Existence of positive solutions and hydrodynamic limit of the steady Boltzmann equation with in-flow boundary condition

Chenghong Wang[1], Fujun Zhou[2], Weijun Wu[3,*]

[1] School of Mathematics, South China University of Technology, Guangzhou 510640, P. R. China. E-mail: machenghongwang2020@mail.scut.edu.cn

[1] School of Mathematics, South China University of Technology, Guangzhou 510640, P. R. China. E-mail: fujunht@scut.edu.cn

[1] School of Mathematics, South China University of Technology, Guangzhou 510640, P. R. China. E-mail: scutweijunwu@qq.com

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Abstract

This work is devoted to the study of existence of positive solutions and hydrodynamic limit of the steady Boltzmann equation with in-flow boundary condition. The proof is based on a $L^6 - L^\infty$ framework developed by [10] and a refined positivity-preserving scheme in deriving positivity of solutions with in-flow boundary condition and external force. The incompressible Navier–Stokes–Fourier limit with Dirichlet boundary condition is justified for in-flow boundary data as small perturbation of a global Maxwellian.

1 Introduction

In this paper we consider existence of positive solutions and hydrodynamic limit of the following steady Boltzmann equation with in-flow boundary condition

$$v \cdot \nabla_x F_\varepsilon + \varepsilon^2 \Phi \cdot \nabla_v F_\varepsilon = \frac{1}{\varepsilon} Q(F_\varepsilon, F_\varepsilon) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \tag{1.1}$$

$$F_\varepsilon = H_\varepsilon \qquad \text{on} \quad \gamma_-, \tag{1.2}$$

which models the motion of a rarefied gas, subjecting to the action of an external field $\Phi = \Phi(x)$. Here $F_\varepsilon(x, v) \geq 0$ represents the distribution density of the gas molecules with position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$, and $\Omega \subset \mathbb{R}^3$ is a C^3 bounded domain. The collision operator takes the form

$$\begin{aligned} Q(F,G)(v) :&= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \omega)[F(v')G(v'_*) - F(v)G(v_*)] \mathrm{d}\omega \mathrm{d}v_* \\ :&= Q_+(F,G)(v) - Q_-(F,G)(v), \end{aligned}$$

where $v' = v - [(v - v_*) \cdot \omega]\omega$, $v'_* = v_* + [(v - v_*) \cdot \omega]\omega$, and B stands for the hard spheres cross section with Grad's angular cutoff $B(v - v_*, \omega) = |(v - v_*) \cdot \omega|$. The positive parameter

*Corresponding author

$\varepsilon > 0$ represents the Knudsen number, proportional to the mean free time and is very small. Let

$$M_{\rho,u,T} := \rho(2\pi T)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v-u|^2}{2T}}$$

stand for a local Maxwellian with density ρ, bulk velocity u and temperature T, and

$$\mu \equiv M_{1,0,1} := (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}$$

be a global Maxwellian.

The boundary condition (1.2), which is called the *in-flow* boundary condition, represents that the number density on the incoming set is prescribed [24]. More precisely, we denote the phase boundary in the space $\Omega \times \mathbb{R}^3$ by $\gamma = \partial\Omega \times \mathbb{R}^3$, and define the outgoing set γ_+, the incoming set γ_- and the singular set γ_0 as

$$\begin{aligned} \gamma_\pm &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v \gtrless 0\}, \\ \gamma_0 &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v = 0\}. \end{aligned}$$

Then the in-flow function H_ε in (1.2) is given by a local Maxwellian

$$H_\varepsilon = M_{1+\varepsilon^2\tilde{\rho},\varepsilon^2\tilde{u},1+\varepsilon^2\tilde{\theta}}, \tag{1.3}$$

with prescribed functions $\tilde{\rho}$, $\tilde{u}$ and $\tilde{\theta}$ independence of ε. Here, for simplicity, the in-flow data is taken as small perturbation of the global Maxwellian $M_{1,0,1}$ with amplitude of order $O(\varepsilon^2)$, which will lead to homogeneous Dirichlet boundary condition for hydrodynamic equations. In fact, the in-flow data on density and temperature can also be taken as small perturbation with amplitude of order $O(\varepsilon)$, which will eventually export non-homogeneous Dirichlet boundary condition for hydrodynamic equations, cf. Remark 1.2.

The Boltzmann equation is the cornerstone of kinetic theory connecting the microscopic and macroscopic theory of gases and fluids. Therefore, the hydrodynamic limit of the Boltzmann equation has been drawn a lot of attention. Justifying these limiting processes rigorously has been an active research field in the past several decades, and a lot of progress has been made, cf. [2, 3, 4, 5, 7, 13, 14, 16, 17, 19, 22, 23, 25, 26].

However, less is known for hydrodynamic limit of the Boltzmann equation with in-flow boundary condition. Recently, Jiang and Zhang [20, 21] studied global renormalized solutions and Navier–Stokes limit of the Boltzmann equation for long rang interaction and cutoff collision kernels. However, due to the lack of L^1 and entropy estimates, this framework of DiPerna–Lions' renormalized solutions [8] is not available to the steady Boltzmann equation, which is indeed an important topic [15]. Particularly, Esposito, Guo, Kim and Marra [10] developed a L^6–L^∞ framework to study hydrodynamic limit of the steady Boltzmann equation with diffuse boundary condition, and proved the positivity of steady solutions through analyzing the asymptotic behavior of solutions to the corresponding time-dependent Boltzmann equation. Then Esposito, Guo and Marra extended the study to exterior domain [11] and constructed a positivity-preserving scheme for steady solutions similarly as Arkeryd and Nouri [1].

In this paper, inspired by the works [1, 10, 11, 20, 21], we consider the steady Boltzmann equation with in-flow boundary condition (1.1)–(1.2) and seek for its positive solutions and hydrodynamic limit. More precisely, we show that (1.1)–(1.2) has a unique positive solution of the form

$$F_\varepsilon = \mu + \varepsilon\sqrt{\mu} f_\varepsilon, \tag{1.4}$$

and the perturbation f_ε converges to

$$f_* = [\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2}]\sqrt{\mu} \tag{1.5}$$

weakly in $L^2(\Omega \times \mathbb{R}^3)$. Here (ρ, u, θ) solves the steady incompressible Navier–Stokes–Fourier system (INSF for short) with Dirichlet boundary condition

$$u \cdot \nabla_x u + \nabla_x p = \eta \Delta_x u + \Phi, \ \ \nabla_x \cdot u = 0 \quad \text{in} \quad \Omega, \tag{1.6}$$

$$u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \ \ \nabla_x(\rho + \theta) = 0 \quad \text{in} \quad \Omega, \tag{1.7}$$

$$u(x) = 0, \ \ \theta(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{1.8}$$

where η, κ and p represent viscosity, heat conductivity and pressure, respectively. Moreover, we construct a refined positivity-preserving scheme directly for the solution of the steady Boltzmann equation with in-flow boundary condition, even in the presence of an external force.

