Memory kernel reconstruction problems in the integro–differential equation of rigid heat conductor

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Abstract. The inverse problems of determining the energy-temperature relation $\alpha(t)$ and the heat conduction relation $k(t)$ functions in the one-dimensional integro–differential heat equation are investigated. The direct problem is the initial-boundary problem for this equation. The integral terms have the time convolution form of unknown kernels and direct problem solution. As additional information for solving inverse problems, the solution of the direct problem for $x = x_0$ is given. At the beginning an auxiliary problem, which is equivalent to the original problem is introduced. Then the auxiliary problem is reduced to an equivalent closed system of Volterra-type integral equations with respect to unknown functions. Applying the method of contraction mappings to this system in the continuous class of functions with weighted norms, we prove the main result of the article, which is a global existence and uniqueness theorem of inverse problem solutions.

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1. Introduction and Setting up the Problem

Integro–differential equations arise in many fields of physics and applied mathematics for modeling the processes of heat transfer with finite propagation speed, systems with thermal memory, viscoelasticity problems and acoustic waves in composite media. In [1] Gurtin and Pipkin derived the integro-differential equation

$$u_{tt} = \triangle u(x,\tau) + \int\limits_0^t K'(t-\tau)\triangle u(x,\tau)d\tau + h(x,t), \tag{1.1}$$

describing propagation of heat in media with memory at a finite speed. Here Δ is the Laplace operator in the variables $x = (x_1, ..., x_n)$. Along with equation (1.1), in the literatures it is considered the equation

$$u_t(x,t) = \int\limits_0^t K(t-\tau)\triangle u(x,\tau)d\tau + g(x,t) \tag{1.2}$$

of the first order in the time variable t. Nowadays, equations (1.1) and (1.2) are referred to as the Gurtin - Pipkin equations. It can readily be seen that equation (1.1) is derived from (1.2) by differentiating with respect to variable t if we set $K(0) = 1$ and $h(x,t) = g_t(x,t)$.

In [2] Miller studied existence, uniqueness, and continuous dependence on parameters for solutions of the certain initial–boundary value problem for following system of integro–differential equations:

$$e(t,x) = e_0 + \alpha(0)\theta(t,x) + \int_0^t \alpha'(t-\tau)\theta(\tau,x)d\tau,$$

$$q(t,x) = -k(0)\theta_x(t,x) - \int_0^t k'(t-\tau)\theta_x(\tau,x)d\tau, \tag{1.3}$$

$$e_t(t,x) = -q_x(t,x) + r(t,x),$$

where $0 \le t < \infty$, $x \in (0;l)$, $e_t = (\partial/\partial t)\, e$, $q_x = (\partial/\partial x)\, q$. In (1.3) $\alpha(t)$ and $k(t)$ are relaxation functions of internal energy and heat flow, respectively. The first and second equalities in equations (1.3) are linearized (with respect to certain constant e_0 energy) constitutive equations for internal energy and heat flow, respectively. And the third relation in (1.3) expresses the fundamental law of thermal conductivity - Fourier's law. For $k(0) = 0$ these equations represent the linearized theory for heat flow in a rigid, isotropic, homogeneous material as proposed by Gurtin and Pipkin (see e.g., [1], [3]). For $k(0) > 0$ the equations represent an alternate linearized theory proposed by Coleman and Gurtin [4]. For the direct problem consisting in determining the distribution of heat from some initial-boundary value problem for equation (1.3) Grabmueller [5] gave a very general uniqueness proof for generalized solutions in a Sobolev space and proved existence theorems in certain special situations.

The determination of the integral operator from the observable information about the solutions of the corresponding equations is a new class of inverse problems that has not yet been sufficiently studied. In view of a wide range of applications, the theory of inverse problems for integro–differential equations is one of the most urgent and rapidly developing fields of world science.

The problem of determining the kernel $K(t)$ of the integral term in equation (1.1) were studied in many publications [6]–[21] (see also references in them), in which both one- and multidimensional inverse problems were investigated. In these works, the questions of correctness of the considered problems were studied. The numerical solutions for this problems were considered in the works [22]–[24].

In the present paper, we study the inverse problems about determining the kernels of an integral convolution-type terms in the system of integro-differential equations (1.3) by the single observation at the point $x = x_0$ from below equations (1.5)-(1.7).

