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RESEARCH ARTICLE

A Study of Elliptic Biquaternionic Angular Momentum and Dirac Equation†

Zülal Derin[1] | Mehmet Ali Güngör*[2]

[1]Department of Mathematics, Sakarya University, Sakarya, Turkey
[2]Department of Mathematics, Sakarya University, Sakarya, Turkey

Correspondence
*Mehmet Ali Güngör, Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, Turkey.
Email: agungor@sakarya.edu.tr

Present Address
Department of Mathematics, Faculty of Arts and Sciences, Sakarya, Turkey

In this article, we deal with the Dirac equation and angular momentum, which have an important place in physics, in terms of elliptic biquaternions. Thanks to the elliptic biquaternionic representation of angular momentum, we have expressed some useful mathematical and physical results. We have obtain the solutions of the Dirac equation with elliptic Dirac matrices. Then, we have express the elliptic biquaternionic rotational Dirac equation. This equation could be interpreted as the combination of rotational energy and angular momentum of the particle and anti particle. Therefore, we also discuss the rotational energy momentum in the Euclidean space, the elliptic biquaternionic form of the relativistic mass. Further, we expressed the spinor wave function with elliptic biquaternions. Accordingly, we also have show elliptic biquaternionic rotational Dirac energy-momentum solutions through this function.

KEYWORDS:
elliptic biquaternions, elliptic Dirac equation

MSC CLASSIFICATION
11R52, 35Q41

1 | INTRODUCTION

Quantum mechanics has an important place in physics. The Dirac equation is the wave equation of relative quantum mechanics. The Dirac equation has provided the beginning of quantum field theory. Quaternions and biquaternions have the necessary algebraic structure to explain physical and mathematical event. There are many studies on quaternions and biquaternions in physics. Some mathematical and physical studies about quaternions and biquaternions are as follows: Quaternionic Lorentz Group and Dirac Equation[1], On a Generalization of Quantum Mechanics by Quaternions[2-10] are examples. A.W. Conway studied Dirac's relativistic equation and used complex quaternions. The negative energy particles seen in the Dirac equation are not actually negative, they are actually positive energy anti-particles by quantum field theory. Elliptic biquaternions include biquaternions. In this way, it can be used in many areas in physics such as quantum mechanics, special and general relativity. Therefore, they are useful numbers in physics. Elliptic biquaternionic physical studies will be investigated by us for the first time. The organization of the paper is as follows: In section 2, we give basic concepts, features of elliptic biquaternion algebra. In section 3, we define elliptic Pauli matrices and elliptic base matrices of elliptic biquaternions. In section 4, we express the elliptic biquaternionic angular momentum. In section 5, we discuss an elliptic biquaternionic Dirac-like new equation for free particle and its solutions .Then, we express the elliptic biquaternionic Dirac equation for the rotating particle. In the last section, we give conclusions of the study. The conclusion and discussion section of the study is in the last section.

†This is an example for title footnote.

2 | ELLIPTIC BIQUATERNIONS

The set of elliptic biquaternions is defined as follows:

$$\mathbb{HC}_p = \left\{ \mathbf{Q} = \mathbf{q} + I\mathbf{q}' = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 \,:\, A_0, A_1, A_2, A_3 \in \mathbb{C}_p, \quad I^2 = p<0 \right\} \tag{1}$$

where $A_i = a_i + I a_i', \quad 0 \leq \mathrm{i} \leq 3$ denote elliptic numbers. The base elements of the elliptic biquaternion are followed the relations

$$e_0{}^2 = e_0 = I_4, \quad e_1{}^2 = e_2{}^2 = e_3{}^2 = -I_4, \quad e_j e_k = \delta_{jk} e_0 - \varepsilon_{jkl} e_l$$

where the symbols δ_{jk} and ε_{jkl} are Kronecker, Levi Civita symbols, respectively. An elliptic biquaternion is expressed as:

$$\mathbf{Q} = \left(a_0 + I a'_0\right) e_0 + \left(a_1 + I a'_1\right) e_1 + \left(a_2 + I a'_2\right) e_2 + \left(a_3 + I a'_3\right) e_3 \tag{2}$$

where a_0,a_1,a_2, $a_3 \in \mathbb{R}$ and $I^2 = p < 0$. Any elliptic biquaternion can be written as scalar and vectorial parts as follows:

$$\mathbf{Q} = S(\mathbf{Q}) + V(\mathbf{Q}) = A_0 + \boldsymbol{A}. \tag{3}$$

The quaternionic product of two elliptic biquaternions $\mathbf{Q}$ and $\mathbf{P}$ as,

$$\mathbf{Q} \otimes \mathbf{P} = A_0 B_0 + A_0 \boldsymbol{B} + B_0 \boldsymbol{A} - \boldsymbol{A}.\boldsymbol{B} + \boldsymbol{A} \wedge \boldsymbol{B} \tag{4}$$

where $\mathbf{Q} = A_0 + \boldsymbol{A}$ and $\mathbf{P} = B_0 + \boldsymbol{B}$ and are also defined. Here "." and "$\wedge$" show the inner product (scalar) and the cross product respectively. The result of this product is the elliptic biquaternion. In addition, this product is non-comutative, but it is associative. The conjugate of an elliptic biquaternion is defined in three ways as follows:

$$\bar{\mathbf{Q}} = A_0 e_0 - A_1 e_1 - A_2 e_2 - A_3 e_3, \qquad \text{(quaternionic conjugate)} \tag{5}$$

$$\mathbf{Q}^* = (A_0)^* e_0 + (A_1)^* e_1 + (A_2)^* e_2 + (A_3)^* e_3, \qquad \text{(complex conjugate)} \tag{6}$$

$$\mathbf{Q}^\dagger = \left(\bar{\mathbf{Q}}\right)^* = \overline{(\mathbf{Q}^*)} = (A_0)^* e_0 - (A_1)^* e_1 - (A_2)^* e_2 - (A_3)^* e_3. \qquad \text{(total conjugate)} \tag{7}$$

