Inverse problem of determining the coefficient and kernel in an integro - differential equation of parabolic type

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Abstract. This article is concerned with the study of the unique solvability of inverse boundary value problem for integro-differential heat equation. To study the solvability of the inverse problem, we first reduce the considered problem to an auxiliary system with trivial data and prove its equivalence (in a certain sense) to the original problem. Then using the Banach fixed point principle, the existence and uniqueness of a solution to this system is shown.

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1. Introduction

Mathematical physics usually studies well-posed problems, that is, the problems that have solutions which are unique and stable to small changes in the data in suitable functional spaces. Such problems, as a rule, are called direct problems. In this case, it is assumed that a differential equation is specified, as well as certain initial and boundary conditions. However, in applications there are interesting problems, where the differential (or integro-differential) equation is only partially specified, namely, some functions that are part of the differential equation (either in the right-hand side, integrant in integro-differential equations or the initial and boundary conditions) remain unknown. The problems, in which these unknowns are to be determined on some information about the solutions of direct problems for differential equations, are called inverse problems.

Inverse problems are widely used to solve practical problems in many branches of science and engineering. The study of inverse kernel determination problems for hyperbolic and parabolic integro-differential equations with an integral term of convolution type is very interesting from both the practical and theoretical viewpoint. Such equations in the case of a parabolic equation arise in problems of heat propagation in media whose state at a given time instant depends on their state at all previous time instants.

Inverse problems in this direction can be found in papers [1]-[12] and the problems studied in them are close to the problem considered in this article. In the above papers, the existence and uniqueness theorems were proved for the solution of the problem of finding the kernel for various overdetermination conditions.

In the present paper, we investigate the inverse problem of the simultaneous determination of two unknowns: the coefficient $a(t)$ and the heat relaxation function $k(t)$ in the integro-differential heat equation. For this, two simple observations are given at two different points.

2. Formulation of problem and auxiliary constractions

Consider the problem of determining of functions $u(x,t)$, $a(t), k(t)$, from the following equations:

$$u_t - u_{xx} + a(t)u = \int_0^t k(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_T, \tag{1.1}$$

$$u|_{t=0} = \varphi(x), \quad x \in [0,1], \tag{1.2}$$

$$u|_{x=0} = u|_{x=1} = 0, \quad t \in [0, T], \tag{1.3}$$

$$u(x_i, t) = h_i(t), x_i \in (0,1), \varphi(0) = \varphi(1) = 0, \quad \varphi(x_i) = h_i(0), i = 0, 1, \tag{1.4}$$

where $D_T = \{(x,t)|x \in (0,1),\ 0 < t \leq T\}$, $T > 0$ are arbitrary fixed numbers.

In equation (1.1) on the left side there is a heat conduction operator acting on the function $u(x,t)$, and on the right side there is a convolution type integral. In fact, if a and the kernel k of the integral are known in equation (1.1), then the problem of finding the function u from equation (1.1), based on conditions (1.2),(1.3), is called the direct problem. Note that direct problem in this case is initial-boundary problem for the equation (1.1).

Since after finding a and the kernel k the solution of the direct problem becomes known, it is convenient to call the solution of the inverse problem (1.1)-(1.4) the problem of finding functions u, k, a.

In the inverse problem, it is required to find a and the kernel k of the integral in equation (1.1) if the additional conditions (1.4) are known with respect to the solution of the direct problem. The functions φ in condition (1.2) and the function h_i in (1.4) are called the data of the direct and inverse problems, respectively. The last conditions in (1.4) are matching conditions for given functions.

