathcal{R}_0 < 1$, then the disease-free equilibrium point E^0 of model (3) is locally asymptotically stable; (ii) If $\mathcal{R}_0 > 1$, then the endemic equilibrium point E^* of model (3) is locally asymptotically stable.

Proof. (i) The Jacobian matrix $J(E^0)$ of disease-free equilibrium point can be calculated as follows

$$J(E^0) = \begin{bmatrix} J(E_1^0) & 0 \\ C^0 & D^0 \end{bmatrix},$$

where

$$C^0 = \begin{bmatrix} 0 & \cdots & 0 & -\lambda_h(1) S_{i,1}^{h0} h(1) & \cdots & -\lambda_h(1) S_{i,1}^{h0} h(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\lambda_h(n) S_{i,n}^{h0} h(1) & \cdots & -\lambda_h(n) S_{i,n}^{h0} h(n) \\ 0 & \cdots & 0 & \lambda_h(1) S_{i,1}^{h0} h(1) & \cdots & \lambda_h(1) S_{i,1}^{h0} h(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_h(n) S_{i,n}^{h0} h(1) & \cdots & \lambda_h(n) S_{i,n}^{h0} h(n) \end{bmatrix},$$

$$D^0 = \begin{bmatrix} -\mu_h & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_h & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \omega_h & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \omega_h \end{bmatrix}.$$

The characteristic equation of Jacobian matrix $J(E^0)$ is

$$(z + \mu_a)^n (z + \mu_h)^n (z + \omega_h)^n |zE - F| = 0, \quad (10)$$

obviously, the equation (10) has $3n$ negative roots, $-\mu_a$, $-\mu_h$, $-\omega_h$, respectively. In addition,

$$\begin{aligned} |zE - F| &= (z + \delta_a + \mu_a)^{n-1} (z + \delta_a + \mu_a - \sum_{i=1}^n \lambda_a(i) S_{i,j}^{a0} f(i)) \\ &= (z + \delta_a + \mu_a)^{n-1} [z - (\delta_a + \mu_a) (\frac{\Lambda_a}{\mu_a(\delta_a + \mu_a)} \frac{1}{\langle k \rangle_a} \sum_{i=1}^n \lambda_a(i) i p_a(i, \cdot) - 1)] \\ &= (z + \delta_a + \mu_a)^{n-1} [z - (\delta_a + \mu_a) (\mathcal{R}_0 - 1)], i = 1, 2, \dots, n, \end{aligned}$$

we can easily see that the disease-free equilibrium point E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$.

(ii) The Jacobian matrix $J(E^*)$ of endemic equilibrium point can be calculated as follows

$$J(E^*) = \begin{bmatrix} J(E_1^*) & 0 \\ C^* & D^* \end{bmatrix},$$

where

$$\begin{aligned} C^* &= \begin{bmatrix} 0 & \cdots & 0 & -m_1^o h(1) & \cdots & -m_1^o h(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -m_n^o h(1) & \cdots & -m_n^o h(n) \\ 0 & \cdots & 0 & m_1^o h(1) & \cdots & -m_1^o h(n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & m_n^o h(1) & \cdots & -m_n^o h(n) \end{bmatrix}, \\ D^* &= \begin{bmatrix} -\mu_h - p_1^o & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\mu_h - p_n^o & 0 & \cdots & 0 \\ p_1^o & \cdots & 0 & \omega_h & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & p_n^o & 0 & \cdots & \omega_h \end{bmatrix}, \end{aligned}$$

where $p_j^o = \lambda_h(j) \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)}$, $m_j^o = \lambda_h(j) \frac{S_{i,j}^{h*}}{[1 + \alpha_2 \Theta_{ah}^*(t)]^2}$, $\omega_h = -(\delta_h + \gamma_h + \mu_h)$, $h(j) = \frac{j p_a(\cdot, j)}{\langle k \rangle_{ah}}$. The characteristic equation of Jacobian matrix $J(E^*)$ is

$$(z + \mu_a)^n (z + \omega_a)^n \mid zE - H \mid \Pi_{j=1}^n (z + \mu_h + p_j^o) = 0, \quad (11)$$

we can easily get that all the eigenvalues of equation (11) are negative according to [Theorem 3.1](#). Therefore, the endemic equilibrium point E^* is locally asymptotically stable. \square

4.2. Globally asymptotical stability of model (3)

Theorem 4.2. (i) If $\mathcal{R}_0 < 1$, then the disease-free equilibrium point E^0 of model (3) is globally asymptotically stable; (ii) If $\mathcal{R}_0 > 1$, then the endemic equilibrium point E^* of model (3) is globally asymptotically stable.