The inner product of $\mathbf{Q}$ and $\mathbf{P}$ elliptic biquaternions is defined as follows:

$$\langle \mathbf{Q}, \mathbf{P} \rangle = \frac{1}{2}\left(\bar{\mathbf{Q}} \otimes \mathbf{P} + \bar{\mathbf{P}} \otimes \mathbf{Q}\right) = \frac{1}{2}\left(\mathbf{Q} \otimes \bar{\mathbf{P}} + \mathbf{P} \otimes \bar{\mathbf{Q}}\right). \tag{8}$$

On the other hand, the inner product of these two elliptic biquaternions is defined in the following way: Using this inner product semi-norm of $\mathbf{Q}$ is expressed as follows:

$$N(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q} \rangle = \mathbf{Q} \otimes \bar{\mathbf{Q}} = \bar{\mathbf{Q}} \otimes \mathbf{Q} = A_0{}^2 + A_1{}^2 + A_2{}^2 + A_3{}^2. \tag{9}$$

Since the above equation is $A_i = q_i + I q_i' \in \mathbb{C}_p$ $(0 \leqslant i \leqslant 3)$ it is seen from the above equation that $N(\mathbf{Q})$ can be equal to zero while. Accordingly, in the algebra of $\mathbf{Q} \neq 0$ elliptic biquaternions there are $\mathbf{Q}$ elliptic biquaternion which provide $\mathbf{Q} \otimes \bar{\mathbf{Q}} = \bar{\mathbf{Q}} \otimes \mathbf{Q}$ equality while, and the algebra of $\mathbb{HC}_p$ elliptic biquaternions contains zero divisors. Therefore, semi-norm is used instead of norm in elliptic biquaternion space in order to be appropriateness to the general literature [11]. Provided the semi-norm is different from zero the inverse of an elliptic biquaternion is given $\mathbf{Q}^{-1} = \frac{\bar{\mathbf{Q}}}{N(\mathbf{Q})}$. The module of this elliptic biquaternion $\mathbf{Q}$ is defined as

$$N(\mathbf{Q}) = \mathbf{Q} \otimes \bar{\mathbf{Q}} = \bar{\mathbf{Q}} \otimes \mathbf{Q} = |\mathbf{Q}|^2$$

and indicated by $|\mathbf{Q}|$. In the case of

$$N(\mathbf{Q}) = \langle \mathbf{Q}, \mathbf{Q} \rangle = \mathbf{Q} \otimes \bar{\mathbf{Q}} = \bar{\mathbf{Q}} \otimes \mathbf{Q} = A_0{}^2 + A_1{}^2 + A_2{}^2 + A_3{}^2 = 1$$

of this elliptic biquaternion, it is defined as unit elliptic biquaternions [11]. In equation (8), if A_0 and B_0 are zero, then the following equations can be written as

$$\frac{1}{2}\left(\mathbf{Q} \otimes \bar{\mathbf{P}} + \mathbf{P} \otimes \bar{\mathbf{Q}}\right) = \frac{1}{2}\left(\bar{\mathbf{Q}} \otimes \mathbf{P} + \bar{\mathbf{P}} \otimes \mathbf{Q}\right) = \langle \boldsymbol{B}, \boldsymbol{A} \rangle \tag{10}$$

and

$$\frac{1}{2}\left(\bar{\mathbf{P}} \otimes \mathbf{Q} - \bar{\mathbf{Q}} \otimes \mathbf{P}\right) = \frac{1}{2}\left(\mathbf{P} \otimes \bar{\mathbf{Q}} - \mathbf{Q} \otimes \bar{\mathbf{P}}\right) = \boldsymbol{A} \wedge \boldsymbol{B}. \tag{11}$$

Theorem 1. If an elliptic biquaternion $\mathbf{Q}$ is defined as $\mathbf{Q} = \cosh(p\frac{\theta_p}{2}) + \frac{1}{I}\hat{\boldsymbol{q}}\sinh(p\frac{\theta_p}{2})$ then the $\mathbf{Q}$ is unit elliptic biquaternion.

Proof. Let $\vec{v} = v_1e_1 + v_2e_2 + v_3e_3$ be a velocity vector and pure unit elliptic biquaternion $\hat{\boldsymbol{q}} = \sqrt{|p|}\frac{v_1e_1+v_2e_2+v_3e_3}{\sqrt{<\vec{v},\vec{v}>}}$. Then $\mathbf{Q} = \cosh(py)+\frac{1}{I}\hat{\boldsymbol{q}}\sinh(py)$ can be written as unit elliptic biquaternion. The conjugate of the elliptic biquaternion $\mathbf{Q}$ is $\bar{\mathbf{Q}} = \cosh(py)-\frac{1}{I}\hat{\boldsymbol{q}}\sinh(py)$. Here, if an elliptic angular y is chosen as $y = \frac{\theta_p}{2}$ then semi-norm of the elliptic biquaternion is written as $\mathbf{Q}\otimes\bar{\mathbf{Q}} = N(\mathbf{Q}) = \cosh^2(p\frac{\theta_p}{2}) - \sinh^2(p\frac{\theta_p}{2}) = 1$. Thus, it is seen that $\mathbf{Q} = \cosh(p\frac{\theta_p}{2}) + \frac{1}{I}\hat{q}\sinh(p\frac{\theta_p}{2})$ is a unit elliptic biquaternion[12]. □

3 | MATRIX REPRESENTATIONS OF ELLIPTIC BIQUATERNIONS

3.1 | The 2x2 Matrix Representation of Elliptic Biquaternions

For the matrix representation of elliptic biquaternions, the following isomorphism has been described by Ozen et al.[13]

$$\sigma : \mathbb{HC}_p \to M_2(\mathbb{C}_p), \mathbf{Q} = A_0 + A_1 i + A_2 j + A_3 k \to \sigma(\mathbf{Q}) = \begin{bmatrix} A_0 + \frac{1}{\sqrt{|p|}} I A_1 & -A_2 - \frac{1}{\sqrt{|p|}} I A_3 \\ A_2 - \frac{1}{\sqrt{|p|}} I A_3 & A_0 - \frac{1}{\sqrt{|p|}} I A_1 \end{bmatrix}.$$