The following assertation is true:

Lemma-2.1. *Problem (1.1)–(1.4) is equivalent to the following auxiliary problem of determining functions $u(x,t), k(t), a(t)$:*

$$\omega_t - \omega_{xx} + a'(t)\varphi''(x) + a'(t)\int_0^t \omega(x,\tau)d\tau + a(t)\omega(x,t) = k(t)\varphi''(x) + \int_0^t k(\tau)\omega(x,t-\tau)d\tau,\ (x,t) \in D_T, \tag{1.5}$$

$$\omega|_{t=0} = \varphi^{(IV)}(x) - a(0)\varphi'(x),\ x \in (0,1), \tag{1.6}$$

$$\omega|_{x=0} = 0, \omega|_{x=1} = 0, \tag{1.7}$$

$$\omega|_{x=x_i} = h_i''(t) + a'(t)h_i(t) + a(t)h_i'(t) - k(t)\varphi(x_i) - \int_0^t k(\tau)h'(t-\tau)d\tau. \tag{1.8}$$

where $\omega(x,t) = u_{txx}(x,t)$,

$$a(0) = \frac{\varphi''(x_0) - h_0'(0)}{\varphi(x_0)}, \tag{1.9}$$

$\varphi^{(IV)}$ is the fourth derivative of function $\varphi(x)$.

Proof. By setting $\vartheta(x,t) = u_t(x,t)$and differentiating in t , we reduce (1.1)-(1.4) to the problem

$$\vartheta_t - \vartheta_{xx} + a'(t)\int_0^t \vartheta(x,\tau)d\tau + a'(t)\varphi(x) + a(t)\vartheta(x,t) = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau,\ (x,t) \in D_T, \tag{1.10}$$

$$\vartheta|_{t=0} = \varphi''(x) - a(0)\varphi(x),\ x \in (0,1), \tag{1.11}$$

$$\vartheta|_{x=0} = 0, \vartheta|_{x=1} = 0, \tag{1.12}$$

$$\vartheta(x_0, t) = h_0'(t), \quad \vartheta(x_1, t) = h_1'(t). \tag{1.13}$$

From condition (1.11) and (1.13),requiring the matching condition at the points $(0, x_0)$ and $(0, x_1)$ we obtain

$$a(0) = \frac{\varphi''(x_0) - h_0'(0)}{\varphi(x_0)} = \frac{\varphi''(x_1) - h_1'(0)}{\varphi(x_1)}.$$

Hence it follows that if (u,k,a) is solution of problem (1.1)-(1.4), then (1.10)-(1.13) has a solution (ϑ,k,a) with the same k,a as well. Let us prove the converse. Let (ϑ,k,a) satisfy relations(1.10)-(1.13); then

$$u(x,t)=\int_0^t \vartheta(x,\tau)d\tau+\varphi(x).$$

Let us show that relation (1.1) holds. It follows from (1.10)-(1.12) that

$$u_t-u_{xx}+a(t)u-\int_0^t k(t-\tau)u(x,\tau)d\tau=$$

$$=\vartheta(x,t)-\int_0^t \vartheta_{xx}(x,\tau)d\tau-\varphi''(x)+a(t)\int_0^t \vartheta(x,\tau)d\tau+a(t)\varphi(x)-$$
$$-\int_0^t k(\tau)\int_0^{t-\tau}\vartheta(x,\alpha)d\alpha d\tau-\int_0^t k(\tau)\varphi(x)d\tau=$$
$$\int_0^t \Big[\vartheta_\tau(x,\tau)-\vartheta_{xx}(x,\tau)+a'(\tau)\int_0^\tau \vartheta(x,\alpha)d\alpha+a(\tau)\vartheta(x,\tau)+a'(\tau)\varphi(x)-k(\tau)\varphi(x)-$$
$$-\int_0^\tau k(x,\alpha)\vartheta(x,\tau-\alpha)d\alpha\Big]d\tau=0$$

This completes the proof of the equivalence of problems (1.1)-(1.4) and (1.10)-(1.13).

Now consider the second auxiliary problem. It is obtained from problem (1.10)-(1.13) for the function $p(x,t)=\vartheta_x(x,t)$,

$$p_t-p_{xx}+a'(t)\int_0^t p(x,\tau)d\tau+a'(t)\varphi'(x)+a(t)p(x,t)=k(t)\varphi'(x)+\int_0^t k(\tau)p(x,t-\tau)d\tau, \quad (1.14)$$

$$p|_{t=0}=\varphi'''(x)-a(0)\varphi'(x); \quad (1.15)$$

$$p_x|_{x=0}=0, \quad p_x|_{x=1}=0; \quad (1.16)$$

$$p_x|_{x=x_i}=h_i''(t)+a'(t)h_i(t)+a(t)h_i'(t)-k(t)\varphi(x_i)-\int_0^t k(\tau)h'(t-\tau)d\tau. \quad (1.17)$$

Therefore, if problem (1.10)-(1.13) has a solution (ϑ,k,a), then problem (1.14)-(1.17) has a solution (p,k,a) with the same k,a; moreover, $p(x,t)=\vartheta_x(x,t)$. Conversely, let (p,k,a) satisfy relations (1.14)-(1.17).