To state the main result accurately, we introduce some notations. Let

$$Lf := -\frac{1}{\sqrt{\mu}}\big[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\big], \quad \Gamma(f, g) := \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}f, \sqrt{\mu}g)$$

stand for the linearized collision operator and nonlinear collision operator, respectively. The null space of L, which we denote by $\mathrm{N}(L)$, is a five-dimensional subspace of $L^2(\mathbb{R}^3_v)$

$$\mathrm{N}(L) = \mathrm{span}\{\sqrt{\mu}, v\sqrt{\mu}, \frac{|v|^2 - 3}{2}\sqrt{\mu}\}.$$

Let $\mathbf{P}$ represent the orthogonal projection of $L^2(\mathbb{R}^3_v)$ onto $\mathrm{N}(L)$, that is,

$$\mathbf{P}f = a\sqrt{\mu} + b \cdot v\sqrt{\mu} + c\frac{|v|^2 - 3}{2}\sqrt{\mu} \quad \text{for } f \in L^2(\mathbb{R}^3_v).$$

Let $(\mathbf{I} - \mathbf{P})f = f - \mathbf{P}f$ be the projection on the orthogonal complement of $\mathrm{N}(L)$. It is known that $Lf = \nu f - Kf$, where $\nu = \nu(v)$ is collision frequency defined by

$$\nu(v) := \frac{1}{\sqrt{\mu}}Q_-(\sqrt{\mu}, \mu) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |(v - v_*) \cdot \omega|\mu(v_*)\mathrm{d}\omega\mathrm{d}v_*, \tag{1.9}$$

and

$$Kf := \int_{\mathbb{R}^3} \mathbf{k}(v, v_*)f(v_*)\mathrm{d}v_* = \frac{1}{\sqrt{\mu}}[Q_+(\mu, \sqrt{\mu}f) + Q_+(\sqrt{\mu}f, \mu) - Q_-(\mu, \sqrt{\mu}f)]$$

is a compact operator on $L^2(\mathbb{R}^3_v)$. In addition, for hard sphere cross section, there are $C_0, C_1 > 0$, such that

$$C_0\langle v\rangle \leq \nu(v) \leq C_1\langle v\rangle$$

with $\langle v\rangle := \sqrt{1 + |v|^2}$.

By substituting (1.4) into (1.1) and expanding (1.3), we find that the perturbation f_ε satisfies

$$v\cdot\nabla_x f_\varepsilon+\varepsilon^2\Phi\cdot\nabla_v f_\varepsilon-\frac{1}{2}\varepsilon^2\Phi\cdot vf_\varepsilon+\frac{1}{\varepsilon}Lf_\varepsilon=\Gamma(f_\varepsilon,f_\varepsilon)+\varepsilon\Phi\cdot v\sqrt{\mu}\quad\text{in }\ \Omega\times\mathbb{R}^3,\tag{1.10}$$

$$f_\varepsilon=h_\varepsilon\qquad\text{on }\ \gamma_-,\tag{1.11}$$

where

$$h_\varepsilon=\varepsilon(\tilde{\rho}+\tilde{u}\cdot v+\tilde{\theta}\frac{|v|^2-3}{2})\sqrt{\mu}+O(\varepsilon^2).\tag{1.12}$$

For notations simplicity, we use $A\lesssim B$ to denote $A\le CB$, where $C>0$ is a constant not depending on A, B. We use $\langle\cdot\ ,\ \cdot\rangle$ to represent the $L^2(\mathbb{R}^3_v)$ inner product. Define $\|\cdot\|_p:=\|\cdot\|_{L^p(\Omega\times\mathbb{R}^3)}$ for $1\le p\le\infty$ and a weighted L^2 norm $\|\cdot\|_\nu:=\|\nu^{\frac{1}{2}}\cdot\|_{L^2(\Omega\times\mathbb{R}^3)}$ with ν given in (1.9). For the phase boundary integration, we use $|\cdot|$–norm, that is, $|f|_p^p:=\int_\gamma|f(x,v)|^p\mathrm{d}\gamma$ for $1\le p<\infty$, where $\mathrm{d}\gamma=|n(x)\cdot v|\mathrm{d}S(x)\mathrm{d}v$ and $\mathrm{d}S(x)$ is the surface measure. Similarly, let $|f|_\infty$ represent $\sup_{(x,v)\in\gamma}|f(x,v)|$. We also denote $|f|_{p,\pm}=|f\mathbf{1}_{\gamma_\pm}|_p$ for $1\le p<\infty$.

Then we can state our main results.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with C^3 boundary $\partial\Omega$. Assume that $\tilde{\rho},\tilde{u},\tilde{\theta}\in L^\infty(\partial\Omega)$, $\Phi\in C^1(\Omega)$ and $\|\Phi\|_2\ll1$. Then for each $0<\varepsilon\ll1$, the steady Boltzmann equation (1.1)–(1.2) has a unique positive solution $F_\varepsilon=\mu+\varepsilon\sqrt{\mu}f_\varepsilon$, where f_ε satisfies (1.10)–(1.11) and*

$$\|\mathbf{P}f_\varepsilon\|_6+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu+\varepsilon^{-\frac{1}{2}}|f_\varepsilon|_{2,+}+\varepsilon^{\frac{1}{2}}\|wf_\varepsilon\|_\infty\ll1\tag{1.13}$$

with a weight $w(v)=\mathrm{e}^{\beta|v|^2}$ for $0<\beta\ll1$. Moreover,

$$f_\varepsilon\to f_*\ \text{weakly in}\ L^2(\Omega\times\mathbb{R}^3)\ \text{as}\ \varepsilon\to0,\tag{1.14}$$

where f_* is given by (1.5) and (ρ,u,θ) solves the steady INSF with Dirichlet boundary condition (1.6)–(1.8).

The proof of Theorem 1.1 is based on the L^6-L^∞ framework developed by [10] and a refined positivity-preserving scheme in deriving positivity of solutions.

One difficulty comes from the well-known challenge – showing the positivity of solutions to the steady Boltzmann equation. An indirect treatment is through analyzing the asymptotic behavior of solutions to the corresponding time-dependent Boltzmann equation, as shown in [9, 10]. Inspired by [1, 11], we construct a refined positivity-preserving scheme directly for the steady Boltzmann solution in the case of in-flow boundary condition and external force, cf. Lemma 3.1.

Another feature lies in the in-flow boundary condition, where the total boundary norm $\varepsilon^{-\frac{1}{2}}|f_\varepsilon|_{2,+}$, rather than its dissipation part $\varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f_\varepsilon|_{2,+}$ for the model with diffusive boundary condition, is controlled in the uniform estimate (1.13). This further leads to different treatment when taking limit for the boundary condition, cf. the proof of Theorem 1.1.

Remark 1.2. *If the in-flow function H_ε in (1.2) is given as small perturbation of the global Maxwellian $M_{1,0,1}$ with amplitude of order $O(\varepsilon)$, that is,*

$$H_\varepsilon=M_{1+\varepsilon\bar{\rho},0,1+\varepsilon\bar{\theta}}$$

with prescribed functions $\bar{\rho}$, $\bar{\theta} \in H^{\frac{1}{2}}(\partial\Omega) \cap W^{1,\infty}(\partial\Omega)$ independence of ε, then positive solution F_ε can still be obtained, but with non-homogeneous Dirichlet boundary condition in the limiting Fourier equation

$$\theta(x) = \bar{\theta} \qquad on \qquad \partial\Omega. \tag{1.15}$$

In fact, this can be shown by constructing an auxiliary function

$$f_w = [\rho_w + \theta_w \frac{|v|^2 - 3}{2}]\sqrt{\mu}$$

where (ρ_w, θ_w) is an extension of $(\bar{\rho}, \bar{\theta})$ in Ω, and translating the remainder equation and boundary condition through $F_\varepsilon = \mu + \varepsilon\sqrt{\mu}(f_\varepsilon + f_w)$ as done in [10].

The rest of this paper is organized as follows. Section 2 concentrates on deducing linear estimate for the remainder equation (1.10)–(1.11). In Section 3 , we construct a refined positivity-preserving scheme for solutions to (1.1)–(1.2) and give the proof of Theorem 1.1.