Among the works which are close to the problem under study below we note [25]–[29]. In [25] there was proven the uniqueness theorem for solution of kernel determination problem for one-dimensional heat conduction equation. The papers [26]–[29] deal with the inverse problems of determining the kernel depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1, ..., x_{n-1})$. While the main part of the considered integro-differential equation is n-dimensional heat conduction operator and the integral term has a convolution type form with respect to unknown functions: the solutions of direct and inverse problem. In these works the theorems of existence and uniqueness of problems solutions were obtained.

It is supposed the rigid body will occupy a fixed open interval $(0, l)$ (one dimensional case). The energy-temperature relation function $\alpha(t)$ and the heat conduction relation $k(t)$ are both assumed sufficiently continuously differentiable functions.

From (1.3) it follows that

$$\theta_t(t,x) = -\frac{\alpha'(0)}{\alpha(0)}\theta(t,x) + \frac{k(0)}{\alpha(0)}\theta_{xx}(t,x)+$$

$$+\int_0^t \left[\frac{k'(t-\tau)}{\alpha(0)}\theta_{xx}(\tau,x) - \frac{\alpha''(t-\tau)}{\alpha(0)}\theta(\tau,x)\right] d\tau + \frac{r(t,x)}{\alpha(0)}. \tag{1.4}$$

Let $k(0) > 0$ and $\alpha(0) \neq 0$. Rewrite the equation (1.4) in the compact form:

$$\theta_t(t,x) = f(t,x)+C\theta_{xx}(t,x)-a(0)\theta(t,x)+\int_0^t \left[Cb(t-\tau)\theta_{xx}(\tau,x) - a'(t-\tau)\theta(\tau,x)\right] d\tau \tag{1.5}$$

for all $t \geq 0, x \in (0;l)$ and consider the initial-boundary value problem with

$$\theta(0,x) = \theta_0(x), \tag{1.6}$$

$$\theta(t,0) = \mu_1(t); \quad \theta(t,l) = \mu_2(t); \quad \theta_0(0) = \mu_1(0); \quad \theta_0(l) = \mu_2(0); \tag{1.7}$$

the initial and boundary conditions, where

$$C = \frac{k(0)}{\alpha(0)},\ a(t) = \frac{\alpha'(t)}{\alpha(0)},\ b(t) = \frac{k'(t)}{k(0)},\ f(t,x) = \frac{r(t,x)}{\alpha(0)}.$$

In equalities (1.6) and (1.7) $\theta_0(x)$, $\mu_1(t)$ and $\mu_1(t)$ are given functions. If $r(t,x)$, θ_0, $\alpha(t)$, $k(t)$, $\mu_1(t)$, $\mu_2(t)$ are given functions, then finding the function $\theta(t,x)$ from (1.5)-(1.7) is called as a direct problem.

We pose the inverse problems:

Inverse problem 1. For given functions $r(t,x)$, $\theta_0(x)$, $k(t)$, $\mu_1(t)$, $\mu_2(t)$ it is required to determine the function $\alpha(t)$, $t > 0$ of the integral term in (1.5) using additional information about the solution of the direct problem (1.5)-(1.7):

$$\theta\mid_{x=x_0} = \psi(t),\ x_0 \in (0,l),\ t > 0 \tag{1.8}$$

In this case $\psi(t)$, $t > 0$ are assumed to be given functions.

Inverse problem 2. For given functions $r(t,x)$, $\theta_0(x)$, $\alpha(t)$, $\mu_1(t)$, $\mu_2(t)$ it is required to determine the function $k(t)$, $t > 0$ of the integral term in (1.5) using additional information (1.8) on the solution of the direct problem (1.5)-(1.7).

Since the method for studying the inverse problems allow to find simultaneously the solution to the inverse problem and the solution to the direct problem, then in the sequel, we will call the inverse problem 1 as a problem of determining functions $\theta(t,x), \alpha(t)$ from equations (1.5)-(1.8).

2. Preliminaries

Let $C^m(0;l)$ be the class of m times continuously differentiable with all derivatives up to the $m-$th order (inclusive) in $(0;l)$ functions. In the case $m = 0$ this space coincides with the class of continuous functions. $C^{m,k}(D_T)$ is the class of m times continuously differentiable with respect to t and k times continuously differentiable with respect to x all derivatives in the domain D_T functions.