We define the elliptic Pauli spin matrices as follows:

$$\sigma(e_0) = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma(e_1) = \sigma_1 = \begin{bmatrix} \frac{p}{\sqrt{|p|}} & 0 \\ 0 & -\frac{p}{\sqrt{|p|}} \end{bmatrix}, \ \sigma(e_2) = \sigma_2 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \ \sigma(e_3) = \sigma_3 = \begin{bmatrix} 0 & -\frac{p}{\sqrt{|p|}} \\ -\frac{p}{\sqrt{|p|}} & 0 \end{bmatrix} \tag{12}$$

where $I^2 = p < 0$. The unit bases of the elliptic biquaternion obtained with the help of the matrices described above are defined as follows:

$$e_0 \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ e_1 \simeq \begin{bmatrix} \frac{I}{\sqrt{|p|}} & 0 \\ 0 & -\frac{I}{\sqrt{|p|}} \end{bmatrix}, \ e_2 \simeq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ e_3 \simeq \begin{bmatrix} 0 & -\frac{I}{\sqrt{|p|}} \\ -\frac{I}{\sqrt{|p|}} & 0 \end{bmatrix}. \tag{13}$$

Thus the elliptic biquaternion $\mathbf{Q} = A_0 + A_1e_1 + A_2e_2 + A_3e_3$ can be represented in type 2×2 as

$$\mathbf{Q} \cong \begin{bmatrix} A_0 + \frac{I}{\sqrt{|p|}} A_1 & -A_2 - \frac{I}{\sqrt{|p|}} A_3 \\ A_2 - \frac{I}{\sqrt{|p|}} A_3 & A_0 - \frac{I}{\sqrt{|p|}} A_1 \end{bmatrix} \tag{14}$$

The determinant of the elliptic biquaternion given in here as

$$\mathbf{Q} \cong \begin{bmatrix} A_0 + \frac{I}{\sqrt{|p|}} A_1 & -A_2 - \frac{I}{\sqrt{|p|}} A_3 \\ A_2 - \frac{I}{\sqrt{|p|}} A_3 & A_0 - \frac{I}{\sqrt{|p|}} A_1 \end{bmatrix}.$$

Thus, we get

$$\det \mathbf{Q} = N(\mathbf{Q}).$$

3.2 | The Matrix Representation of Elliptic Biquaternions

Elliptic biquaternions can be represented by 4×4 matrices. The base of an elliptic biquaternion and $p-$ complex number I can be written in matrix form as follows:

$$\begin{gathered} e_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ e_1 = \begin{bmatrix} \frac{-I}{\sqrt{|p|}} & 0 & 0 & 0 \\ 0 & \frac{I}{\sqrt{|p|}} & 0 & 0 \\ 0 & 0 & \frac{-I}{\sqrt{|p|}} & 0 \\ 0 & 0 & 0 & \frac{I}{\sqrt{|p|}} \end{bmatrix} \\ e_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & \frac{-I}{\sqrt{|p|}} & 0 & 0 \\ \frac{-I}{\sqrt{|p|}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-I}{\sqrt{|p|}} \\ 0 & 0 & \frac{-I}{\sqrt{|p|}} & 0 \end{bmatrix}, \ I_4 = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}. \end{gathered} \tag{15}$$

The elliptic biquaternion **Q** with the help of these matrices is obtained in the form of 4×4 matrix as follows:

$$\mathbf{Q} = \begin{bmatrix} A_0 - \frac{I}{\sqrt{|p|}}A_1 & A_2 - \frac{I}{\sqrt{|p|}}A_3 & 0 & 0 \\ -A_2 - \frac{I}{\sqrt{|p|}}A_3 & A_0 + \frac{I}{\sqrt{|p|}}A_1 & 0 & 0 \\ 0 & 0 & A_0 - \frac{I}{\sqrt{|p|}}A_1 & A_2 - \frac{I}{\sqrt{|p|}}A_3 \\ 0 & 0 & -A_2 - \frac{I}{\sqrt{|p|}}A_3 & A_0 + \frac{I}{\sqrt{|p|}}A_1 \end{bmatrix}.$$

We express in 4×4 matrices types the elliptic biquaternion given in (2) as follows:

$$\mathbf{Q} = \begin{bmatrix} a_0 - \frac{I}{\sqrt{|p|}}a_1 & a_2 - \frac{I}{\sqrt{|p|}}a_3 & I(a'_0 - \frac{I}{\sqrt{|p|}}a'_1) & I(a'_2 - \frac{I}{\sqrt{|p|}}a'_3) \\ -a_2 - \frac{I}{\sqrt{|p|}}a_3 & a_0 + \frac{I}{\sqrt{|p|}}a_1 & -I(a_2 + \frac{I}{\sqrt{|p|}}a_3) & I(a'_0 + \frac{I}{\sqrt{|p|}}a'_1) \\ I(a'_0 - \frac{I}{\sqrt{|p|}}a'_1) & I(a'_2 - \frac{I}{\sqrt{|p|}}a'_3) & a_0 - \frac{I}{\sqrt{|p|}}a_1 & a_2 - \frac{I}{\sqrt{|p|}}a_3 \\ -I(a_2 + \frac{I}{\sqrt{|p|}}a_3) & I(a'_0 + \frac{I}{\sqrt{|p|}}a'_1) & -a_2 - \frac{I}{\sqrt{|p|}}a_3 & a_0 + \frac{I}{\sqrt{|p|}}a_1 \end{bmatrix}.$$

The elliptic biquaternion can be represented by 8×8 matrices. For this, let's define the $\Gamma_0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3$ base matrices as,

$$\Gamma_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} \frac{1}{I}\sigma_2 & 0 \\ 0 & -\frac{1}{I}\sigma_2 \end{bmatrix} = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&-1&0 \end{bmatrix}$$
$$\Gamma_2 = \begin{bmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{bmatrix} = \begin{bmatrix} 0&0&-1&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix},\ \Gamma_3 = \begin{bmatrix} 0 & \frac{1}{I}\sigma_2 \\ \frac{1}{I}\sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} 0&0&0&-1 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 1&0&0&0 \end{bmatrix}. \tag{16}$$