2 Linear Estimate

In this section, we deduce the linear estimate for the remainder equation (1.10)–(1.11). For this, we consider the corresponding linear problem

$$\begin{cases} v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f - \dfrac{1}{2}\varepsilon^2 \Phi \cdot v f + \varepsilon^{-1} L f = g \quad \text{in} \;\; \Omega \times \mathbb{R}^3, \\ f = r \quad \text{on} \;\; \gamma_-. \end{cases} \tag{2.1}$$

Firstly, we present the following L^∞ estimate for the linear equation (2.1).

Lemma 2.1. *Assume that f solves the equation (2.1). Then there holds*

$$\|wf\|_\infty \lesssim |wr|_\infty + \varepsilon\|\langle v\rangle^{-1} wg\|_\infty + \varepsilon^{-\frac{1}{2}}\|\mathbf{P}f\|_6 + \varepsilon^{-\frac{3}{2}}\|(\mathbf{I} - \mathbf{P})f\|_2, \tag{2.2}$$

where $w(v) = \mathrm{e}^{\beta|v|^2}$ with $0 < \beta \ll 1$.

Proof. The spirit of proving this result comes from [9, 10], but the in-flow boundary condition here leads to different treatment along the characteristic. We just give the derivation for the difference.

In fact, by letting

$$h(x, v) := w(v) f(x, v), \tag{2.3}$$

it follows from (2.1) and Lemma 3 of [18] that

$$\big[v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v - \frac{1}{2}\varepsilon^2 \Phi \cdot v + \varepsilon^{-1} C_0 \langle v\rangle\big]|h| \le \varepsilon^{-1}\int_{\mathbb{R}^3} k_{\tilde{\beta}}(v, v_*)|h(x, v_*)|\mathrm{d}v_* + |wg|, \tag{2.4}$$

$$|h| \le w(v)|r| \quad \text{for } (x, v) \in \gamma_-, \tag{2.5}$$

where $\tilde{\beta} = \tilde{\beta}(\beta)$ and

$$k_{\tilde{\beta}}(v, v_*) := \{|v - v_*| + |v - v_*|^{-1}\}\exp\left[-\tilde{\beta}|v - v_*|^2 - \tilde{\beta}\frac{||v|^2 - |v_*|^2|^2}{|v - v_*|^2}\right]. \tag{2.6}$$

Clearly, $\varepsilon^{-1}C_0\langle v\rangle - \frac{1}{2}\varepsilon^2\Phi\cdot v \sim \varepsilon^{-1}C_0\langle v\rangle$.

Define the characteristics

$$\dot{X} = V, \quad \dot{V} = \varepsilon^2\Phi; \quad X(t;t,x,v) = x, \quad V(t;t,x,v) = v. \tag{2.7}$$

Then for $\max\{0,t_1\} < s \le t \le T_0$, there holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathrm{e}^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}|h(X(s;t,x,v),V(s;t,x,v))|\right]\\ \le&\mathrm{e}^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}\frac{1}{\varepsilon}\int_{\mathbb{R}^3} k_{\tilde{\beta}}(V(s;t,x,v),v')|h(X(s;t,x,v),v')|\mathrm{d}v'\\ &+\mathrm{e}^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}|wg(X(s;t,x,v),V(s;t,x,v))|.\end{aligned} \tag{2.8}$$

Along with the characteristics, (2.4) has the following expression

$$\begin{aligned}&|h(x,v)|\\ &\le \mathbf{1}_{\{t_1<0\}}\mathrm{e}^{-\int_0^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}|h(X(0;t,x,v),V(0;t,x,v))| &(2.9)\\ &\quad+\int_{\max\{0,t_1\}}^t \mathrm{d}s\ \frac{1}{\varepsilon}\mathrm{e}^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}\\ &\qquad\times\int_{\mathbb{R}^3} k_{\tilde{\beta}}(V(s;t,x,v),v')|h(X(s;t,x,v),v')|\mathrm{d}v' &(2.10)\\ &\quad+\int_{\max\{0,t_1\}}^t \mathrm{d}s\ \frac{1}{\varepsilon}\mathrm{e}^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}|\varepsilon wg(X(s;t,x,v),V(s;t,x,v))| &(2.11)\\ &\quad+\mathbf{1}_{\{t_1\ge 0\}}\mathrm{e}^{-\int_{t_1}^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau}|h(X(t_1;t,x,v),V(t_1;t,x,v))|, &(2.12)\end{aligned}$$

where $X(t_1;t,x,v) = x_b(x,v)$ and $V(t_1;t,x,v) = v_b(x,v)$.

Then (2.9), (2.11) and (2.12) are bounded directly by

$$\mathrm{e}^{-\frac{C_0}{\varepsilon}t}\|h\|_\infty + \|wr\|_\infty + \varepsilon\|\langle v\rangle^{-1}wg\|_\infty. \tag{2.13}$$

With the help of the second iteration and Duhamel principle, (2.10) is bounded by

$$\varepsilon^{-\frac{1}{2}}\|\mathbf{P}f\|_6 + \varepsilon^{-\frac{3}{2}}\|(\mathbf{I}-\mathbf{P})f\|_2 + \|wr\|_\infty + \varepsilon\|\langle v\rangle^{-1}wg\|_\infty + o(1)\|h\|_\infty, \tag{2.14}$$

similarly as that of Proposition 2.6 in [10]. The detail is omitted for simplicity. □

In order to acquire the estimate for $\mathbf{P}f$, we need the following lemma.

Lemma 2.2. *Assume that $g \in L^2(\Omega\times\mathbb{R}^3)$, $r \in L^2(\gamma_-)$, $\Phi \in L^\infty(\Omega)$. Let f^τ be a solution of the following equation*

$$\begin{cases} [(1-\tau)\varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v]f^\tau + v\cdot\nabla_x f^\tau + \varepsilon^2\Phi\cdot\nabla_v f^\tau + \varepsilon^{-1}\tau Lf^\tau = g \quad \text{in} \quad \Omega\times\mathbb{R}^3,\\ f^\tau = r \qquad\qquad \text{on} \quad \gamma_- \end{cases} \tag{2.15}$$

in the sense of distribution. Then for all τ sufficiently close to 1, there holds

$$\begin{aligned}\|\mathbf{P}f^\tau\|_2 \lesssim &\ \varepsilon^{-1}\tau\|(\mathbf{I}-\mathbf{P})f^\tau\|_\nu + \|\frac{g}{\sqrt{\nu}}\|_2 + |f^\tau|_{2,+} + |r|_{2,-}\\ &+ \{o(1)+\varepsilon^2\|\Phi\|_\infty\}\|(\mathbf{I}-\mathbf{P})f^\tau\|_2,\end{aligned} \tag{2.16}$$

$$\|\mathbf{P}f^\tau\|_6 \lesssim \varepsilon^{-1}\tau\|(\mathbf{I}-\mathbf{P})f^\tau\|_\nu + \|\frac{g}{\sqrt{\nu}}\|_2 + C_\eta\varepsilon^{-\frac12}|f^\tau|_{2,+} + \eta\varepsilon^{\frac12}\|wf^\tau\|_\infty \\ + \varepsilon^{-\frac12}|r|_{2,-} + \varepsilon^{\frac12}|wr|_\infty + \{o(1)+\varepsilon^2\|\Phi\|_\infty\}\|(\mathbf{I}-\mathbf{P})f^\tau\|_6. \tag{2.17}$$

Proof. The proof is similar to that of Lemma 2.12 in [10], except for dealing with boundary terms carefully due to the in-flow boundary condition. We skip the detail for simplicity. □

Define a norm

$$|||\, f||| := \|\mathbf{P}f\|_6 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \varepsilon^{-\frac12}|f|_{2,+} + \varepsilon^{\frac12}\|wf\|_\infty. \tag{2.18}$$

Then we can state the following main result of this section.