We need the following assertion:

Lemma 1. **(see [2])** *Suppose $\alpha(0) > 0, \alpha \in C^3[0,T], k \in C^2[0,T]$, $T > 0$ is an arbitrary fixed number, are true with $k(0) > 0$. Then equation (1.3) is equivalent to the following integrodifferential equation:*

$$\frac{\partial \theta}{\partial t}(t,x) = F(t,x) + C\triangle\theta(t,x) + y(0)\theta(t,x) + \int_0^t y'(t-\tau)\theta(\tau,x)d\tau, \tag{2.1}$$

where F is defined as

$$F(t,x) = f(t,x) - \int_0^t D(t-\tau)f(\tau,x)d\tau + D(t)\theta(0,x),$$

and where $D(t)$ and $y(t)$ satisfy the scalar equations

$$D(t) = b(t) - \int_0^t b(t-\tau)D(\tau)d\tau, \tag{2.2}$$

$$y(t) = b(t) - a(t) - \int_0^t b(t-\tau)y(\tau)d\tau. \tag{2.3}$$

If $b(t)$ function is continuously for $t > 0$ then the solution to the integral equations (2.2) exists and unique. Note that for given equation (2.3) it can be considered to be an integral Volterra equation of the second kind with respect to $y(t)$ with the kernel $b(t)$,

$$y(t) = -\int_0^t b(t-\tau)y(\tau)d\tau + [b(t) - a(t)]. \tag{2.4}$$

It follows from the general theory of integral equations (see, e.g., [30, p.39-44]) that the solution of this equation is expressed by the formula

$$y(t) = b(t) - a(t) + \int_0^t R(t-\tau)\left[b(\tau) - a(\tau)\right]d\tau, \tag{2.5}$$

where the kernels $R(t)$ and $b(t)$ are related by

$$b(t) = -R(t) - \int_0^t R(t-\tau)b(\tau)d\tau. \tag{2.6}$$

If $b(0)$ is a known number, from relation (2.6) we find $R(0) = -b(0)$.

Everywhere in this paper it is supposed $\alpha(0)$ and $k(0)$ are given numbers such that $\alpha(0) \neq 0$, $k(0) > 0$.

In the next sections we will use the contraction mapping principle to proof the unique solvability of inverse problems.

Definition. *Let F be an operator defined on a closed set Ω which is a subset of a Banach space. F is called a contraction mapping operator in Ω if it satisfies the following two properties:*

1) if $y \in \Omega$, then $Fx \in \Omega$ (i.e. F maps Ω into itself);

2) if $y, z \in \Omega$, then $\|Fy - Fz\| \leq \rho \|y - z\|$ with $\rho < 1$ (ρ - is a constant independent of y and z).

Lemma 2.(contraction mapping principle [[31], pp. 87-97]). *If F is a contraction mapping operator from Ω to Ω, then the equation*

$$y = Fy$$

has a unique solution $y_0 \in \Omega$.

3. Problem of determining the functions $\theta(t,x), \alpha(t)$

In this section existence and uniqueness for the inverse problem (2.1), (1.6)-(1.8) is proved using the contraction mapping principle [Lemma 2]. The idea is to write the integral equations for unknown functions $\theta(x,t)$, $a(t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator. The existence and uniqueness then follow immediately.

The solution of the initial-boundary problem (2.1), (1.6), (1.7) satisfies the integral equation [[32], pp. 200-221]:

$$\theta(t,x) = \Psi(t,x) + \int_0^t \int_0^l G(t-\tau, x, \xi) \left(y(0)\theta(\tau,\xi) + \int_0^\tau y'(\tau-\alpha)\theta(\alpha,\xi)d\alpha \right) d\xi d\tau =$$

$$= \Psi(t,x) + \int_0^t \int_0^l G(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(t-\tau,x,\xi) \int_0^\tau y(\alpha)\theta_\alpha(\tau-\alpha,\xi)d\alpha d\xi d\tau. \tag{3.1}$$

where

$$\Psi(t,x) = \int_0^l G(t,x,\xi)\theta_0(\xi)d\xi + \int_0^t \int_0^l G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau +$$

$$+ \sum_{n=1}^{\infty} \int_0^t \frac{2\pi n}{l^2} \left[\mu_1(\tau) - (-1)^n \mu_2(\tau)\right] e^{-(\frac{\pi n}{l})^2(t-\tau)} sin\left(\frac{\pi n}{l}x\right) d\tau;$$

$$G(t-\tau,x,\xi) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-(\frac{\pi a n}{l})^2(t-\tau)} sin\left(\frac{\pi a n}{l}\xi\right) sin\left(\frac{\pi a n}{l}x\right)$$

is the Green function of the initial-boundary problem for one-dimensional heat equation.