Hence it follows that

$$\vartheta(x,0)=\int_0^x p(y,0)dy=\int_0^t(\varphi'''(y)-a(0)\varphi'(y))dy=\varphi''(x)-a(0)\varphi(x);$$

i.e., condition (1.11) is satisfied. It remains to show that equation (1.10) holds. It follows from relations (1.14)-(1.17) that

$$\vartheta_t-\vartheta_{xx}+a'(t)\int_0^t \vartheta(x,\tau)d\tau+a'(t)\varphi(x)+a(t)\vartheta(x,t)-k(t)\varphi(x)-\int_0^t k(\tau)\vartheta(x,t-\tau)d\tau=$$

$$=\int_0^x \Big[p_t(y,t)-p_{yy}(y,t)+a'(t)\int_0^t p(y,\tau)d\tau+a'(t)\varphi'(y)+a(t)p(y,t)-$$

$$-k(t)\varphi'(y) - \int_0^t k(\tau)p(y,t-\tau)d\tau\Big]dy + a'(t)\varphi(0) - k(t)\varphi(0) = 0.$$

We have thereby proved the equivalence of problems (1.10)-(1.13) and (1.14)-(1.17). In similar way, one can show that problem (1.14)-(1.17) is equivalent to problem (1.5)-(1.8) for the function $\omega = p_x$. This implies the equivalence of problems (1.1)-(1.4) and (1.5)-(1.8). The lemma is proved.

3. Formulation of main result and its proof

In this section existence and uniqueness for the problem (1.5)-(1.8) is proved using the contraction mapping principle. The idea is to write the integral equations for unknown functions $\omega(x,t), k(t), a(t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator for sufficiently small T. The existence and uniqueness then follow immediately.

From problem (1.5)-(1.7), we obtain

$$\omega(x,t) = \omega_0(x,t) + \int_0^t\int_0^1 G(x,\xi,t-\tau)k(\tau)\varphi(\xi)d\xi d\tau - \int_0^t\int_0^1 G(x,\xi,t-\tau)a(\tau)\omega(\xi,\tau)d\xi d\tau -$$

$$-\int_0^t\int_0^1 G(x,\xi,t-\tau)a'(\tau)\varphi(\xi)d\xi d\tau - \int_0^t\int_0^1 G(x,\xi,t-\tau)a'(\tau)\int_0^\tau \omega(\xi,\alpha)d\alpha d\xi d\tau +$$

$$+\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha d\xi d\tau, \tag{2.1}$$

where

$$\omega_0(x,t) = \int_0^1 G(x,\xi,t)(\varphi^{(IV)}(\xi) - a(0)\varphi'(\xi))d\xi,$$

$$G(t-\tau,x,\xi) = 2\sum_{n=1}^\infty e^{-(\pi n)^2(t-\tau)} sin\,(\pi n\xi)\, sin\,(\pi nx)$$

is the Green function of the initial-boundary problem for one-dimensional heat equation. By setting $x = x_0, x = x_1$ in integral equation (2.1) and by taking into account condition (1.8), for the functions $k(t), a'(t))$, we obtain the integral equations

$$k(t) = \frac{1}{\Delta}\Big[h_1(t)\Big(\omega_0(x_0,t) - h_0''(t)\Big) - h_0(t)(\omega_0(x_1,t) - h_1''(t))\Big] +$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1 \Big(h_1(t)G(x_0,\xi,t-\tau) - h_0(t)G(x_1,\xi,t-\tau)\Big)\Big[k(\tau)\varphi(\xi) - a(\tau)\omega(\xi,\tau) - a'(\tau)\varphi''(\xi)\Big]d\xi d\tau +$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1 \Big(h_1(t)G(x_0,\xi,t-\tau) - h_0(t)G(x_1,\xi,t-\tau)\Big)\Big[a'(\tau)\int_0^\tau \omega(\xi,\alpha)d\alpha - \int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau -$$