Theorem 2.3. *Assume that $g \in L^2(\Omega\times\mathbb{R}^3)$, $r \in L^2(\gamma_-)$ and $\Phi \in L^\infty(\Omega)$. Then for $0<\varepsilon\ll 1$, the linear problem (2.1) has a unique solution f, which satisfies*

$$|||\, f||| \lesssim \varepsilon^{-\frac12}|r|_{2,-} + \varepsilon^{\frac12}|wr|_\infty + \varepsilon^{\frac32}\|\langle v\rangle^{-1}wg\|_\infty + \varepsilon^{-1}\|\mathbf{P}g\|_2 + \|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2. \tag{2.19}$$

Proof. Firstly, we consider the following auxiliary problem

$$\begin{cases} \mathscr{L}f := (\varepsilon^{-1}\nu - \frac12\varepsilon^2\Phi\cdot v)f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f = g & \text{in } \ \Omega\times\mathbb{R}^3, \\ f = r & \text{on } \ \gamma_-. \end{cases} \tag{2.20}$$

We claim that (2.20) has a unique solution f for sufficiently small $\varepsilon \ll 1$, which satisfies

$$\varepsilon^{-1}\|f\|_\nu^2 + |f|_{2,+}^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.21}$$

In fact, denote $\sigma := \varepsilon^{-1}\nu - \frac12\varepsilon^2\Phi\cdot v$. Obviously, $\sigma \ge \frac12\varepsilon^{-1}\nu_0\langle v\rangle$. From the characteristics (2.7) and (2.20), for $-t_b(x,v) < t < t_f(x,v)$, we deduce

$$f(x,v) = r(x_b(x,v),v_b(x,v))\mathrm{e}^{-\int_{-t_b}^0\sigma(\tau)\mathrm{d}\tau} + \int_{-t_b}^0 g(X(s),V(s))\mathrm{e}^{-\int_s^0\sigma(\tau)\mathrm{d}\tau}\mathrm{d}s,$$

where

$$t_f(x,v) := \sup\{t\ge 0 : X(t;0,x,v)\in\Omega\}, \quad t_b(x,v) := \sup\{t\ge0 : X(-t;0,x,v)\in\Omega\},$$
$$\big(X(s),V(s)\big) := \big(X(s;0,x,v),V(s;0,x,v)\big), \quad \sigma(\tau) := \sigma\big(X(\tau;0,x,v),V(\tau;0,x,v)\big).$$

Existence and uniqueness of solution of (2.21) are thus achieved. Further, the inequality (2.21) follows from standard L^2 energy estimate. Thus the operator $\mathscr{L}^{-1}$ is well-defined and bounded in L^2.

Next, from the definition of $\mathscr{L}$ in (2.20), a solution to (2.1) is a fixed point of the map

$$f \mapsto \mathscr{L}^{-1}[\varepsilon^{-1}Kf + g]. \tag{2.22}$$

Hence for any $f\in L^2$, there is $h\in L^2$ such that $f = \mathscr{L}^{-1}h$. Thus, the fixed point problem (2.22) for f is equivalent to the following fixed point problem for h

$$h \mapsto \varepsilon^{-1}K\mathscr{L}^{-1}h + g. \tag{2.23}$$

The operator $K\mathscr{L}^{-1}$ is in fact a compact operator, which can be proved similarly as Appendix A.2 in [10]. Moreover, we claim that

$$\begin{aligned}&\text{if } h^\tau \text{ solves } h^\tau = \tau\varepsilon^{-1}K\mathscr{L}^{-1}h^\tau + g \text{ for some } \ \tau\in[1^-,1],\\&\text{then } \|h^\tau\| \text{ is bounded uniformly in } \tau.\end{aligned} \tag{2.24}$$

In fact, since $\mathscr{L}^{-1}$ is bounded, it suffices to show a uniform bound of $f^\tau = \mathscr{L}^{-1}h^\tau$, which is a solution to (2.15). Standard L^2 energy estimate of (2.15) leads to

$$\begin{aligned}\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f^\tau\|_\nu^2 + |f^\tau|_{2,+}^2 \lesssim & |r|_{2,-}^2 + \varepsilon^2\|\Phi\|_\infty\|f^\tau\|_\nu^2 + o(1)\|f^\tau\|_\nu^2 + \varepsilon\|f^\tau\|_2^2\\ & + o(1)\varepsilon^{-1}\|f^\tau\|_\nu^2 + \varepsilon^{-1}\|\mathbf{P}g\|_2^2 + \varepsilon\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2.\end{aligned} \tag{2.25}$$

Combining this with the macroscopic estimate of $\mathbf{P}f^\tau$ given in Lemma 2.2, we have

$$\delta\|\mathbf{P}f^\tau\|_2^2 + \varepsilon^{-2}\tau\|(\mathbf{I}-\mathbf{P})f^\tau\|_\nu^2 + \varepsilon^{-1}|f^\tau|_{2,+}^2 \lesssim \varepsilon^{-1}|r|_{2,-}^2 + \|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2 + \varepsilon^{-2}\|\mathbf{P}g\|_2^2. \tag{2.26}$$

Therefore, we have shown the uniform boundedness of $\|f^\tau\|_2$ in τ. Due to $f^\tau = \mathscr{L}^{-1}h^\tau$ and (2.24), we have

$$h^\tau = \tau\varepsilon^{-1}Kf^\tau + g. \tag{2.27}$$

Then $\|h^\tau\|_2$ is bounded uniformly in τ and the claim (2.24) is thus proved.

Hence, by the Schaefer's fixed point theorem [12], (2.23) has a fixed point h, which further indicates that (2.22) has a fixed point $f = \mathscr{L}^{-1}h$. Thus, the existence of a unique solution f to (2.1) is proved.

Finally, applying the L^2 energy estimate to $f - f^\tau$, we have

$$\begin{aligned}&\delta\|\mathbf{P}(f-f^\tau)\|_2^2 + \varepsilon^{-2}\tau\|(\mathbf{I}-\mathbf{P})(f-f^\tau)\|_\nu^2 + \varepsilon^{-1}|f-f^\tau|_{2,+}^2\\ \lesssim&\varepsilon^{-2}(1-\tau)\|f^\tau\|_\nu^2 \to 0 \quad \text{as } \tau\to 1.\end{aligned} \tag{2.28}$$

Therefore, f^τ converges to f strongly in L^2. From (2.26), as $\tau \to 1$, we obtain

$$\|\mathbf{P}f\|_2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \varepsilon^{-\frac12}|f|_{2,+} \lesssim \|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2 + \varepsilon^{-1}\|\mathbf{P}g\|_2 + \varepsilon^{-\frac12}|r|_{2,-}. \tag{2.29}$$

Similarly, we can conclude

$$\|\mathbf{P}f\|_6 \lesssim \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \|\frac{g}{\sqrt{\nu}}\|_2 + \varepsilon^{-\frac12}|f|_{2,+} + \varepsilon^{-\frac12}|r|_{2,-} + \varepsilon^{\frac12}|wr|_\infty + \varepsilon^{\frac32}\|\langle v\rangle^{-1}wg\|_\infty. \tag{2.30}$$

Then (2.19) follows from (2.29), (2.30) and (2.2). This completes the proof. □

3 Validity of the nonlinear problem

In this section, we improve the argument from [1, 11] and construct a non-negative solution to the problem (1.1)–(1.2) for the case of in-flow boundary condition and external force.