We differentiate the equation (3.1) with respect to t. Introducing the notation $\vartheta(t,x) := \theta_t(t,x)$ and taking into account the following relations:

$$\lim_{t\to 0} G(t,\xi,x) = \delta(x-\xi), \quad \lim_{t\to 0} \int_0^l G(t,x,\xi)\theta_0(\xi)d\xi = \theta_0(x),$$

where $\delta(\cdot)$ is the Dirac's delta function, we rewrite the result in the form

$$\vartheta(t,x) = \Psi_t(t,x) + \theta_0(x)y(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x)d\alpha +$$

$$+\int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau. \tag{3.2}$$

Further, using the condition (1.8), we obtain:

$$\psi'(t) = \Psi_t(t,x_0) + \theta_0(x_0)y(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x_0)d\alpha +$$

$$+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau.$$

Next we write this equality as the integral equation of the second order with respect to unknown function $y(t)$

$$y(t) = -\frac{1}{\theta_0(x_0)}\big[\Psi_t(t,x_0) - \psi'(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x_0)d\alpha +$$

$$+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau\big]. \tag{3.3}$$

Replacing $t = 0$ in integral equation (3.3), the unknown function $y(0)$ is found as follows:

$$y(0) = \frac{\psi'(0) - \Psi_t(0,x_0)}{\theta_0(x_0)};$$

In what follows we assume $\theta_0(x_0) \neq 0$.

We represent the system of equations (3.2), (3.3) in the form

$$Ag = g, \tag{3.4}$$

where $g = (g_1,\ g_2) = (\vartheta(x,t) - \theta_0(x)y(t),\ y(t))$ is the vector-function and unknown functions are represented by g_1, g_2 functions as follows:

$$\vartheta(t,x) = \theta_t(t,x) = g_1(t,x) + \theta_0(x)g_2(t);$$

$$y(t) = g_2(t).$$

$A = (A_1,\ \ A_2)$ is defined by the right sides of equations (3.2), (3.3):

$$A_1 g = g_{01} + \int_0^t g_2(\alpha)\left(g_1(t-\alpha,x) + \theta_0(x)g_2(t-\alpha)\right)d\alpha +$$

$$+\int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)g_2(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x,\xi)\times$$

$$\times \int_0^\tau g_2(\alpha)\big(g_1(\tau-\alpha,\xi)+\theta_0(\xi)g_2(\tau-\alpha)\big)d\alpha d\xi d\tau; \tag{3.5}$$

$$A_2 g = g_{02} - \frac{1}{\psi(0)}\int_0^t g_2(\alpha)\left(g_1(t-\alpha,x_0)+\theta_0(x_0)g_2(t-\alpha)\right)d\alpha -$$

$$-\frac{1}{\psi(0)}\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)g_2(\tau)d\xi d\tau -$$

$$-\frac{1}{\psi(0)}\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau g_2(\alpha)\big(g_1(\tau-\alpha,\xi)+\theta_0(\xi)g_2(\tau-\alpha)\big)d\alpha d\xi d\tau. \tag{3.6}$$

The following notations were introduced in the equalities (3.1), (3.2):

$$g_0(t,x) = (g_{01}(t,x),\ \ g_{02}(t)) = (\Psi_t(t,x),\ \ -\frac{1}{\psi(0)}\big(\Psi_t(t,x_0)-\psi'(t)\big)).$$

Theorem 1(existence and uniqueness). *Assume the conditions $\theta_0(x)\in C\,(0,l)$, $\psi(t)\in C[0;T]$, $r(t,x)\in C(D_T)$, $\mu_i(t)\in C[0,T]$, $i=1,2$, $k(t)\in C^2[0,T]$, $\theta_0(0)=\psi(0)$, $\theta_0(x_0)\neq 0$, $\theta_0(0)=\mu_1(0)$, $\theta_0(l)=\mu_2(0)$ are hold. Then there exists sufficiently small number $T^*\in(0,T)$ that the solution to the integral equations (3.1), (3.2) in the class of functions $\vartheta(t,x)\in C^{1,2}\,(D_{T^*})$, $y(t)\in C[0;T^*]$ exist and unique, where $D_{T^*}=\{(x,t)|x\in(0,l),t\in[0,T^*]\}$.*