These matrices are satisfy multiplication relations basis of elliptic biquaternion in the following: ${\Gamma_0}^2 = \Gamma_0 = I,\ \Gamma_j\Gamma_k = \delta_{jk}\Gamma_0 - \varepsilon_{jkl}\Gamma_l$ where the δ and ε expressions show the Kronecker Delta and Levi-Civita symbols, respectively. Thus for an elliptic biquaternion $\mathbf{Q} = A_0\Gamma_0 + A_1\Gamma_1 + A_2\Gamma_2 + A_3\Gamma_3$ the known left Hamilton matrix using quaternionic bases given in (16) can be given as:

$$\mathrm{H}^{-}(\mathbf{Q}) = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & A_3 & -A_2 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & A_2 & -A_1 & A_0 \end{bmatrix}. \tag{17}$$

We can show left Hamilton matrix with $\mathrm{H}^{-}(\mathbf{Q})$. To find the 8×8 matrix representation expression of the elliptic biquaternion **Q**, let's first define the following matrices with the help of the matrices Γ as,

$$T = \mu \times \Gamma_0 = \begin{bmatrix} 0 & \Gamma_0\sqrt{|p|} \\ -\Gamma_0\sqrt{|p|} & 0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&\sqrt{|p|}&0&0&0 \\ 0&0&0&0&0&\sqrt{|p|}&0&0 \\ 0&0&0&0&0&0&\sqrt{|p|}&0 \\ 0&0&0&0&0&0&0&\sqrt{|p|} \\ -\sqrt{|p|}&0&0&0&0&0&0&0 \\ 0&-\sqrt{|p|}&0&0&0&0&0&0 \\ 0&0&-\sqrt{|p|}&0&0&0&0&0 \\ 0&0&0&-\sqrt{|p|}&0&0&0&0 \end{bmatrix} \tag{18}$$

where quaternionic units provide the following multiplication rules as,

$${\xi_0}^2 = -{\xi_j}^2 = I_8 = \xi_0,\ \xi_1\xi_2 = -\xi_3,\ \xi_2\xi_3 = -\xi_1,\ \xi_3\xi_1 = -\xi_2,\ \xi_2\xi_1 = \xi_3,\ \xi_3\xi_2 = \xi_1,\ \xi_1\xi_3 = \xi_2.$$

also let we define matrices of ξ_j, $j = 0, 1, 2, 3$ and μ elliptic matrix, 2×2 as,

$$\xi_j = \sigma_0 \times \Gamma_{_j} = \begin{bmatrix} \Gamma_{_j} & 0 \\ 0 & \Gamma_{_j} \end{bmatrix} \ (j = 1, 2, 3) \tag{19}$$

and

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mu = \sqrt{|p|} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{20}$$

where K and $\times$ denote generator matrix, kronecker product, respectively. From the equation (2) can be written the following equation as,

$$\mathrm{H}^{-}(\mathbf{Q}) = (a_0 + Ta_0')\xi_0 + (a_1 + Ta_0')\xi_1 + (a_2 + Ta_0')\xi_2 + (a_3 + Ta_0')\xi_3 \tag{21}$$

We obtain the real matrix representation from the expression (17) - (22) as follows:

$$\mathbb{H}^{-}(\mathbf{Q}) \cong \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & \sqrt{|p|}a'_0 & -\sqrt{|p|}a'_1 & -\sqrt{|p|}a'_2 & -\sqrt{|p|}a'_3 \\ a_1 & a_0 & a_3 & -a_2 & \sqrt{|p|}a'_1 & \sqrt{|p|}a'_0 & \sqrt{|p|}a'_3 & -\sqrt{|p|}a'_2 \\ a_2 & -a_3 & a_0 & a_1 & \sqrt{|p|}a'_2 & -\sqrt{|p|}a'_3 & \sqrt{|p|}a'_0 & \sqrt{|p|}a'_1 \\ a_3 & a_2 & -a_1 & a_0 & \sqrt{|p|}a'_3 & \sqrt{|p|}a'_2 & -\sqrt{|p|}a'_1 & \sqrt{|p|}a'_0 \\ -\sqrt{|p|}a'_0 & \sqrt{|p|}a'_1 & \sqrt{|p|}a'_2 & \sqrt{|p|}a'_3 & a_0 & -a_1 & -a_2 & -a_3 \\ -\sqrt{|p|}a'_1 & -\sqrt{|p|}a'_0 & -\sqrt{|p|}a'_3 & \sqrt{|p|}a'_2 & a_1 & a_0 & a_3 & -a_2 \\ -\sqrt{|p|}a'_2 & \sqrt{|p|}a'_3 & -\sqrt{|p|}a'_0 & -\sqrt{|p|}a'_1 & a_2 & -a_3 & a_0 & a_1 \\ -\sqrt{|p|}a'_3 & -\sqrt{|p|}a'_2 & \sqrt{|p|}a'_1 & -\sqrt{|p|}a'_0 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Also, we can show this matrix in 2×2 type as follows:

$$\mathbb{H}^{-}(\mathbf{Q}) \cong \begin{bmatrix} \sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}) & \sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}') \\ -\sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}') & \sqrt{|p|}\mathbb{H}^{-}(\mathbf{Q}) \end{bmatrix}. \tag{22}$$

This matrix is antisymmetric matrix.

4 | ELLIPTIC BIQUATERNIONIC REPRESENTATION OF ANGULAR MOMENTUM

Angular momentum is a vector quantity in the following manner

$$\vec{L} = \vec{r} \wedge \vec{P} \tag{23}$$

where $\vec{r}$ and $\vec{P}$ are position and momentum vectors, respectively. If $\vec{r}$ and $\vec{P}$ vectors are written in the elliptic biquaternionic notation, the following equations are obtained as,

$$\mathbf{R} = r_0e_0 + Ir_1e_1 + Ir_2e_2 + Ir_3e_3 = r_0 + I\boldsymbol{r}$$

and

$$\mathbf{P} = P_0e_0 + IP_1e_1 + IP_2e_2 + IP_3e_3 = P_0 + I\boldsymbol{P}.$$

Using by eq. (23), the elliptic biquaternionic angular momentum can be written as

$$\mathbf{L} = \mathbf{R} \otimes \mathbf{P}^* \ . \tag{24}$$

Here, "$\mathbf{R}$" represents the elliptic biquaternionic representation of the position, while "$\mathbf{P}$" is the elliptic biquaternionic representation of momentum. Thus, from the equation (24) is obtained as

$$\begin{aligned} \mathbf{L} = &\left[r_0P_0 + I^2(r_1P_1 + r_2P_2 + r_3P_3)\right]e_0 + I\left[-r_0P_1 + P_0r_1 + I\left(r_2P_3 - r_3P_2\right)\right]e_1 + I\left[-r_0P_2 + P_0r_2 + I\left(r_3P_1 - r_1P_3\right)\right]e_2 \\ &+ I\left[-r_0P_3 + P_0r_3 + I\left(r_1P_2 - r_2P_1\right)\right]e_3. \end{aligned}$$