$$-\frac{a(t)}{\Delta}\Big(h_1(t)h_0'(t) - h_0(t)h_1'(t)\Big) + \frac{1}{\Delta}\int_0^t k(\tau)\Big(h_1(t)h_0'(t-\tau) - h_0(t)h_1'(t-\tau)\Big)d\tau, \tag{2.2}$$

where

$$\Delta = \varphi(x_1)h_0(t) - \varphi(x_0)h_1(t).$$

$$a'(t) = \frac{1}{\Delta}\Big[\varphi(x_1)(\omega_0(x_0,t) - h_0''(t)) - \varphi(x_0)(\omega_0(x_1,t) - h_1''(t))\Big] +$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(\varphi(x_1)G(x_0,\xi,t-\tau)-\varphi(x_0)G(x_1,\xi,t-\tau)\Big)\Big[k(\tau)\varphi(\xi)-a(\tau)\omega(\xi,\tau)-a'(\tau)\varphi''(\xi)\Big]d\xi d\tau+$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(\varphi(x_1)G(x_0,\xi,t-\tau)-\varphi(x_0)G(x_1,\xi,t-\tau)\Big)\Big[a'(\tau)\int_0^\tau\omega(\xi,\alpha)d\alpha-\int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau-$$

$$-\frac{a(t)}{\Delta}\Big(\varphi(x_1)h_0'(t)-\varphi(x_0)h_1'(t)\Big)+\frac{1}{\Delta}\int_0^t k(\tau)\Big(\varphi(x_1)h_0'(t-\tau)-\varphi(x_0)h_1'(t-\tau)\Big)d\tau. \quad (2.3)$$

$$a(t)=a(0)+\int_0^t a'(\tau)d\tau. \quad (2.4)$$

Equations (2.1)-(2.4) form a complete system of integral equations for the unknown functions $\omega(x,t),k(t),a'(t),a(t)$. We represent this system in the form of the operator equation

$$\psi=A\psi, \quad (2.5)$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4):=(\omega(x,t),k(t)+a(t)\beta(t),a'(t)+a(t)\gamma(t),a(t))$, $\Big(\beta(t)=\frac{1}{\Delta}\Big(h_1(t)h_0'(t)-h_0(t)h_1'(t)\Big),\gamma(t)=\frac{1}{\Delta}\Big(\varphi(x_1)h_0'(t)-\varphi(x_0)h_1'(t)\Big)\Big)$ is the vector-function and unknown functions are represented by functions $\psi_1,\psi_2,\psi_3,\psi_4$ as follows:

$$\omega(x,t)=\psi_1(x,t),\quad k(t)=\psi_2(t)-\psi_4(t)\beta(t),$$

$$a'(t)=\psi_3(t)-\psi_4(t)\gamma(t),\quad a(t)=\psi_4(t).$$

$A=(A_1,\ \ A_2,\ \ A_3,\ \ A_4)$ is defined by the right sides of equations (2.1)-(2.4):

$$(A\psi)_1(x,t)=\omega_0(x,t)+\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)d\xi d\tau-$$

$$-\int_0^t\int_0^1 G(x,\xi,t-\tau)\psi_4(\tau)\psi_1(\xi,\tau)d\xi d\tau-\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau-$$

$$-\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\alpha)d\alpha d\xi d\tau+$$

$$+\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau\Big(\psi_2(\alpha)-\psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha d\xi d\tau,$$

$$(A\psi)_2(t)=\frac{1}{\Delta}\Big[h_1(t)(\omega_0(x_0,t)-h_0''(t))-h_0(t)(\omega_0(x_1,t)-h_1''(t))\Big]+$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(h_1(t)G(x_0,\xi,t-\tau)-h_0(t)G(x_1,\xi,t-\tau)\Big)\times$$