To **prove** the theorem 1,we define for the unknown vector-function $g(x,t)\in C(D_T)$ the following weight norm:

$$\|g\|_\sigma = \max\left\{\sup_{(x,t)\in D_T}\left|g_1(x,t)e^{-\sigma t}\right|,\ \sup_{t\in[0,T]}\left|g_2(t)e^{-\sigma t}\right|\right\} =$$

$$= \max\left\{\|g_1\|_\sigma, \|g_2\|_\sigma\right\},\ \sigma\geq 0.$$

At $\sigma=0$ this norm coincides with the usual norm

$$\|g\| = \max\left\{\sup_{(x,t)\in D_T}|g_1(x,t)|,\ \sup_{t\in[0,T]}|g_2(t)|\right\}.$$

The number $\sigma>0$ will be chosen later. Denote by $S(g_0,\rho)$ the ball of vector-functions g with center at the point g_0 and radius $\rho>0$, i.e. $S(g_0,\rho)=\{g:\|g-g_0\|_\sigma\leq\rho\}$. The number $\rho>0$ will be also chosen later.

Obviously, $\|g\|\leq\rho+\|g_0\|$ for $g(x,t)\in S(g_0,\rho)$. We prove that the operator A is contracting in the Banach space $S(g_0,\rho)$ if the numbers σ and ρ will be chosen in suitable way.

Note that the weight norm $\|\cdot\|_\sigma$ is equivalent to the usual norm $\|\cdot\|$:

$$\|\cdot\|_\sigma \leq \|\cdot\| \leq e^{\sigma T}\|\cdot\|_\sigma,\ \sigma\geq 0. \tag{3.7}$$

The convolution operator is commutative and invariant with respect to multiplication by $e^{-\sigma t}$:

$$\left(h_1 * h_2\right)(t) = \int_0^t h_1(t-s)h_2(s)ds = \int_0^t h_1(s)h_2(t-s)ds = \left(h_2 * h_1\right)(t), \tag{3.8}$$

$$e^{-\sigma t}\left(h_1 * h_2\right)(t) = (e^{-\sigma t}h_1(t)) * (e^{-\sigma t}h_2(t)). \tag{3.9}$$

The last formula implies the estimation

$$\|h_1 * h_2\|_\sigma \le \|h_1\|_\sigma \|h_2\|_\sigma T. \tag{3.10}$$

Moreover, since

$$\int_0^t e^{-\sigma s} ds = \int_0^t e^{-\sigma(t-s)} ds \le \frac{1}{\sigma},\ \sigma \ge 0 \tag{3.11}$$

we have

$$\|h_1 * h_2\|_\sigma \le \frac{1}{\sigma}\|h_1\|\|h_2\|_\sigma \le \frac{1}{\sigma}\|h_1\|\|h_2\|,\ \sigma \ge 0 \tag{3.12}$$

using (3.7) and the results of [10].

Now we write two properties of Green function [see [32], pp.200-221] which will be needed in the future.

Remark 1. *The integral of the Green function does not exceed 1:*

$$\int_0^l G(x,\xi,t)d\xi \le 1, x \in (0,l), t \in (0,T].$$

Remark 2. *The function $G(x,\xi,t)$ is infinitely continuously differentiable with respect to x, ξ, t and $G_t(x,\xi,t)$ is bounded for $0 < x < l$, $0 < \xi < l$, $0 < t \le T$, i.e.*

$$|G_t(x,\xi,t-\tau)| \le \frac{2}{l}.$$

Now we check the first condition of contractive mapping for operator A.