If momentum values are written, then we get

$$\begin{aligned} \mathbf{L} = &\left[r_0m_0c + I^2(r_1m_1v_1 + r_2m_2v_2 + r_3m_3v_3)\right]e_0 + I\left[-r_0m_1v_1 + m_0cr_1 + I\left(r_2m_3v_3 - r_3m_2v_2\right)\right]e_1 \\ &+ I\left[-r_0m_2v_2 + m_0cr_2 + I\left(r_3m_1v_1 - r_1m_3v_3\right)\right]e_2 + I\left[-r_0m_3v_3 + m_0cr_3 + I\left(r_1m_2v_2 - r_2m_1v_1\right)\right]e_3 \end{aligned} \tag{25}$$

To put it simply, the elliptic biquaternionic angular momentum can be expressed as follows:

$$\mathbf{L} = L_0 e_0 + I L_1 e_1 + I L_2 e_2 + I L_3 e_3 \tag{26}$$

where $L_0 \sim E_0$ denotes elliptic biquaternionic energy and $\boldsymbol{L}_j \quad (j = 1,2,3)$ elliptic biquaternionic angular momentum. The elliptic biquaternionic energy can be definde as

$$E_0 \sim L_0 = r_0 P_0 + I^2 (\boldsymbol{r}.\boldsymbol{P}) \tag{27}$$

where $r_0 P_0$ denotes rest mass-energy of a particle and $I^2 (\boldsymbol{r}.\boldsymbol{P})$ represents moving the projective energy. Also, the elliptic biquaternionic angular momentum which the coefficient of e_j can be written as following:

$$\boldsymbol{L}_j = \left(-r_0 \boldsymbol{P}_j + P_0 \boldsymbol{r}_j\right) + I^2(\left(\boldsymbol{r}_j \wedge \boldsymbol{P}_j\right) \quad)(\forall j = 1,2,3)\,. \tag{28}$$

Here, the first term is the longitudinal component, the second term is the transverse component of the elliptical biquaternionic momentum. Then, the elliptic biquaternionic angular momentum can be expressed as follows:

$$\mathbf{L} = E_0 e_0 + I \boldsymbol{L}_j \ \ (\forall j = 1,2,3) \tag{29}$$

If necessary to make a few comments mathematically and physically; If $r_0 P_0 = 0$ is, $L_0 = -I^2 (\boldsymbol{r}.\boldsymbol{p})$ then the three-dimensional state of energy and angular momentum is considered. if $\boldsymbol{r} = \boldsymbol{P} = 0$ is taken in equation (27), since it will be $E_0 \simeq r_0 P_0$ and $L_0 \simeq 0$ only a particle with rest mass energy and no rotational motion can be mentioned. Also, if $\boldsymbol{r} \wedge \boldsymbol{p} = 0$ is the case, it can be expressed as pure energy. But if $\boldsymbol{r}.\boldsymbol{p} = 0$ is then the pure angular momentum expression is valid. Elliptic biquaternionic angular momentum since the scalar part expresses elliptic biquaternionic energy and the vector part expresses pure angular momentum, a few situations described above occur conditionally. It is possible to talk about a particle or antiparticle that exhibits an elliptical behavior with rotational energy and angular momentum if the statements given are not equal to zero depending on the above situations.

5 | ELLIPTIC BIQUATERNIONIC DIRAC EQUATION

It is predicted that wave mechanics can explain more events than classical mechanics. However, the wave equation does not include the electron spin given by Shrödinger. In addition, the wave equation is not invariant compared to Lorentz transformations. As a result, the speed of the Shrödinger equation is valid for low energy systems that are too small compared to the speed of light. In order to incorporate spin into the wave mechanics and attribute spins to the particles through the Shrödinger equation, Wolfgang Pauli first thought of wave function $\psi_{(\vec{r})}$ as a two-component vector, which corresponds to one of the possible orientations of the spin. Although the concept of this new wave function constitutes progress, it is not invariant that is, it is not relativist according to the Lorentz transformation for the new wave equation. In 1928, P.A.M. Dirac generalized the Schrödinger wave equation to both spin and remain invariant according to Lorentz transformations. It is known from Dirac's theory that the presence of positron, the anti-particle of the electron, is one of the greatest achievements for conceptual physics. Now the new elliptical bicuaternionic definition of the Dirac equation will be investigated for the free electron. The expression of energy in relativity is known as follows:

$$E^2 = {m_0}^2 c^4 + p^2 c^2. \tag{30}$$

Thus, it was possible to write the Dirac equation for a free particle. In this case, Hamiltonian independent of the position vector $\vec{r}$ and the time t as a linear expression of energy and momentum is defined as follows:

$$H = c\vec{\alpha}.\vec{P} + \beta m c^2.$$

Here is $\vec{P} = -i\hbar \frac{\partial}{\partial t}$ according to Correspondance Principle. β and $\vec{\alpha}(\alpha_1, \alpha_2, \alpha_3)$ are of independent $\vec{r}, t, \vec{P}$ and E.
$E = H = i\hbar \frac{\partial}{\partial t}$ can be written as,

$$\left(E - c\vec{\alpha}.\vec{P} - \beta m c^2\right)\psi = 0 \tag{31}$$

or $\left(i\hbar \frac{\partial}{\partial t} + i\hbar c\vec{\alpha}.\vec{\nabla} - \beta m c^2\right)\psi = 0$ where $\alpha_1, \alpha_2, \alpha_3$ and β are $n \times n$ matrices and ψ is the spin wave function of the particle. E, $\vec{P}$ and m represent energy, momentum and mass, respectively. On the other hand, let the pure elliptic biquaternions α and $\boldsymbol{P}$ be given as follows:

$$\boldsymbol{\alpha} = \frac{1}{I}\alpha_1 e_1 + \frac{1}{I}\alpha_2 e_2 + \frac{1}{I}\alpha_3 e_3, \ \ \boldsymbol{P} = I P_1 + I P_2 + I P_3. \tag{32}$$