Proof. (i) To prove the globally asymptotically stability of the disease-free equilibrium point E^0 , we only need to discuss the following system

$$\begin{cases} \frac{dS_{i,j}^h(t)}{dt} = \Lambda_h - \mu_h S_{i,j}^h(t), \\ \frac{dI_{i,j}^h(t)}{dt} = -\gamma_h I_{i,j}^h(t) - \delta_h I_{i,j}^h(t) - \mu_h I_{i,j}^h(t). \end{cases} \quad (12)$$

Consider the following Lyapunov function

$$V(t) = \sum_{j=1}^n h(j) (S_{i,j}^h - S_{i,j}^{h0} - S_{i,j}^{h0} \ln \frac{S_{i,j}^h}{S_{i,j}^{h0}}) + \sum_{j=1}^n h(j) I_{i,j}^h(t). \quad (13)$$

The derivative of $V(t)$ can be written as

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{j=1}^n h(j) \left(1 - \frac{S_{i,j}^{h0}}{S_{i,j}^h}\right) \frac{dS_{i,j}^h(t)}{dt} + \sum_{j=1}^n h(j) \frac{dI_{i,j}^h}{dt} \\ &= \sum_{j=1}^n h(j) \left(1 - \frac{S_{i,j}^{h0}}{S_{i,j}^h}\right) (\Lambda_h - \mu_h S_{i,j}^h(t)) - \sum_{j=1}^n h(j) (\gamma_h + \delta_h + \mu_h) I_{i,j}^h(t) \\ &= -\mu_a \frac{(S_{i,j}^h - S_{i,j}^{h0})^2}{S_{i,j}^h} - \sum_{j=1}^n h(j) (\gamma_h + \delta_h + \mu_h) I_{i,j}^h(t) < 0, \end{aligned}$$

we have $S_{i,j}^h(t) \rightarrow S_{i,j}^{h0}(t)$, $I_{i,j}^h(t) \rightarrow 0$ ($t \rightarrow \infty$). So, model (3) is globally asymptotically stable when $\mathcal{R}_0 < 1$.

(ii) To prove the globally asymptotically stability of the endemic equilibrium point E^* , we only need to discuss the following system.

$$\begin{cases} \frac{dS_{i,j}^h(t)}{dt} = \Lambda_h - \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \mu_h S_{i,j}^h(t), \\ \frac{dI_{i,j}^h(t)}{dt} = \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \gamma_h I_{i,j}^h(t) - \delta_a I_{i,j}^h(t) - \mu_h I_{i,j}^h(t). \end{cases} \quad (14)$$

Consider the following Lyapunov function

$$V(t) = \sum_{j=1}^n c_j S_{i,j}^{h*} \left(\frac{S_{i,j}^h}{S_{i,j}^{h*}} - \ln \frac{S_{i,j}^h}{S_{i,j}^{h*}} \right) + \sum_{j=1}^n c_j I_{i,j}^{h*} \left(\frac{I_{i,j}^h}{I_{i,j}^{h*}} - \ln \frac{I_{i,j}^h}{I_{i,j}^{h*}} \right). \quad (15)$$