$$\times\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi''(\xi)\Big]d\xi d\tau+$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(h_1(t)G(x_0,\xi,t-\tau)-h_0(t)G(x_1,\xi,t-\tau)\Big)\times$$

$$\times\Big[\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\alpha)d\alpha-\int_0^\tau\Big(\psi_2(\alpha)-\psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau+$$

$$+\frac{1}{\Delta}\int_0^t\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\Big(h_1(t)h_0'(t-\tau)-h_0(t)h_1'(t-\tau)\Big)d\tau,$$

$$(A\psi)_3(t)=\frac{1}{\Delta}\Big[\varphi(x_1)(\omega_0(x_0,t)-h_0''(t))-\varphi(x_0)(\omega_0(x_1,t)-h_1''(t))\Big]+$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(\varphi(x_1)G(x_0,\xi,t-\tau)-\varphi(x_0)G(x_1,\xi,t-\tau)\Big)\times$$

$$\times\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi''(\xi)\Big]d\xi d\tau+$$

$$+\frac{1}{\Delta}\int_0^t\int_0^1\Big(\varphi(x_1)G(x_0,\xi,t-\tau)-\varphi(x_0)G(x_1,\xi,t-\tau)\Big)\times$$

$$\times\Big[\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\alpha)d\alpha-\int_0^\tau\Big(\psi_2(\alpha)-\psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau+$$

$$+\frac{1}{\Delta}\int_0^t\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\Big(\varphi(x_1)h_0'(t-\tau)-\varphi(x_0)h_1'(t-\tau)\Big)d\tau,$$

$$(A\psi)_4(t)=a(0)+\int_0^t\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)d\tau.$$

Let $\psi_0:=(\psi_{01},\psi_{02},\psi_{03},\psi_{04})$, where

$$\psi_{01}=\omega_0(x,t),$$

$$\psi_{02}=\frac{1}{\Delta}\Big[h_1(t)(\omega_0(x_0,t)-h_0''(t))-h_0(t)(\omega_0(x_1,t)-h_1''(t))\Big],$$

$$\psi_{03}=\frac{1}{\Delta}\Big[\varphi(x_1)(\omega_0(x_0,t)-h_0''(t))-\varphi(x_0)(\omega_0(x_1,t)-h_1''(t))\Big],$$

$$\psi_{04}=a(0).$$

Theorem (existence and uniqueness). *If the conditions $\varphi(x)\in C^2(0,1)$, $h_i(t)\in C^2\big[0,T\big], i=0,1,$ $\varphi(0)=\varphi(1)=0,$ $\varphi(x_i)=h_i(0), i=0,1, \Delta\neq 0, \varphi(x_0)\neq 0, \varphi(x_1)\neq 0$ are met,then there exists sufficiently small number $T^*\in(0,T)$ that the solution to the system of integral equations (2.1)–(2.4) in the class of functions $\omega(x,t)\in C^{2,1}(D_T]$, $k(t)\in C(0,T]$, $a(t)\in C^1(0,T]$ exists is unique. Thus, there is the unique classical solution to the problem (1.1)-(1.4).*

Proof. Consider the functional space of vector functions $\psi\in C(D_T)$ with the norm given by the relation

$$\|\psi\|=\max\left\{\sup_{(x,t)\in\overline{D}_T}|\psi_1(x,t)|\,,\ \sup_{t\in[0,T]}|\psi_2(t)|\,,\ \sup_{t\in[0,T]}|\psi_3(t)|\,,\ \sup_{t\in[0,T]}|\psi_4(t)|\right\}=$$

$$=\max\Big\{\|\psi_1\|,\|\psi_2\|,\|\psi_3\|,\|\psi_4\|\Big\}.$$

In this space, by $B(\psi_0,\|\psi_0\|)$ we denote the ball with center ψ_0 and radius $\|\psi_0\|$, i.e. $B(\psi_0,\|\psi_0\|)=\{\psi:\|\psi-\psi_0\|\le\|\psi_0\|\}$. Obviously,