If the equation (31) is multiplied by $\left(E + c\vec{\alpha}.\vec{P} - \beta mc^2\right)\psi = 0$ from the left, the elliptic expression of energy in relativity is obtained as

$$\left(E^2 - |p|\, c^2 \left({P_x}^2 + {P_y}^2 + {P_z}^2\right) - m^2c^4\right)\psi = 0 \tag{33}$$

which satisfy the following situations:

$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0$$
$$\alpha_2\alpha_3 + \alpha_3\alpha_2 = 0$$
$$\alpha_3\alpha_1 + \alpha_1\alpha_3 = 0$$

and

$$\alpha_1\beta + \beta\alpha_1 = 0$$
$$\alpha_2\beta + \beta\alpha_2 = 0$$
$$\alpha_3\beta + \beta\alpha_3 = 0\ .$$

Also $\alpha = \left\{\alpha_1, \alpha_2, \alpha_3\right\}$ and β are 4×4 elliptic Dirac matrices, which can be written as follows:

$$\alpha_1 = \begin{bmatrix} 0 & 0 & \frac{p}{\sqrt{|p|}} & 0 \\ 0 & 0 & 0 & \frac{-p}{\sqrt{|p|}} \\ \frac{p}{\sqrt{|p|}} & 0 & 0 & 0 \\ 0 & \frac{-p}{\sqrt{|p|}} & 0 & 0 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix},\ \alpha_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{-p}{\sqrt{|p|}} \\ 0 & 0 & \frac{-p}{\sqrt{|p|}} & 0 \\ 0 & \frac{-p}{\sqrt{|p|}} & 0 & 0 \\ \frac{-p}{\sqrt{|p|}} & 0 & 0 & 0 \end{bmatrix},$$
$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},\ I = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \tag{34}$$

where I is elliptic number in the matrix form and the rules ${\alpha_1}^2 = {\alpha_2}^2 = {\alpha_3}^2 = -p = |p|$, $\beta^2 = 1$ and $I^2 = p < 0$ is satisfied. The α matrices can be defined with elliptic biquaternionic unit bases as follows:

$$\begin{aligned} \alpha_1 &= -I_4 e_1 \\ \alpha_2 &= -I_4 e_2 \\ \alpha_3 &= I_4 e_3. \end{aligned} \tag{35}$$

Here, Dirac matrices for the free particle are associated with quaternionic bases. eliptic Biquaternionic Dirac equation for the free particle can be written as follows

$$\left(E - c\boldsymbol{P}.\boldsymbol{\alpha} - \beta mc^2\right) \otimes e_0 \otimes \psi = 0 \tag{36}$$

where can write as $\boldsymbol{P}.\boldsymbol{\alpha} = P_1\alpha_1 + P_2\alpha_2 + P_3\alpha_3$. and spinor wafe function $\psi = \psi_0 e_0 + I\psi_1 e_1 + I\psi_2 e_2 + I\psi_3 e_3$. If the elliptic terms $\boldsymbol{P}$ and $\boldsymbol{\alpha}$ in the equation (32) are written in equation (36) then is obtained as

$$\left(E - cP_1\alpha_1 - cP_2\alpha_2 - cP_3\alpha_3 - mc^2\beta\right) \otimes e_0 \otimes \psi = 0. \tag{37}$$

Thus, from the equation (35) is obtained as

$$\left(\left(E - mc^2\beta\right) e_0 + IcP_1 e_1 + IcP_2 e_2 - IcP_3 e_3\right) \otimes \psi = 0.$$

The expression matrix form of equation (37) is

$$E\begin{bmatrix} \psi_1 \\ I\psi_2 \\ I\psi_3 \\ I\psi_4 \end{bmatrix} = \begin{bmatrix} mc^2 & 0 & \frac{p}{\sqrt{|p|}}cP_1 & -IcP_2 - \frac{p}{\sqrt{|p|}}cP_3 \\ 0 & mc^2 & \frac{-p}{\sqrt{|p|}}cP_3 + IcP_2 & \frac{-p}{\sqrt{|p|}}cP_1 \\ \frac{p}{\sqrt{|p|}}cP_1 & -IcP_2 - \frac{p}{\sqrt{|p|}}cP_3 & -mc^2 & 0 \\ IcP_2 - \frac{p}{\sqrt{|p|}}cP_3 & -\frac{p}{\sqrt{|p|}}cP_1 & 0 & -mc^2 \end{bmatrix}\begin{bmatrix} \psi_1 \\ I\psi_2 \\ I\psi_3 \\ I\psi_4 \end{bmatrix}.$$

In this case we get the solutions of new like elliptic Dirac equations

$$\begin{aligned}
&\left(E - mc^2\right)\psi_1 - \frac{p}{\sqrt{|p|}} cP_x I\psi_3 + \left(IcP_y + \frac{p}{\sqrt{|p|}} cP_z\right) I\psi_4 = 0\\
&\left(E - mc^2\right) I\psi_2 + \left(\frac{p}{\sqrt{|p|}} cP_z - IcP_y\right) I\psi_3 + \frac{p}{\sqrt{|p|}} cP_x I\psi_4 = 0\\
&\left(E + mc^2\right) I\psi_3 - \frac{p}{\sqrt{|p|}} cP_x \psi_1 + \left(IcP_y + \frac{p}{\sqrt{|p|}} cP_z\right) I\psi_2 = 0\\
&\left(E + mc^2\right) I\psi_4 - \frac{p}{\sqrt{|p|}} cP_x I\psi_2 + \left(IcP_y - \frac{p}{\sqrt{|p|}} cP_z\right) I\psi_3 = 0\,.
\end{aligned}$$