The derivative of $V(t)$ is as follows

$$\begin{aligned} \frac{dV}{dt} &= \sum_{j=1}^n c_j \left(1 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} \right) \frac{dS_{i,j}^h(t)}{dt} + \sum_{j=1}^n c_j \left(1 - \frac{I_{i,j}^{h*}}{I_{i,j}^h} \right) \frac{dI_{i,j}^h(t)}{dt} \\ &= \sum_{j=1}^n c_j \left(1 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} \right) \left[\Lambda_h - \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \mu_h S_{i,j}^h(t) \right] \\ &\quad + \sum_{j=1}^n c_j \left(1 - \frac{I_{i,j}^{h*}}{I_{i,j}^h} \right) \left[\lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \gamma_h I_{i,j}^h(t) - \delta_a I_{i,j}^h(t) - \mu_h I_{i,j}^h(t) \right] \\ &= \sum_{j=1}^n c_j \left(1 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} \right) \left[\lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} + \mu_h S_{i,j}^{h*}(t) - \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \mu_h S_{i,j}^h(t) \right] \\ &\quad + \sum_{j=1}^n c_j \left(1 - \frac{I_{i,j}^{h*}}{I_{i,j}^h} \right) \left[\lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} \frac{I_{i,j}^h}{I_{i,j}^{h*}} \right] \\ &= \sum_{j=1}^n c_j \left[\lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} + \mu_h S_{i,j}^{h*}(t) - \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \mu_h S_{i,j}^h(t) \right] \\ &\quad + \sum_{j=1}^n c_j \left[\lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} \frac{I_{i,j}^h}{I_{i,j}^{h*}} \right] \\ &\quad - \sum_{j=1}^n c_j \frac{S_{i,j}^{h*}}{S_{i,j}^h} \left[\lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} + \mu_h S_{i,j}^{h*}(t) - \lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \mu_h S_{i,j}^h(t) \right] \\ &\quad - \sum_{j=1}^n c_j \frac{I_{i,j}^{h*}}{I_{i,j}^h} \left[\lambda_h(j) S_{i,j}^h \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} - \lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} \frac{I_{i,j}^h}{I_{i,j}^{h*}} \right] \\ &= \sum_{j=1}^n c_j \lambda_h(j) S_{i,j}^{h*} \frac{\Theta_{ah}^*(t)}{1 + \alpha_2 \Theta_{ah}^*(t)} \left(3 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} - \frac{I_{i,j}^h}{I_{i,j}^{h*}} - \frac{S_{i,j}^h I_{i,j}^{h*}}{S_{i,j}^{h*} I_{i,j}^h} \right) + \sum_{j=1}^n c_j \mu_h S_{i,j}^{h*} \left(2 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} - \frac{S_{i,j}^h}{S_{i,j}^{h*}} \right), \end{aligned}$$

because $3 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} - \frac{I_{i,j}^h}{I_{i,j}^{h*}} - \frac{S_{i,j}^h I_{i,j}^{h*}}{S_{i,j}^{h*} I_{i,j}^h} \leq 0$, $2 - \frac{S_{i,j}^{h*}}{S_{i,j}^h} - \frac{S_{i,j}^h}{S_{i,j}^{h*}} \leq 0$. we have $\frac{dV(t)}{dt} \leq 0$ if $\mathcal{R}_0 > 1$, then model (3) is globally asymptotically stable. \square

5. Optimal Control Strategies

In this section, we will establish an optimal control problem of system (3) and get an optimal control strategy in theory. We introduce control variables $(u_i(t), v_j(t)) \in \mathcal{U} = \{u_i(t), v_j(t) \text{ measurable} : 0 \leq u_i(t) \leq 1, 0 \leq v_j(t) \leq 1, i, j = 1, 2, \dots, n\}$. $u_i(t)$ denotes the proportion of slaughtered susceptible poultry and infected poultry, $v_j(t)$ denotes the proportion of treatment for infected humans.

Now, we obtain an optimal control system as follows

$$\begin{cases} \frac{dS_{i,j}^a(t)}{dt} = \Lambda_a - \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \mu_a S_{i,j}^a(t) - u_i(t)S_{i,j}^a(t), \\ \frac{dI_{i,j}^a(t)}{dt} = \lambda_a(i)S_{i,j}^a(t)\frac{\Theta_a(t)}{1 + \alpha_1\Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t) - u_i(t)I_{i,j}^a(t), \\ \frac{dS_{i,j}^h(t)}{dt} = \Lambda_h - \lambda_{ah}(j)S_{i,j}^h(t)\frac{\Theta_{ah}(t)}{1 + \alpha_2\Theta_{ah}(t)} - \mu_h S_{i,j}^h(t), \\ \frac{dI_{i,j}^h(t)}{dt} = \lambda_{ah}(j)S_{i,j}^h(t)\frac{\Theta_{ah}(t)}{1 + \alpha_2\Theta_{ah}(t)} - \gamma_h I_{i,j}^h(t) - \delta_h I_{i,j}^h(t) - \mu_h I_{i,j}^h(t) - \frac{cv_j(t)I_{i,j}^h(t)}{1 + \alpha_3 I_{i,j}^h(t)}. \end{cases} \quad (16)$$