Define $F_\varepsilon^+ := \max\{F_\varepsilon, 0\}$, $F_\varepsilon^- := \max\{-F_\varepsilon, 0\}$. Then $F_\varepsilon = F_\varepsilon^+ - F_\varepsilon^-$. Let us consider the following equation

$$\begin{aligned}& v\cdot\nabla_x F_\varepsilon + \varepsilon^2\Phi\cdot\nabla_v F_\varepsilon - \varepsilon^2\Phi\cdot\nabla_v(\frac{1}{\sqrt{\mu}})\sqrt{\mu}F_\varepsilon^-\\ &\quad = \varepsilon^{-1}\big[Q(F_\varepsilon^+,F_\varepsilon^+) - Q(\mu,F_\varepsilon^-) - Q(F_\varepsilon^-,\mu)\big] \quad \text{in } \ \Omega\times\mathbb{R}^3, \end{aligned}\tag{3.1}$$

$$F_\varepsilon = M_{1+\varepsilon^2\tilde\rho,\varepsilon^2\tilde u,1+\varepsilon^2\tilde\theta} \quad \text{on } \ \gamma_-. \tag{3.2}$$

We have the following result.

Lemma 3.1. *Let $F_\varepsilon \in L^\infty(\Omega\times\mathbb{R}^3)$ be a solution of (3.1)–(3.2). Then $F_\varepsilon^- = 0$ and F_ε^+ solves the Boltzmann equation (1.1)–(1.2).*

Proof. Observe that $F_\varepsilon^- \neq 0$ implies $F_\varepsilon^+ = 0$. It follows that

$$\mathbf{1}_{F_\varepsilon^-\neq 0} Q^-(F_\varepsilon^+, F_\varepsilon^+) = \mathbf{1}_{F_\varepsilon^-\neq 0} F_\varepsilon^+ \nu\big(F_\varepsilon^+\big) = 0.$$

Thus we get the following equation for F_ε^-

$$\begin{aligned}
&- v\cdot\nabla_x F_\varepsilon^- - \varepsilon^2\Phi\cdot\nabla_v F_\varepsilon^- - \varepsilon^2\Phi\cdot\nabla_v\big(\frac{1}{\sqrt{\mu}}\big)\sqrt{\mu}F_\varepsilon^- \\
=&\varepsilon^{-1}\mathbf{1}_{F_\varepsilon^-\neq 0}\big[Q^+(F_\varepsilon^+, F_\varepsilon^+) - Q(\mu, F_\varepsilon^-) - Q(F_\varepsilon^-,\mu)\big] \quad \text{in } \ \Omega\times\mathbb{R}^3.
\end{aligned} \tag{3.3}$$

By (3.2), we know $F_\varepsilon > 0$ on γ_-. Then by the definition of F_ε^-, we have

$$F_\varepsilon^- = 0 \quad \text{on} \ \ \gamma_-. \tag{3.4}$$

Multiplying (3.3) with $-\mu^{-1}F_\varepsilon^-$ and integrating on $\Omega\times\mathbb{R}^3$, we get

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} \mu^{-1} v\cdot\nabla_x \frac{(F_\varepsilon^-)^2}{2}\mathrm{d}x\mathrm{d}v \\
=&\iint_{\Omega\times\mathbb{R}^3} \varepsilon^{-1}\mathbf{1}_{F_\varepsilon^-\neq 0}\mu^{-1}F_\varepsilon^-[Q(\mu, F_\varepsilon^-) + Q(F_\varepsilon^-,\mu)]\mathrm{d}x\mathrm{d}v \\
&- \iint_{\Omega\times\mathbb{R}^3} \varepsilon^{-1}\mathbf{1}_{F_\varepsilon^-\neq 0} Q^+(F_\varepsilon^+, F_\varepsilon^+)F_\varepsilon^-\mu^{-1}\mathrm{d}x\mathrm{d}v,
\end{aligned} \tag{3.5}$$

where we have used the following simplification

$$\begin{aligned}
&\varepsilon^2\iint_{\Omega\times\mathbb{R}^3} \mu^{-1}\Phi\cdot\nabla_v\frac{(F_\varepsilon^-)^2}{2}\mathrm{d}x\mathrm{d}v + \varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\Phi\cdot\nabla_v\Big(\frac{1}{\sqrt{\mu}}\Big)\mu^{-1/2}(F_\varepsilon^-)^2\mathrm{d}x\mathrm{d}v \\
=&\ \varepsilon^2\iint_{\Omega\times\mathbb{R}^3} \mu^{-1}\Phi\cdot\nabla_v\frac{(F_\varepsilon^-)^2}{2}\mathrm{d}x\mathrm{d}v + \varepsilon^2\iint_{\Omega\times\mathbb{R}^3}\Phi\cdot\nabla_v\Big(\frac{1}{\mu}\Big)\frac{(F_\varepsilon^-)^2}{2}\mathrm{d}x\mathrm{d}v \\
=&\ 0.
\end{aligned}$$

By the spectral inequality, cf. Theorem 7.2.5 in [6], there holds

$$-\iint_{\Omega\times\mathbb{R}^3}\mathbf{1}_{F_\varepsilon^-\neq 0}\mu^{-1}F_\varepsilon^-[Q(\mu, F_\varepsilon^-) + Q(F_\varepsilon^-,\mu)]\mathrm{d}x\mathrm{d}v \gtrsim \Big\|(\mathbf{I}-\mathbf{P})\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big)\Big\|_\nu^2. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{aligned}
&\iint_{\partial\Omega\times\mathbb{R}^3}\mu^{-1}v\cdot n(x)\frac{(F_\varepsilon^-)^2}{2}\mathrm{d}S(x)\mathrm{d}v + \varepsilon^{-1}\Big\|(\mathbf{I}-\mathbf{P})\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big)\Big\|_\nu^2 \\
\lesssim& -\iint_{\Omega\times\mathbb{R}^3}\varepsilon^{-1}\mathbf{1}_{F_\varepsilon^-\neq 0}Q^+(F_\varepsilon^+, F_\varepsilon^+)F_\varepsilon^-\mu^{-1}\mathrm{d}x\mathrm{d}v \leq 0.
\end{aligned} \tag{3.7}$$

It implies $(\mathbf{I}-\mathbf{P})\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big) = 0$. According to (3.4), this indicates $F_\varepsilon^- = 0$ on γ_+ and then

$$F_\varepsilon^- = 0 \quad \text{on} \ \ \gamma.$$

Owing to $\mu^{-1}F_\varepsilon^- > 0$ and (3.7), we obtain

$$Q(\mu, F_\varepsilon^-) + Q(F_\varepsilon^-, \mu) = 0. \tag{3.8}$$

From (3.3) and (3.8), we find

$$-v\cdot\nabla_x F_\varepsilon^- - \varepsilon^2\Phi\cdot\nabla_v\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big)\sqrt{\mu} = \varepsilon^{-1}\mathbf{1}_{F_\varepsilon^-\neq 0}Q^+(F_\varepsilon^+, F_\varepsilon^+) \geq 0.$$

In summary, we conclude

$$\begin{cases} v\cdot\nabla_x\Big(\dfrac{F_\varepsilon^-}{\sqrt{\mu}}\Big) + \varepsilon^2\Phi\cdot\nabla_v\Big(\dfrac{F_\varepsilon^-}{\sqrt{\mu}}\Big) \leq 0 \quad \text{in} \quad \Omega\times\mathbb{R}^3, \\ \hfill \dfrac{F_\varepsilon^-}{\sqrt{\mu}} = 0 \quad \text{on} \quad \gamma. \end{cases} \tag{3.9}$$

Obviously,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big) = (v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v)\Big(\frac{F_\varepsilon^-}{\sqrt{\mu}}\Big).$$

Then integrating the first inequality of (3.9) on $[t-t_b,\ t]$, one has

$$\frac{F_\varepsilon^-}{\sqrt{\mu}}(t,x,v) \leq \frac{F_\varepsilon^-}{\sqrt{\mu}}(t-t_b, x_b, v_b) = 0.$$

This gives $F_\varepsilon^- \leq 0$. While $F_\varepsilon^- \geq 0$ by the definition of F_ε^-, we have $F_\varepsilon^- = 0$. Then $F_\varepsilon = F_\varepsilon^+$ and F_ε^+ solves the problem (1.1)–(1.2). The proof is completed. $\square$

With the help of Lemma 3.1, to prove existence and positivity of a solution to (1.1)–(1.2), it suffices to show existence of a solution to (3.1)–(3.2).