Taking equal to $E = 0$ in the new elliptic biquaternionic formula of the Dirac equation known for the free particle in the equation (31) which gives energy momentum relations depending on the rotational motion of the electron in space-time can be extended as follows:

$$\left(c\boldsymbol{\alpha} \otimes \mathbf{L} + \beta \mathbf{M} c^2\right) \otimes \psi = 0\,. \tag{38}$$

Here $\boldsymbol{\alpha}, \mathbf{L}, \beta$ and $\mathbf{M}$ are elliptic biquaternionic variables. We can think of c as the maximum speed for electrons. Relativistic elliptic biquaternionic mass $\mathbf{M}$ which can be expressed as:

$$\mathbf{M} = \frac{\varepsilon_0}{c^2} e_0 + I\left|\frac{p_1}{v_1}\right| e_1 + I\left|\frac{p_2}{v_2}\right| e_2 + I\left|\frac{p_3}{v_3}\right| e_3 \simeq m_0 e_0 + I m_1 e_1 + I m_2 e_2 + I m_3 e_3 \tag{39}$$

where c is the rest mass and m_1, m_2 and m_3 are the mass of the moving particle of the elliptic biquaternion having velocities v_1, v_2 and v_3 corresponding to unit bases e_1, e_2 and e_3. The 4×4 elliptic Dirac matrices $\boldsymbol{\alpha}$, the matrix β and the elliptic number in the matrix form I_4 can be defined as:

$$\begin{aligned}
&\alpha_0 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{bmatrix},\ \alpha_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix},\ (\forall j = 1,2,3)\\
&\beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix},\ I_4 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.
\end{aligned} \tag{40}$$

Now, using the quaternionic product $\boldsymbol{\alpha} \otimes \mathbf{L}$, which is the first term in the expression (38) can be expressed together with the equations in (32) as follows:

$$\begin{aligned}
\boldsymbol{\alpha} \otimes \mathbf{L} &= \left(\alpha_0 e_0 + \frac{1}{I}\alpha_1 e_1 + \frac{1}{I}\alpha_2 e_2 + \frac{1}{I}\alpha_3 e_3\right) \otimes \left(L_0 e_0 + I L_1 e_1 + I L_2 e_2 + I L_3 e_3\right)\\
\boldsymbol{\alpha} \otimes \mathbf{L} &= \left(\alpha_0 L_0 - \alpha_1 L_1 - \alpha_2 L_2 - \alpha_3 L_3\right) e_0 + \left(I\alpha_0 L_1 + \frac{1}{I} L_0 \alpha_1 + \frac{1}{I^2}\alpha_2 L_3 - \frac{1}{I^2}\alpha_3 L_2\right) e_1\\
&\quad + \left(I\alpha_0 L_2 + \frac{1}{I} L_0 \alpha_2 + \frac{1}{I^2}\alpha_3 L_1 - \frac{1}{I^2}\alpha_1 L_3\right) e_2 + \left(I\alpha_0 L_3 + \frac{1}{I} L_0 \alpha_3 + \frac{1}{I^2}\alpha_1 L_2 - \frac{1}{I^2}\alpha_2 L_1\right) e_3
\end{aligned}$$

which can be further simplfy as

$$\boldsymbol{\alpha} \otimes \mathbf{L} = \left[\alpha_0 E_0 - \frac{1}{I^2}\boldsymbol{\alpha}.\boldsymbol{L}\right] e_0 + \left[I\alpha_0 \boldsymbol{L}_j + \frac{1}{I} L_0 \alpha_j + \frac{1}{I^2}\left(\alpha_j \wedge \boldsymbol{L}_j\right)\right] e_j\ \ (\forall j = 1,2,3)\,. \tag{41}$$

Correspondingly, the second term in equation (38) i.e. $\beta \mathbf{M} c^2$ can be expressed as,

$$\beta \mathbf{M} c^2 = \beta m_0 c^2 e_0 + I\beta m_1 c^2 e_1 + I\beta m_2 c^2 e_2 + I\beta m_3 c^2 e_3. \tag{42}$$

Therefore, from equations (41) and (42), the elliptic rotational Dirac equation can be written as

$$\begin{aligned}
\left(c\boldsymbol{\alpha} \otimes \mathbf{L} + \beta \mathbf{M} c^2\right) \otimes \psi = \left(\alpha_0 E_0 - \frac{1}{I^2}\boldsymbol{\alpha}.\boldsymbol{L}\right) e_0 + \left(I\alpha_0 \boldsymbol{L}_j + \frac{1}{I} L_0 \alpha_j + \frac{1}{I^2}\left(\alpha_j \wedge \boldsymbol{L}_j\right)\right) e_j + \beta m_0 c^2 e_0\\
+ I\beta m_1 c^2 e_1 + I\beta m_2 c^2 e_2 + I\beta m_3 c^2 e_3 = 0\ \ (\forall j = 1,2,3)\,.
\end{aligned} \tag{43}$$

The elliptic biquaternionic equation (43) contains both scalar and vectorial components that give biquaternionic energy and angular momentum of electrons. The real part corresponding to e_0 gives the elliptic Dirac rotational energy and the part corresponding to $e_j\ \ (j = 1,2,3)$ gives the elliptic rotational momentum. The elliptic biquaternionic spinor is defined as follows:

$$\boldsymbol{\psi} = \psi_0 e_0 + I\psi_1 e_1 + I\psi_2 e_2 + I\psi_3 e_3 = \left(\psi_0 + I\psi_1 e_1\right) + \left(I\psi_2 - I\psi_3 e_1\right) e_2 = \psi_a + I\psi_b e_2\,. \tag{44}$$

This spin wave function can be expressed as two-component and four-component follows

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix}$$

and

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_0 \\ I\psi_1 \\ I\psi_2 \\ I\psi_3 \end{bmatrix}.$$