Here, we take saturated treatment rate $\frac{cv_j(t)I_{i,j}^h(t)}{1 + \alpha_3 I_{i,j}^h(t)}$ (α_3 denotes saturation constant) because of the medical resources are limited. We intend to get an optimal pair of slaughter and treatment, which seeks to minimize the number of infected poultry, the number of infected humans, and the cost during the implementing these two control strategies. Therefore, we establish the following objective function

$$\begin{aligned} J(u_i(t), v_j(t)) = & \int_0^T \sum_{i=1}^n [A_i S_{i,j}^a(t) + B_i I_{i,j}^a(t) + C_i u_i(t)(S_{i,j}^a(t) + I_{i,j}^a(t)) + \frac{1}{2} D_i u_i^2(t)] dt \\ & + \int_0^T \sum_{j=1}^n [G_j I_{i,j}^h(t) + K_j v_j(t) I_{i,j}^h(t) + \frac{1}{2} L_j v_j^2(t)] dt, \end{aligned}$$

where $A_i, B_i, G_j, C_i, K_j, D_i, L_j$ are regarded as positive weight constants to make the terms of integrand keep balance in objective functional J .

Theorem 5.1. *There exists an optimal control pair $(u_i^*(t), v_j^*(t)) \in \mathcal{U}$ such that*

$$J(u_i^*(t), v_j^*(t)) = \min\{ J(u_i(t), v_j(t)) | u_i(t), v_j(t) \in \mathcal{U}, i, j = 1, \dots, n \}.$$

Proof. Because the control variables $u_i(t), v_j(t)$ and the state variables $S_{i,j}^a(t), I_{i,j}^a(t), S_{i,j}^h(t), I_{i,j}^h(t)$ are both non-negative, the objective function satisfy the convex condition with respect to the control variables, so the allowable control set \mathcal{U} is a closed convex set. We get the existence of optimal control. In addition, the integrand function $\sum_{i=1}^n [A_i S_{i,j}^a(t) + B_i I_{i,j}^a(t) + C_i u_i(t)(S_{i,j}^a(t) + I_{i,j}^a(t)) + \frac{1}{2} D_i u_i^2(t)] + \sum_{j=1}^n [G_j I_{i,j}^h(t) + K_j v_j(t) I_{i,j}^h(t) + \frac{1}{2} L_j v_j^2(t)]$ is convex on the control set, there exists a constant κ and positive numbers ϖ_1, ϖ_2 such that

$$J(u_i(t), v_j(t)) \geq \varpi_2 + \varpi_1(|u_i(t)|^2 + |v_j(t)|^2)^{\frac{\kappa}{2}}$$

□

Next, we characterize this optimal control by definitioning a Hamiltonian function.

Theorem 5.2. Let $u_i(t)$ and $v_j(t)$ be optimal control variables, $S_{i,j}^a(t), I_{i,j}^a(t), S_{i,j}^h(t), I_{i,j}^h(t)$ be corresponding optimal state variables of the control model. Then there exists adjoint variable $\xi(t) = (\xi_{1i}(t), \xi_{2i}(t), \xi_{1j}(t), \xi_{2j}(t)) \in R^{4n}$ that satisfies the following adjoint equations:

$$\begin{aligned}
\frac{d\xi_{1i}(t)}{dt} &= -A_i - C_i u_i^*(t) - \xi_{1i}(t) [-\lambda_a(i) \frac{\Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))}{1 + \alpha_1 \Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))} - \mu_a - u_i^*(t)] \\
&\quad - \xi_{2i}(t) \lambda_a(i) \frac{\Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))}{1 + \alpha_1 \Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))}, \\
\frac{d\xi_{2i}(t)}{dt} &= -B_i - C_i u_i^*(t) - \xi_{1i}(t) [-\lambda_a(i) f(i) \frac{S_{i,j}^{a*}(t)}{[1 + \alpha_1 \Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))]^2}] \\
&\quad - \xi_{2i}(t) [\lambda_a(i) f(i) \frac{S_{i,j}^{a*}(t)}{[1 + \alpha_1 \Theta_a(I_{1,j}^{a*}(t), I_{2,j}^{a*}(t), \dots, I_{n,j}^{a*}(t))]^2} - \delta_a - \mu_a - u_i^*(t)], \\
\frac{d\xi_{1j}(t)}{dt} &= -\xi_{1j}(t) [-\lambda_{ah}(j) \frac{\Theta_{ah}(I_{i,1}^{a*}(t), I_{i,2}^{a*}(t), \dots, I_{i,n}^{a*}(t))}{1 + \alpha_1 \Theta_{ah}(I_{i,1}^{a*}(t), I_{i,2}^{a*}(t), \dots, I_{i,n}^{a*}(t))} - \mu_h] \\
&\quad - \xi_{2j}(t) \lambda_{ah}(j) \frac{\Theta_{ah}(I_{i,1}^{a*}(t), I_{i,2}^{a*}(t), \dots, I_{i,n}^{a*}(t))}{1 + \alpha_1 \Theta_{ah}(I_{i,1}^{a*}(t), I_{i,2}^{a*}(t), \dots, I_{i,n}^{a*}(t))}, \\
\frac{d\xi_{2j}(t)}{dt} &= -G_j - K_j v_j^*(t) + \xi_{2j}(t) [\mu_h + \delta_h + \gamma_h + \frac{c v_j^*(t)}{(1 + \alpha_3 I_{i,j}^h)^2}],
\end{aligned} \tag{17}$$

with transversality conditions

$$\xi_{1i}(T) = \xi_{2i}(T) = \xi_{1j}(T) = \xi_{2j}(T) = 0, i, j = 1, 2, \dots, n.$$

Furthermore, the corresponding optimal controls are give as follows

$$\begin{cases} u_i^*(t) = \min\{\max\{\frac{(\xi_{1i}(t) - C_i) S_{i,j}^{a*}(t) + (\xi_{2i}(t) - C_i) I_{i,j}^{a*}(t)}{D_i}, 0\}, 1\}, i = 1, 2, \dots, n, \\ v_j^*(t) = \min\{\max\{\frac{c \xi_{2j}(t) I_{i,j}^{h*}(t) - K_j I_{i,j}^{h*}(t) (1 + \alpha_3 I_{i,j}^{h*}(t))}{(1 + \alpha_3 I_{i,j}^{h*}(t)) L_j}, 0\}, 1\}, j = 1, 2, \dots, n. \end{cases} \tag{18}$$

Proof. Define the Hamiltonian function of the control system as follows

$$\begin{aligned}
H &= \sum_{i=1}^n [A_i S_{i,j}^a(t) + B_i I_{i,j}^a(t) + C_i u_i(t) (S_{i,j}^a(t) + I_{i,j}^a(t)) + \frac{1}{2} D_i u_i^2(t)] \\
&\quad + \sum_{j=1}^n [G_j I_{i,j}^h(t) + K_j v_j(t) I_{i,j}^h(t) + \frac{1}{2} L_j v_j^2(t)] \\
&\quad + \sum_{i=1}^n \xi_{1i}(t) [\Lambda_a - \lambda_a(i) S_{i,j}^a(t) \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \mu_a S_{i,j}^a(t) - u_i(t) S_{i,j}^a(t)] \\
&\quad + \sum_{i=1}^n \xi_{2i}(t) [\lambda_a(i) S_{i,j}^a(t) \frac{\Theta_a(t)}{1 + \alpha_1 \Theta_a(t)} - \delta_a I_{i,j}^a(t) - \mu_a I_{i,j}^a(t) - u_i(t) I_{i,j}^a(t)]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \xi_{1j}(t) [\Lambda_h - \lambda_{ah}(j) S_{i,j}^h(t) \frac{\Theta_{ah}(t)}{1 + \alpha_2 \Theta_{ah}(t)} - \mu_h S_{i,j}^h(t)] \\
& + \sum_{j=1}^n \xi_{2j}(t) [\lambda_{ah}(j) S_{i,j}^h(t) \frac{\Theta_{ah}(t)}{1 + \alpha_2 \Theta_{ah}(t)} - \gamma_h I_{i,j}^h(t) - \delta_h I_{i,j}^h(t) - \mu_h I_{i,j}^h(t) - \frac{cv_j(t) I_{i,j}^h(t)}{1 + \alpha_3 I_{i,j}^h(t)}].
\end{aligned}$$