For this, define

$$\bar{f}_\varepsilon := \begin{cases} f_\varepsilon, & \text{if } \mu + \varepsilon\sqrt{\mu}f_\varepsilon \geq 0, \\ -\varepsilon^{-1}\sqrt{\mu}, & \text{if } \mu + \varepsilon\sqrt{\mu}f_\varepsilon < 0, \end{cases} \tag{3.10}$$

and

$$\widetilde{f}_\varepsilon := \bar{f}_\varepsilon - f_\varepsilon. \tag{3.11}$$

For fixed $(x,v)\in\Omega\times\mathbb{R}^3$, if $F_\varepsilon = \mu + \varepsilon\sqrt{\mu}f_\varepsilon \geq 0$ then

$$F_\varepsilon^+ = F_\varepsilon = \mu + \varepsilon\sqrt{\mu}f_\varepsilon = \mu + \varepsilon\sqrt{\mu}\bar{f}_\varepsilon.$$

If $F_\varepsilon = \mu + \varepsilon\sqrt{\mu}f < 0$ then

$$F_\varepsilon^+ = 0 = \mu + \varepsilon\sqrt{\mu}\bar{f}_\varepsilon.$$

Moreover,

$$F_\varepsilon^- = F_\varepsilon^+ - F_\varepsilon = \mu + \varepsilon\sqrt{\mu}\bar{f}_\varepsilon - (\mu + \varepsilon\sqrt{\mu}f_\varepsilon) = \varepsilon\sqrt{\mu}\widetilde{f}_\varepsilon.$$

Hence, we have

$$F_\varepsilon^+ = \mu + \varepsilon\sqrt{\mu}\bar{f}_\varepsilon, \tag{3.12}$$

$$F_\varepsilon^- = \varepsilon\sqrt{\mu}\widetilde{f}_\varepsilon. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.1), we conclude

$$v\cdot\nabla_x f_\varepsilon+\frac{1}{\sqrt{\mu}}\varepsilon^2\Phi\cdot\nabla_v(\sqrt{\mu}f_\varepsilon)+\varepsilon^{-1}Lf_\varepsilon=\Gamma(\bar{f}_\varepsilon,\bar{f}_\varepsilon)+\varepsilon\sqrt{\mu}(\Phi\cdot v)+\frac{1}{2}\varepsilon^2(\Phi\cdot v)\widetilde{f}_\varepsilon, \tag{3.14}$$

$$f_\varepsilon|_{\gamma_-}=h_\varepsilon, \tag{3.15}$$

where h_ε has been given in (1.12).

We need the following lemma on the relations among $\bar{f}_\varepsilon$, $\widetilde{f}_\varepsilon$ and f_ε.

Lemma 3.2. *Let Ω be a bounded domain in $\mathbb{R}^3$. Then the following inequalities hold:*

$$|\bar{f}_\varepsilon|\le|f_\varepsilon|, \tag{3.16}$$

$$|\widetilde{f}_\varepsilon|\le\mathbf{1}_{\{\mu+\varepsilon\sqrt{\mu}f_\varepsilon<0\}}2|f|, \tag{3.17}$$

$$|\bar{f}^1_\varepsilon-\bar{f}^2_\varepsilon|\le|f^1_\varepsilon-f^2_\varepsilon|, \tag{3.18}$$

$$|\widetilde{f}^1_\varepsilon-\widetilde{f}^2_\varepsilon|\le\big(\mathbf{1}_{\{\mu+\varepsilon\sqrt{\mu}f^1_\varepsilon<0\}}+\mathbf{1}_{\{\mu+\varepsilon\sqrt{\mu}f^2_\varepsilon<0\}}\big)2|f^1_\varepsilon-f^2_\varepsilon|, \tag{3.19}$$

where $\bar{f}^i_\varepsilon$ and $\widetilde{f}^i_\varepsilon$ are defined in (3.10) and (3.11), respectively.

Proof. Firstly, if $F_\varepsilon=\mu+\varepsilon\sqrt{\mu}f_\varepsilon\ge0$ then $\bar{f}_\varepsilon=f_\varepsilon$. If $F_\varepsilon=\mu+\varepsilon\sqrt{\mu}f_\varepsilon<0$ then $\bar{f}_\varepsilon=-\varepsilon^{-1}\sqrt{\mu}<0$. Hence

$$\varepsilon|\bar{f}_\varepsilon|=-\varepsilon\bar{f}_\varepsilon=\sqrt{\mu}<-\varepsilon f_\varepsilon=\varepsilon|f_\varepsilon|,$$

which indicates (3.16).

Next, it follows from $F_\varepsilon=\mu+\varepsilon\sqrt{\mu}f_\varepsilon\ge0$ that

$$f_\varepsilon=\bar{f}_\varepsilon,\qquad|\widetilde{f}_\varepsilon|=|\bar{f}_\varepsilon-f_\varepsilon|=0.$$

In the case $F_\varepsilon=\mu+\varepsilon\sqrt{\mu}f_\varepsilon<0$, thank to the triangle inequality and (3.16), we have

$$|\widetilde{f}_\varepsilon|=|\bar{f}_\varepsilon-f_\varepsilon|\le(|\bar{f}_\varepsilon|+|f_\varepsilon|)\mathbf{1}_{\{\mu+\varepsilon\sqrt{\mu}f_\varepsilon<0\}}\le2|f_\varepsilon|\mathbf{1}_{\{\mu+\varepsilon\sqrt{\mu}f_\varepsilon<0\}},$$

which proves (3.17). Finally, (3.18) and (3.19) can be proved similarly as that of Lemma 6.3 in [11]. □

We are now in the position to present existence and uniqueness of the solution to (3.14)–(3.15).

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ with $\partial\Omega\in C^3$. Assume that $\Phi(x)\in C^1(\Omega)$ and $\|\Phi\|_2\ll1$. Then the problem (3.14)–(3.15) has a unique solution f_ε.*

Proof. We define a sequence $\{f^n_\varepsilon\}_{n=1}^\infty$, where f^{n+1}_ε is the solution to linear problem

$$v\cdot\nabla_x f^{n+1}_\varepsilon+\frac{1}{\sqrt{\mu}}\varepsilon^2\Phi\cdot\nabla_v(\sqrt{\mu}f^{n+1}_\varepsilon)+\varepsilon^{-1}Lf^{n+1}_\varepsilon$$
$$=\Gamma(\bar{f}^n_\varepsilon,\bar{f}^n_\varepsilon)+\varepsilon\sqrt{\mu}(\Phi\cdot v)+\frac{1}{2}\varepsilon^2(\Phi\cdot v)\widetilde{f}^n_\varepsilon\quad\text{in }\ \Omega\times\mathbb{R}^3, \tag{3.20}$$

$$f^{n+1}_\varepsilon=h_\varepsilon\quad\text{on }\ \gamma_-, \tag{3.21}$$

with $f^0:=0$.