We can use these matrices to define elliptic biquaternionic energy and angular momentum solutions. For energy solutions of the elliptic biquaternionic rotational Dirac equation, we can express the scalar part in equation (43) as follows:

$$c\left(\alpha_0 E_0 - \frac{1}{I^2}\boldsymbol{\alpha}.\boldsymbol{L} + \beta m_0 c^2 e_0\right) = 0 \tag{45}$$

and with the help of the matrices in equation (40) we get the following matrix expression as,

$$\begin{bmatrix} c\left(E_0 - m_0 c^2\right) & \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right) \\ \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right) & c\left(E_0 + m_0 c^2\right) \end{bmatrix} \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} \tag{46}$$

where $\boldsymbol{e}$ is defined as $\boldsymbol{e} = I(-e_1, -e_2, e_3)$.

$$c\left(E_0 - m_0 c^2\right)\psi_a + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_b = 0. \tag{47}$$

$$c\left(E_0 + m_0 c^2\right)\psi_b + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_a = 0. \tag{48}$$

The equations (47) and (48) obtained represent the positive and negative energy solutions of the particles. If ψ_a and ψ_b values are taken as a unified function of energy and momentum in these equations then the following solutions are obtained as

$$\begin{aligned}
c\left(E_0 - m_0 c^2\right)\psi_0\left(E_0, \boldsymbol{L}\right) + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_2\left(E_0, \boldsymbol{L}\right) &= 0 \\
c\left(E_0 - m_0 c^2\right)\psi_1\left(E_0, \boldsymbol{L}\right) + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_3\left(E_0, \boldsymbol{L}\right) &= 0 \\
c\left(E_0 + m_0 c^2\right)\psi_2\left(E_0, \boldsymbol{L}\right) + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_0\left(E_0, \boldsymbol{L}\right) &= 0 \\
c\left(E_0 + m_0 c^2\right)\psi_3\left(E_0, \boldsymbol{L}\right) + \frac{1}{I}\left(\boldsymbol{e}.\boldsymbol{L}\right)\psi_1\left(E_0, \boldsymbol{L}\right) &= 0.
\end{aligned}$$

These statements represent the positive and negative energy solution of the rotating particles. The elliptic solutions obtained here are associated with elliptic biquaternionic bases corresponding to all positive and negative energy spinners particle and anti particle. These solutions demonstrate the elliptic behavior of the elliptic biquaternionic quantum wave spinor function associated with the interaction between elliptic biquaternionic spin and orbital angular momentum.

6 | CONCLUSIONS

In this article, we present a study of the Dirac equation and angular momentum in the elliptic biquaternionic field. Elliptic biquaternion is an algebraic structure consisting of elliptic components, including both biquaternions and quaternions. Dirac equations explain relativistic systems as they are known and have an important place in physics. Therefore, elliptic biquaternions can be used to describe important equations in many physical fields such as quantum mechanics, general and special relativity. In our study and we defined the elliptic matrices of its type. We presented the matrix representation of elliptic biquaternionic expressions with these matrices we have defined. These matrices are useful matrices for the Dirac equation. With these matrices we have defined for elliptic biquaternions, we have defined new elliptic Dirac matrices. We associated the elliptic Dirac matrices with elliptic biquaternionic bases. We obtained the solutions of the Dirac equation in (37) with the matrices given in (34) α. Then, it is given in the new elliptic biquaternionic formula (38), which gives the energy momentum relations depending on the motion of the space-time electron of the known Dirac equation for the free particle. In addition, the elliptic biquaternionic mass is defined in equation (39). The mass here is associated with the scalar part rest mass corresponding to the unit bases of the elliptic biquaternion, and the vectorial part as the moving mass. Moreover, the Dirac equation for the rotating particle was written in a simpler and more compact form with the equation (43), which includes the elliptic biquaternionic rotational energy and angular momentum. Then, elliptic biquaternionic spinor is defined. Thanks to the elliptic biquaternionic definition of this wave function, elliptic biquaternionic energy and angular momentum solutions have been expressed. With these solutions, the elliptic behavior of the quantum wave spinor function associated with the interaction between the elliptic biquaternionic spin and the orbital angular momentum is expressed. These expressions are very useful for fields such as quantum mechanics general and specific relativity.

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Author contributions

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Conflict of interest

The authors declare no potential conflict of interests.

Orcid

Zülal Derin ID https://orcid.org/0000-0001-9712-0428
Mehmet Ali Güngör ID https://orcid.org/0000-0003-1863-3183

References

1. De Leo S. Quaternionic Lorentz Group and Dirac Equation. *Foundations of Physics Letters*. 2001;14:37-50 .

2. Conte E. On a Generalization of Quantum Mechanics of Biquaternions. *Haddronnic Journal*. 1993;16:261-275 .

3. Negi OPS, et al. Revisting Quaternion Formulation and Electromagnetism. *Nuovo Cimento*. 1998;11B:1449-1467 .

4. Ward JP. *Quaternions and Cayley Numbers*. London: Kluwer Academic Publishers; 1997.

5. Chanyal BC, Sandhya. A comparative study of quaternionic rotational Dirac equation and its interpretation. *International Journal of Geometric Methods in Modern Physics*. 2020;17:2050018.

6. Yaglom IM. *Complex Numbers in Geometry*. New York: Academic Press; 1968.

7. Kansu ME, Tanışlı M, Demir S. Quaternionic Comparisons of Electromagnetism Using Lorentz Transformations. *The European Physical Journal Plus.* 2020;135:187-203.

8. Harkin AA, Harkin JB. Geometry of Generalized Complex Numbers. *Mathematics Magazine*. 2004;77:118–129.

9. Hacısalihoğlu HH, *Hareket Geometrisi ve Kuaterniyonlar Teorisi*. Ankara: Gazi Üniversitesi Fen Edebiyat Fakültesi Yayınları; 1983.

10. Tanışlı M, Özgür G. Biquaternionic representations of angular momentum and Dirac equation. *Acta Physica Slovaca*. 2003;53:243-252.

11. Özen KE, Tosun M. Elliptic biquaternion algebra. *AIP Conf. Proc.* 2018;1926 020032-1-020032-6.

12. Derin Z, Güngör MA. On Lorentz Transformations with Elliptic Biquaternions. *Tbilisi Mathematical Journal*. 2020;13:121–140.

13. Özen KE, Tosun M. Further Results for Elliptic Biquaternions. International Eurasian Conference on Mathematical Sciences and Application, (IECMSA), 2018. In: 7th International; 2018;1(1):20-27.

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