By the Pontryagin's maximum principle, there exists $\xi(t) = (\xi_{1i}(t), \xi_{2i}(t), \xi_{1j}(t), \xi_{2j}(t)) \in R^{4n}$ such that the first order necessary conditions for the existence of optimal control are given by the following four equations,

$$\begin{aligned}
\frac{d\xi_{1i}(t)}{dt} &= - \frac{\partial H}{\partial S_{i,j}^a(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)}, \\
\frac{d\xi_{2i}(t)}{dt} &= - \frac{\partial H}{\partial I_{i,j}^a(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)}, \\
\frac{d\xi_{1j}(t)}{dt} &= - \frac{\partial H}{\partial S_{i,j}^h(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)}, \\
\frac{d\xi_{2j}(t)}{dt} &= - \frac{\partial H}{\partial I_{i,j}^h(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)}.
\end{aligned}$$

By the optimal conditions, we have

$$\begin{cases} \frac{\partial H}{\partial u_i(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)} = 0, \\ \frac{\partial H}{\partial v_j(t)} \Big|_{S_{i,j}^a(t)=S_{i,j}^{a*}(t), I_{i,j}^a(t)=I_{i,j}^{a*}(t), S_{i,j}^h(t)=S_{i,j}^{h*}(t), I_{i,j}^h(t)=I_{i,j}^{h*}(t), u_i(t)=u_i^*(t), v_j(t)=v_j^*(t)} = 0. \end{cases}$$

Thus, we have

$$u_i^*(t) = \begin{cases} 0, & \Phi_i < 0, \\ \Phi_i, & 0 \leq \Phi_i \leq 1, \\ 1, & \Phi_i > 1, \end{cases} \quad v_j^*(t) = \begin{cases} 0, & \Psi_j < 0, \\ \Psi_j, & 0 \leq \Psi_j \leq 1, \\ 1, & \Psi_j > 1, \end{cases}$$

where $\Phi_i = \frac{(\xi_{1i}(t) - C_i) S_{i,j}^{a*}(t) + (\xi_{2i}(t) - C_i) I_{i,j}^{a*}(t)}{D_i}$, $\Psi_j = \frac{c \xi_{2j}(t) I_{i,j}^{h*}(t) - K_j I_{i,j}^{h*}(t) (1 + \alpha_3 I_{i,j}^{h*}(t))}{(1 + \alpha_3 I_{i,j}^{h*}(t)) L_j}$. Furthermore, we obtain equation (18). Hence, we can acquire an optimal control system. Analysis shows that slaughtering poultry and treatment for humans are useful measures to control the spread of avian influenza. \square

6. Numerical Simulation of the Stability of Equilibrium Points

In this section, we want to verify our theoretical results through numerical simulations. Simulations are based on a scale-free network with $p(k) = (r-1)m^{(r-1)}k^{-r}$, where m represents the smallest degree on a scale-free network nodes, r is power exponent. Let $m = 1, r = 3$, the number

of nodes on a scale-free network is $N = 100$, and we add each new node with 3 new edges. We choose degree k as $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, k_6 = 6, k_7 = 7, k_8 = 8, k_9 = 9$. We get the average degree of complex network structure $\langle k \rangle_a (\langle k^2 \rangle_a) = 3.27(9.04)$ through simple calculation.

Example 6.1. For model (3), all parameters take the following values: $\Lambda_a = 350$ per day, $\lambda_a = 3 \times 10^{-7}$ per day, $\mu_a = 0.01$ per day, $\delta_a = 0.05$ per day, $\alpha_1 = 0.001$ per day, $\Lambda_h = 100$ per day, $\lambda_{ah} = 6.201 \times 10^{-7}$ per day, $\mu_h = 3.91 \times 10^{-3}$ per day, $\delta_h = 0.3$ per day, $\alpha_2 = 0.001$ per day, $\gamma_h = 0.01$ per day, we can get $\mathcal{R}_0 = 0.4838 < 1$, which satisfy condition of Theorem 4.1. Fig. 1 shows the unique disease-free equilibrium point E^0 is globally asymptotically stable, and the virus will die out in the long run.