Define

$$\Xi_n = \sup_{0\le j\le n} |||f_\varepsilon^j|||\,.$$

For $0<\eta_0\ll 1$, let $\Xi_n^2<\eta_0$. We claim that

$$|||\, f_\varepsilon^{n+1}|||^2 < \eta_0 \quad \text{ for all } n\ge 1. \tag{3.22}$$

Indeed, (3.22) can be proved by applying the linear theory given in Section 2. For this, let

$$g := \Gamma(\bar{f_\varepsilon^n}, \bar{f_\varepsilon^n}) + \varepsilon\sqrt{\mu}(\Phi\cdot v) + \frac{1}{2}\varepsilon^2(\Phi\cdot v)\widetilde{f_\varepsilon^n}.$$

From (3.16) and Lemma 2.13 in [10], we have

$$\|\nu^{-\frac12}\Gamma(\bar{f_\varepsilon^n}, \bar{f_\varepsilon^n})\|_2 \lesssim \varepsilon^{\frac12}\big[\varepsilon^{\frac12}\|w\bar{f_\varepsilon^n}\|_\infty\big(\varepsilon^{-1}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})\bar{f_\varepsilon^n}\|_2\big)\big] + \|\mathbf{P}\bar{f_\varepsilon^n}\|_3\|\mathbf{P}\bar{f_\varepsilon^n}\|_6 \lesssim |||f_\varepsilon^n\,|||^2\,.$$

Using (3.17), there holds

$$\begin{aligned}\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})\frac12\varepsilon^2(\Phi\cdot v)\widetilde{f_\varepsilon^n}\|_2 &\lesssim \|\varepsilon^2(\Phi\cdot v)\widetilde{f_\varepsilon^n}\|_2 \lesssim \|\varepsilon^2(\Phi\cdot v)|f_\varepsilon^n|\|_2\\ &\lesssim \varepsilon^2\|\Phi\|_\infty[\|\mathbf{P}f_\varepsilon^n\|_2 + \|(\mathbf{I}-\mathbf{P})f_\varepsilon^n\|_\nu]\\ &\lesssim \varepsilon^2\,|||\, f_\varepsilon^n\,|||\,.\end{aligned}$$

Based on the above estimates, we find

$$\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2 \lesssim |||f_\varepsilon^n\,|||^4 + \varepsilon^4\,|||\, f_\varepsilon^n\,|||^2\,. \tag{3.23}$$

By Lemma 2.5 in [18], we conclude

$$\|\nu^{-1}w\Gamma(\bar{f_\varepsilon^n}, \bar{f_\varepsilon^n})\|_\infty \lesssim \|\nu^{-1}w\Gamma(f_\varepsilon^n, f_\varepsilon^n)\|_\infty \lesssim \|wf_\varepsilon^n\|_\infty^2 \lesssim \varepsilon^{-1}\,|||\, f_\varepsilon^n\,|||^2\,.$$

Due to (3.17), we get

$$\varepsilon^2\|\nu^{-1}w(\Phi\cdot v)\widetilde{f_\varepsilon^n}\|_\infty \lesssim \varepsilon^2\|\Phi\|_\infty\|wf_\varepsilon^n\|_\infty \lesssim \varepsilon^{\frac32}\,|||\, f_\varepsilon^n\,|||\,.$$

Noticing that

$$\varepsilon\|\nu^{-1}w\sqrt{\mu}(\Phi\cdot v)\|_\infty \lesssim \varepsilon\|\Phi\|_\infty,$$

we deduce

$$\varepsilon^3\|\langle v\rangle^{-1}wg\|_\infty^2 \lesssim \varepsilon\,|||\, f_\varepsilon^n\,|||^4 + \varepsilon^5\|\Phi\|_\infty^2 + \varepsilon^6\,|||\, f_\varepsilon^n\,|||^2\,. \tag{3.24}$$

Since $\mathbf{P}g = \varepsilon\Phi\cdot v\sqrt{\mu} + \frac12\varepsilon^2\mathbf{P}[(\Phi\cdot v)\widetilde{f_\varepsilon^n}]$ and

$$\|\varepsilon^2\mathbf{P}[(\Phi\cdot v)\widetilde{f_\varepsilon^n}]\|_2^2 \lesssim \varepsilon^4\|\Phi\|_\infty^2\|f_\varepsilon^n\|_2^2 \lesssim \varepsilon^4\,|||\, f_\varepsilon^n|||^2,$$

we conclude

$$\varepsilon^{-2}\|\mathbf{P}g\|_2^2 \lesssim \varepsilon^2\,|||\, f_\varepsilon^n\,|||^2 + \|\Phi\|_2^2. \tag{3.25}$$

From the boundary term (1.12), there holds

$$\varepsilon^{-1}|h_\varepsilon|_{2,-}^2 \lesssim \varepsilon, \tag{3.26}$$

$$\varepsilon|wh_\varepsilon|_{\infty,-}^2 \lesssim \varepsilon^3. \tag{3.27}$$

Substituting (3.23)–(3.27) into (2.19), we conclude

$$|||f_\varepsilon^{n+1}|||^2 \lesssim \Xi_n^4 + \varepsilon^2\Xi_n^2 + \varepsilon^5\|\Phi\|_\infty^2 + \|\Phi\|_2^2 + \varepsilon \lesssim \eta_0,$$

which proves the claim(3.22).

Taking difference between (3.20) for the $(n+1)$-th step and n-th step, we have

$$\begin{aligned}
&v\cdot\nabla_x(f_\varepsilon^{n+1}-f_\varepsilon^n) + \frac{1}{\sqrt{\mu}}\varepsilon^2\Phi\cdot\nabla_v(\sqrt{\mu}(f_\varepsilon^{n+1}-f_\varepsilon^n)) + \varepsilon^{-1}L(f_\varepsilon^{n+1}-f_\varepsilon^n)\\
&\qquad\qquad = \Gamma(\bar{f}_\varepsilon^n,\bar{f}_\varepsilon^n) - \Gamma(\bar{f}_\varepsilon^{n-1},\bar{f}_\varepsilon^{n-1}) + \frac{1}{2}\varepsilon^2(\Phi\cdot v)(\widetilde{f}_\varepsilon^n - \widetilde{f}_\varepsilon^{n-1}) \quad \text{in } \Omega\times\mathbb{R}^3,\\
&f_\varepsilon^{n+1}-f_\varepsilon^n = 0 \quad \text{on } \gamma_-.
\end{aligned}$$

Repeating the same procedure as above for $f_\varepsilon^{n+1}-f_\varepsilon^n$, we can get

$$|||f_\varepsilon^{n+1}-f_\varepsilon^n||| \lesssim (\Xi_n + \varepsilon^2 + \varepsilon^{\frac{1}{2}}\Xi_n + \varepsilon^3 + \varepsilon)\,|||\, f_\varepsilon^n - f_\varepsilon^{n-1}\,|||\,. \tag{3.28}$$

It follows that there exists $0<\lambda<1$ such that

$$|||f_\varepsilon^{n+1}-f_\varepsilon^n||| \le \lambda\,|||\, f_\varepsilon^n - f_\varepsilon^{n-1}\,|||\,.$$

This means that $\{f_\varepsilon^n\}$ is a Cauchy sequence in $L^\infty\cap L^2$, for any given $\varepsilon>0$. Therefore, $\{f_\varepsilon^n\}$ converges strongly to f_ε in the norm of $|||\cdot|||$ and f_ε is the unique solution of (3.14)–(3.15). The proof is completed. □

Remark 3.4. *By Theorem 3.3 we know that $F_\varepsilon = \mu + \varepsilon\sqrt{\mu}f_\varepsilon$ is the unique solution of (3.1)–(3.2). This, combined with Lemma 3.1, implies that $F_\varepsilon = \mu+\varepsilon\sqrt{\mu}f_\varepsilon$ is the unique positive solution to the Boltzmann equation (1.1)–(1.2).*

Finally, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. In fact, by virtue of Remark 3.4, existence and positivity of a unique solution F_ε to (1.1)–(1.2) has been shown. Thus, it remains to check the hydrodynamic limit equations together with the boundary conditions.