We introduce the notations

$$\theta_0 := \max_{x\in(0;l)} |\theta_0(x)|\,, \quad \psi_0 := \max\left\{\max_{t\in[0,T]} |\psi(t)|\,,\ \max_{t\in[0,T]} |\psi'(t)|\right\}.$$

Let $g(x,t)$ be an element of $S(g_0,\rho)$, i.e. $g \in S(g_0,\rho)$. Then for $(x,t) \in D_T$ we have

$$\|A_1 g - g_{01}\|_\sigma = \sup_{(x,t)\in D_T} |(A_1 g - g_{01})e^{-\sigma t}| \le \sup_{(x,t)\in D_T}\left|\int_0^t g_2(\alpha)(g_1(t-\alpha,x)+\theta_0(x)g_2(t-\alpha))e^{-\sigma t}d\alpha\right| +$$

$$+ \sup_{(x,t)\in D_T}\left|\int_0^t \int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)g_2(\tau)e^{-\sigma t}d\xi d\tau\right| + \sup_{(x,t)\in D_T}\left|\int_0^t \int_0^l G_t(t-\tau,x,\xi)\times\right.$$

$$\left.\times\int_0^\tau g_2(\alpha)\big(g_1(\tau-\alpha,\xi)+\theta_0(\xi)g_2(\tau-\alpha)\big)e^{-\sigma t}d\alpha d\xi d\tau\right| =: I_1 + I_2 + I_3.$$

We estimate each I_i, $i = 1,2,3$, separately:

$$I_1 := \sup_{(x,t)\in D_T}\left|\int_0^t g_2(\alpha)(g_1(t-\alpha,x)+\theta_0(x)g_2(t-\alpha))e^{-\sigma t}d\alpha\right| \le$$

$$\leq \sup_{(x,t)\in D_T}\Big|\int_0^t g_2(\alpha)g_1(t-\alpha,x)e^{-\sigma t}d\alpha\Big| + \sup_{(x,t)\in D_T}\Big|\int_0^t g_2(\alpha)\theta_0(x)g_2(t-\alpha)e^{-\sigma t}d\alpha\Big| \leq$$

$$\leq \sup_{(x,t)\in D_T}\Big|(g_2 * g_1)(t)e^{-\sigma t}\Big| + \theta_0 \sup_{(x,t)\in D_T}\Big|(g_2 * g_2)(t)e^{-\sigma t}\Big| \leq$$

$$\leq \sup_{(x,t)\in D_T}\Big|\Big\{\left[(g_2-g_{02})*(g_1-g_{01})\right](t)+(g_2*g_{01})(t)+(g_1*g_{02})(t)-(g_{02}*g_{01})(t)\Big\}e^{-\sigma t}\Big|+$$

$$+\theta_0 \sup_{(x,t)\in D_T}\Big|\Big\{\left[(g_2-g_{02})*(g_2-g_{02})\right](t)+(g_2*g_{02})(t)+(g_2*g_{02})(t)-(g_{02}*g_{02})(t)\Big\}e^{-\sigma t}\Big| \leq$$

$$\leq \Big(\|g_2-g_{02}\|_\sigma\|g_1-g_{01}\|_\sigma T + \frac{1}{\sigma}\|g_2\|_\sigma\|g_{01}\| + \frac{1}{\sigma}\|g_1\|_\sigma\|g_{02}\| + \frac{1}{\sigma}\|g_{01}\|_\sigma\|g_{02}\|\Big)+$$

$$+\theta_0\Big(\|g_2-g_{02}\|_\sigma\|g_2-g_{02}\|_\sigma T + \frac{1}{\sigma}\|g_2\|_\sigma\|g_{02}\| + \frac{1}{\sigma}\|g_2\|_\sigma\|g_{02}\| + \frac{1}{\sigma}\|g_{02}\|_\sigma\|g_{02}\|\Big) \leq$$

$$\leq (1+\theta_0)(\rho^2 T + \frac{2}{\sigma}(\rho+\|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2);$$

$$I_2 := \sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)g_2(\tau)e^{-\sigma t}d\xi d\tau\Big| \leq \frac{2\theta_0(\rho+\|g_0\|)}{\sigma};$$

$$I_3 := \sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau g_2(\alpha)\big(g_1(\tau-\alpha,\xi)+\theta_0(\xi)g_2(\tau-\alpha)\big)e^{-\sigma t}d\alpha d\xi d\tau\Big| \leq$$

$$\leq \sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau g_2(\alpha)g_1(\tau-\alpha,\xi)e^{-\sigma t}d\alpha d\xi d\tau\Big|+$$

$$+\sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau g_2(\tau-\alpha)\theta_0(\xi)g_2(\alpha)e^{-\sigma t}d\alpha d\xi d\tau\Big| \leq$$

$$\leq \frac{2(\rho+\|g_0\|)^2 T}{\sigma} + \frac{2\theta_0(\rho+\|g_0\|)^2 T}{\sigma} = (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\sigma};$$