$$\|\psi\|\le 2\|\psi_0\|.$$

Let us show that A is a contraction operator in the ball $B(\psi_0, \|\psi_0\|)$ provided that T is sufficiently small number. For simplicity, we denote

$$h_0 := \max\left\{\sup_{t\in(0;T)}|h_i(t)|\,,\ \sup_{t\in(0,T)}|h_i'(t)|\,,\ \sup_{t\in(0,T)}|h_i''(t)|\right\},\, i=0,1,\quad \beta=\max_{t\in[0,T]}|\beta(t)|\,,\gamma=\max_{(t)\in[0,T]}|\gamma(t)|$$

$$\varphi_0 := \max\left\{\sup_{x\in(0,1)}|\varphi(x)|\,,\ \sup_{x\in(0,1)}|\varphi'(x)|\,,\ \sup_{x\in(0,1)}|\varphi''(x)|\right\}.$$

Let us verify the first condition of a fixed point argument. Let $\psi\in B$; then $\|\psi\|\le 2\|\psi_0\|$. In addition, for $(x,t)\in D_T$, we have estimates

$$\|(A\psi)_1-\psi_{01}\| = \sup_{(x,t)\in D_T}|(A\psi)_1-\psi_{01}| \le$$

$$\le \sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)d\xi d\tau\right| +$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\psi_4(\tau)\psi_1(\xi,\tau)d\xi d\tau\right| +$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau\right| +$$

$$\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau \psi_1(\xi,\alpha)d\alpha d\xi d\tau\right| +$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau\Big(\psi_2(\alpha)-\psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha d\xi d\tau\right| \le$$

$$\le 2T\|\psi_0\|\left[(2+\beta+\gamma)\varphi_0+\|\psi_0\|(1+2T+T\beta+T\gamma)\right].$$

$$\|(A\psi)_2-\psi_{02}\| = \sup_{t\in(0,T)}|((A\psi)_2-\psi_{02})| \le$$

$$\le \frac{4Th_0\|\psi_0\|}{\Delta}\left(\varphi_0(2+\beta+\gamma)+\|\psi_0\|(1+2T+T\gamma+T\beta)+h_0(1+\beta)\right);$$

$$\|(A\psi)_3-\psi_{03}\| = \sup_{t\in(0,T)}|((A\psi)_3-\psi_{03})| \le$$

$$\le \frac{4T\varphi_0\|\psi_0\|}{\Delta}\left(\varphi_0(2+\beta+\gamma)+\|\psi_0\|(1+2T+T\gamma+T\beta)+h_0(1+\beta)\right);$$

$$\|(A\psi)_4-\psi_{04}\| = \sup_{t\in(0,T)}|((A\psi)_4-\psi_{04})| \le 2\|\psi_0\|T(1+\gamma),$$

Denote $T_1=\min\{T_{11},T_{12},T_{13},T_{14}\}$, where $T_{1i}, i=\overline{1,4}$ are the positive roots of the following equations, respectively

$$2T\Big[(2+\beta+\gamma)\varphi_0+\|\psi_0\|(1+2T+T\beta+T\gamma)\Big]=1,$$

$$\frac{4Th_0}{\Delta}\Big(\varphi_0(2+\beta+\gamma)+\|\psi_0\|(1+2T+T\gamma+T\beta)+h_0(1+\beta)\Big)=1,$$
$$\frac{4T\varphi_0}{\Delta}\Big(\varphi_0(2+\beta+\gamma)+\|\psi_0\|(1+2T+T\gamma+T\beta)+h_0(1+\beta)\Big)=1,$$
$$2T(1+\gamma)=1.$$

If we choose T so that $T<T_1$, then $A\psi\in B(\psi_0,\|\psi_0\|)$.