Example 6.2. For model (3), all parameters take the following values: $\Lambda_a = 350$ per day, $\lambda_a = 9 \times 10^{-7}$ per day, $\mu_a = 0.01$ per day, $\delta_a = 0.05$ per day, $\alpha_1 = 0.001$ per day, $\Lambda_h = 100$ per day, $\lambda_{ah} = 6.201 \times 10^{-7}$ per day, $\mu_h = 3.91 \times 10^{-3}$ per day, $\delta_h = 0.3$ per day, $\alpha_2 = 0.001$ per day, $\gamma_h = 0.01$ per day, we can get $\mathcal{R}_0 = 1.4514 > 1$, which satisfy condition of Theorem 4.2. Fig. 2 shows the unique endemic equilibrium point E^* is globally asymptotically stable, and the virus will persist.

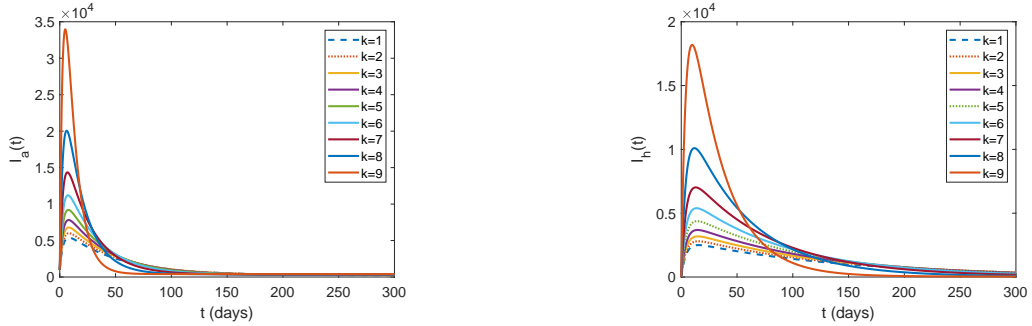


Figure 1: The density of infected nodes with different degree $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$ when $\mathcal{R}_0 < 1$

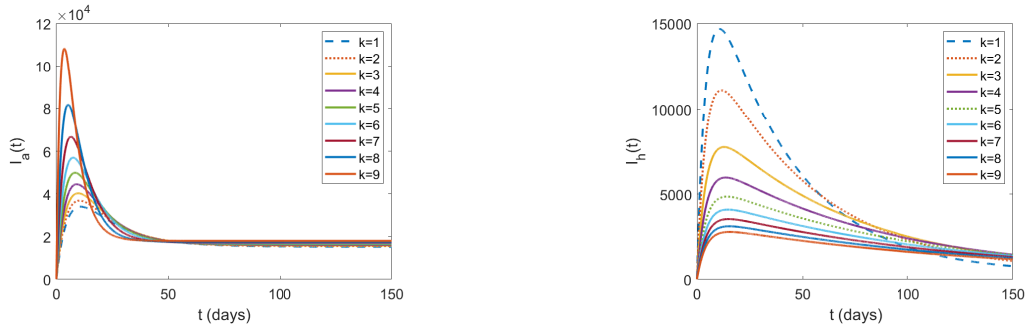


Figure 2: The density of infected nodes with different degree $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$ when $\mathcal{R}_0 > 1$

Furthermore, we can obtain that the density of infected nodes increase with the degree k increase.

In other words, the larger the degree k is, the higher the density of infected nodes is, which indicates that the nodes having lots of relative neighbors are more likely to be infected by contacting frequently.

7. Conclusions and Future Works

In this paper, we have investigated the epidemic dynamics of an avian influenza model in complex heterogeneous networks. The model has two equilibrium points: the disease-free equilibrium point E^0 and the endemic equilibrium point E^* . We have obtained the basic reproduction number \mathcal{R}_0 according to the method discussed in [19]. In the following, we have proven that the disease-free equilibrium point E^0 is globally asymptotically stable if $\mathcal{R}_0 < 1$, and the endemic equilibrium E^* is globally asymptotically stable if $\mathcal{R}_0 > 1$. Furthermore, an optimal control problem is established by taking slaughter rate and treatment for humans. We have also performed a series of numerical experiments to confirm the correctness of the theoretical analysis. Our future work is to propose the stochastic avian influenza models with spatial diffusion on complex networks.

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