It follows from (1.13) that $\|(\mathbf{I}-\mathbf{P})f_\varepsilon\|_\nu \to 0$ as $\varepsilon\to 0$. Moreover, $\mathbf{P}f_\varepsilon$ is bounded in $L^6_{x,v}$ and $\langle v\rangle^{-\frac{1}{2}}\Gamma(f_\varepsilon,f_\varepsilon)$ is bounded in $L^2_{x,v}$. Therefore, with the aid of (1.10), we find

$$v\cdot\nabla_x(\langle v\rangle^{-\frac{1}{2}}f_\varepsilon) + \varepsilon^2\langle v\rangle^{-\frac{1}{2}}\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v[\sqrt{\mu}f_\varepsilon] \in L^2(\Omega\times\mathbb{R}^3). \tag{3.29}$$

Since f_ε is bounded in L^2, passing to the weak limit as $\varepsilon\to 0$, up to a subsequence, we see that $f_\varepsilon \to f_*$ weakly and $\varepsilon^2\langle v\rangle^{-\frac{1}{2}}\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v(\sqrt{\mu}f_\varepsilon)\to 0$ in the sense of distribution. Thus

$$v\cdot\nabla_x(\langle v\rangle^{-\frac{1}{2}}f_\varepsilon) + \varepsilon^2\langle v\rangle^{-\frac{1}{2}}\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v(\sqrt{\mu}f_\varepsilon) \to v\cdot\nabla_x(\langle v\rangle^{-\frac{1}{2}}f_*) \tag{3.30}$$

in the sense of distribution.

Combining (3.29), (3.30) and the uniqueness of the distribution limit, we get

$$f_* = \mathbf{P}f_*, \quad v\cdot\nabla_x(f_*\langle v\rangle^{-\frac{1}{2}}) \in L^2_{x,v}, \quad \|f_*\|_{L^6_{x,v}} \ll 1.$$

It follows that

$$f_* = [\rho + u\cdot v + \theta(|v|^2-3)/2]\sqrt{\mu} \tag{3.31}$$

and $\rho, u, \theta \in H^1(\Omega)$. The limiting fluid equations can be deduced similarly as that in [3]. In fact, applying $\mathbf{P}$ to (1.10) and taking the weak limit, we obtain

$$\mathbf{P}(v\cdot\nabla_x f_*) = 0,$$

which is equivalent to

$$\nabla_x(\rho+\theta) = 0, \ \ \nabla_x\cdot u = 0.$$

Then multipling (1.10) by $\varepsilon^{-1}v\sqrt{\mu}$, integrating on velocity and taking the weak limit, we obtain

$$\begin{aligned}
\Phi &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\nabla_x\cdot\langle\sqrt{\mu}v\otimes v f_\varepsilon\rangle \\
&= \lim_{\varepsilon\to 0}\varepsilon^{-1}\nabla_x\cdot\langle L^{-1}[(v\otimes v - \frac{|v|^2}{3}\mathbb{I})\sqrt{\mu}], Lf_\varepsilon\rangle + \lim_{\varepsilon\to 0}\varepsilon^{-1}\nabla_x\langle\frac{|v|^2}{3}\sqrt{\mu}, f_\varepsilon\rangle \\
&= \nabla_x\cdot\langle L^{-1}[(v\otimes v - \frac{|v|^2}{3}\mathbb{I})\sqrt{\mu}], \Gamma(f_*,f_*) - v\cdot\nabla_x f_*\rangle + \nabla_x p \\
&= u\cdot\nabla_x u - \eta\triangle u + \nabla_x p,
\end{aligned}$$

where

$$p := \lim_{\varepsilon\to 0}\varepsilon^{-1}\langle\frac{|v|^2}{3}\sqrt{\mu}, f_\varepsilon\rangle,$$

and $\mathbb{I}$ is the unit matrix. Hence u is a weak solution to (1.6).

Similarly, we multiply (1.10) by $\varepsilon^{-1}\frac{|v|^2-5}{2}\sqrt{\mu}$, integrate and take the weak limit,

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\langle\frac{|v|^2-5}{2}\sqrt{\mu}, v\cdot\nabla_x f_\varepsilon\rangle \\
&= \lim_{\varepsilon\to 0}\varepsilon^{-1}\nabla_x\cdot\langle L^{-1}[v\frac{|v|^2-5}{2}\sqrt{\mu}], Lf_\varepsilon\rangle \\
&= \nabla_x\cdot\langle L^{-1}[v\frac{|v|^2-5}{2}\sqrt{\mu}], \Gamma(f_*,f_*) - v\cdot\nabla_x f_*\rangle \\
&= \frac{5}{2}\nabla_x\cdot(\kappa\nabla_x\theta - u\theta).
\end{aligned}$$

Thus θ is a weak solution to (1.7).

Next, we check the boundary conditions that (u,θ) satisfies. Taking $\phi(x,\cdot) \in C_0^\infty(\mathbb{R}^3)$, $\phi(\cdot,v) \in C^\infty(\bar{\Omega})$, we have

$$\begin{aligned}
\int_\gamma \nu^{-\frac{1}{2}} f_\varepsilon\phi \mathrm{d}\lambda :&= \iint_{\partial\Omega\times\mathbb{R}^3}\nu^{-\frac{1}{2}} f_\varepsilon\phi n(x)\cdot v\mathrm{d}v\mathrm{d}S_x \\
&= \iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x(\phi f_\varepsilon)\nu^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}v \\
&= \iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\phi f_\varepsilon\nu^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}v + \iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x f_\varepsilon\phi\nu^{-\frac{1}{2}}\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

Since

$$f_\varepsilon \to f_*, \quad v\cdot\nabla_x f_\varepsilon \nu^{-\frac{1}{2}} \to v\cdot\nabla_x f_* \nu^{-\frac{1}{2}}$$

weakly in L^2, we obtain

$$\int_\gamma \nu^{-\frac{1}{2}} f_\varepsilon \phi \mathrm{d}\lambda \to \int_\gamma \nu^{-\frac{1}{2}} f_* \phi \mathrm{d}\lambda.$$

That is to say, f_ε converges to f_* on γ in the sense of distribution, where f_* is given by (3.31).

By the uniform estimate (1.13), one has

$$\varepsilon^{-\frac{1}{2}} |f_\varepsilon|_{2,+}^2 \le C.$$

Combining the boundary (1.12) with the above result, we conclude that $f_\varepsilon \to 0$ strongly in $L^2_{\gamma_+}$ and $f_\varepsilon \to 0$ strongly in $L^2_{\gamma_-}$. In view of the uniqueness of the distribution limit, we deduce

$$f_*\big|_\gamma = \left[\rho\big|_{\partial\Omega} + v\cdot u\big|_{\partial\Omega} + \theta\big|_{\partial\Omega}(|v|^2-3)/2\right]\sqrt{\mu} = 0,$$

for every $v\in\mathbb{R}^3$. This implies

$$u\big|_{\partial\Omega} = \theta\big|_{\partial\Omega} = 0,$$

that is, (1.8) holds.

Finally, observe that $\|u\|_6$ and $\|\theta\|_6$ are small, inheriting from the uniform estimate (1.13) and the weak limit. This, combined with standard L^2 energy estimate, leads to the uniqueness of weak solution to INSF (1.6)–(1.8). This further indicates that all the weak limit points of f_ε must coincide and f_ε converges to f_* weakly in L^2. This completes the proof of Theorem 1.1. □

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