We now check the second condition of a fixed point argument

$$\|(A\psi^1-A\psi^2)_1\|\leq\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big([\psi_2^1(\tau)-\psi_2^2(\tau)]-[\psi_4^1(\tau)-\psi_4^2(\tau)]\beta(\tau)\Big)\varphi(\xi)d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)[\psi_4^1(\tau)\psi_1^1(\xi,\tau)-\psi_4^2(\tau)\psi_1^2(\xi,\tau)]d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big([\psi_3^1(\tau)-\psi_3^2(\tau)]-[\psi_4^1(\tau)-\psi_4^2(\tau)]\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\alpha)\psi_3^1(\tau)-\psi_1^2(\xi,\alpha)\psi_3^2(\tau)]d\alpha d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\alpha)\psi_4^1(\tau)-\psi_1^2(\xi,\alpha)\psi_4^2(\tau)]\gamma(\tau)d\alpha d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\tau-\alpha)\psi_2^1(\alpha)-\psi_1^2(\xi,\tau-\alpha)\psi_2^2(\alpha)]d\alpha d\xi d\tau\right|+$$
$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\tau-\alpha)\psi_4^1(\alpha)-\psi_1^2(\xi,\tau-\alpha)\psi_4^2(\alpha)]\beta(\alpha)\Big)d\alpha d\xi d\tau\right|.$$

The integrand in the second integral can be estimated as follows:

$$\left\|\psi_4^1\psi_1^1-\psi_4^2\psi_1^2\right\|=\left\|(\psi_4^1-\psi_4^2)\psi_1^1+\psi_4^2(\psi_1^1-\psi_1^2\right\|\leq$$
$$\leq 2\left\|\psi^1-\psi^2\right\|\max\left(\left\|\psi_1^1\right\|,\left\|\psi_4^2\right\|\right)\leq 4\left\|\psi_0\right\|\left\|\psi^1-\psi^2\right\|.$$

Therefore,

$$\|(A\psi^1-A\psi^2)_1\|\leq\|\psi^1-\psi^2\|\left(2\|\psi_0\|(2+\beta+\gamma)T^2+((2+\beta+\gamma)\varphi_0+4\|\psi_0\|)T\right);$$

The next components can be estimated in a similar way,

$$\|(A\psi^1-A\psi^2)_2\|\leq\frac{2h_0}{\Delta}\left(((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|)\,T+2\|\psi_0\|(2+\beta+\gamma)T^2\right)\|\psi^1-\psi^2\|;$$
$$\|(A\psi^1-A\psi^2)_3\|\leq\frac{2\varphi_0}{\Delta}\left(((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|)\,T+2\|\psi_0\|(2+\beta+\gamma)T^2\right)\|\psi^1-\psi^2\|;$$
$$\|(A\psi^1-A\psi^2)_4\|\leq(1+\gamma)\,T\|\psi^1-\psi^2\|.$$

Denote $T_2=(T_{21},T_{22},T_{23},T_{24})$; where $T_{2i}, i=\overline{1,4}$ are the positive roots of the following equations, respectively

$$2\|\psi_0\|\Big(2+\beta+\gamma\Big)T^2+\Big((2+\beta+\gamma)\varphi_0+4\|\psi_0\|\Big)T=1,$$

$$\frac{2h_0}{\Delta}\Big(((2+\beta+\gamma)\varphi_0 + h_0(1+\beta) + 4\|\psi_0\|)\, T + 2\|\psi_0\|(2+\beta+\gamma)T^2\Big) = 1,$$
$$\frac{2\varphi_0}{\Delta}\Big(\Big((2+\beta+\gamma)\varphi_0 + h_0(1+\beta) + 4\|\psi_0\|\Big)T + 2\|\psi_0\|(2+\beta+\gamma)T^2\Big) = 1,$$
$$\Big(1+\gamma\Big)T = 1.$$

Therefore, if the number T^* is small enough to ensure that condition $T^* \in (0, \min(T_1, T_2))$ is satisfied, then A is contraction operator on B. Then, by the Banach principle, integral equations (2.1)-(2.4) has a unique solution in B. Theorem is proved.

By found function $\omega(x,t)$, the function $u(x,t)$ is found by virtue of the formula $u_{txx} = \vartheta$, from which follows

$$u(x,t) = \varphi(x) - \varphi(0) + x\varphi'(0) + \int_0^x (x-\xi) \int_0^t \omega(\xi,\tau)d\tau d\xi.$$

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