Accordingly, we get

$$\|A_1 g - g_{01}\|_\sigma \leq (1+\theta_0)(\rho^2 T + \frac{2}{\sigma}(\rho+\|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2) + \frac{2\theta_0(\rho+\|g_0\|)}{\sigma}+$$

$$+(1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\sigma} = (1+\theta_0)T\rho^2 + (2\|g_0\|(1+\theta_0)+2\theta_0)\frac{(\rho+\|g_0\|)}{\sigma}+$$

$$+(1+\theta_0)\frac{1}{\sigma}\|g_0\|^2 + (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\sigma}.$$

Now we can choose ρ, σ such that there hold the inequalities:

$$\begin{cases} (1+\theta_0)T\rho^2 < \frac{1}{4}\rho, \\ (2\|g_0\|(1+\theta_0)+2\theta_0)\frac{(\rho+\|g_0\|)}{\sigma} < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{1}{\sigma}\|g_0\|^2 < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\sigma}) < \frac{1}{4}\rho. \end{cases}$$

It follows that if

$$\begin{cases} \rho < \frac{1}{4(1+\theta_0)T} = \rho_1, \\ \beta_1 = 4(2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho_1 + \|g_0\|)}{\rho_1} < \sigma, \\ \beta_2 = (1+\theta_0)\frac{4}{\rho_1}\|g_0\|^2 < \sigma, \\ \beta_3 = (1+\theta_0)\frac{8(\rho_1+\|g_0\|)^2 T}{\rho_1} < \sigma. \end{cases}$$

then $A_1 g \in S(g_0, \rho)$.

So, if the inequality

$$\sigma > \sigma_1 = \max\{\beta_1,\ \beta_2, \beta_3\}$$

and $\rho \in (0, \rho_1)$ holds, then the operator A_1 maps $S(g_0, \rho)$ into itself, i.e. $A_1 g \in S(g_0, \rho)$.

$$\|A_2 g - g_{02}\|_\sigma = \sup_{t\in[0,T]} \left|(A_2 g - g_{02})e^{-\sigma t}\right| \leq \sup_{t\in[0,T]} \Big|\frac{1}{\theta_0(x_0)} \int_0^t g_2(\alpha)(g_1(t-\alpha, x_0) +$$

$$+\theta_0(x_0) g_2(t-\alpha))e^{-\sigma t} d\alpha\Big| + \sup_{t\in[0,T]} \left|\frac{1}{\theta_0(x_0)} \int_0^t \int_0^l G_t(t-\tau, x_0, \xi)\theta_0(\xi) g_2(\tau) e^{-\sigma t} d\xi d\tau\right| +$$

$$+ \sup_{t\in[0,T]} \left|\frac{1}{\theta_0(x_0)} \int_0^t \int_0^l G_t(t-\tau, x_0, \xi) \int_0^\tau g_2(\alpha)\big(g_1(\tau-\alpha, \xi) + \theta_0(\xi) g_2(\tau-\alpha)\big) e^{-\sigma t} d\alpha d\xi d\tau\right| \leq$$

$$\leq \frac{(1+\theta_0)}{\theta_0(x_0)}(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2) + \frac{2\theta_0(\rho + \|g_0\|)}{\theta_0(x_0)\sigma} + (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\theta_0(x_0)\sigma} =$$

$$= \frac{(1+\theta_0)T\rho^2}{\theta_0(x_0)} + (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho + \|g_0\|)}{\theta_0(x_0)\sigma} + (1+\theta_0)\frac{1}{\theta_0(x_0)\sigma}\|g_0\|^2 + (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\theta_0(x_0)\sigma}.$$

Now we can choose ρ, σ such that there hold the inequalities:

$$\begin{cases} \frac{(1+\theta_0)T\rho^2}{\theta_0(x_0)} < \frac{1}{4}\rho, \\ (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho+\|g_0\|)}{\theta_0(x_0)\sigma} < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{1}{\theta_0(x_0)\sigma}\|g_0\|^2 < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\theta_0(x_0)\sigma} < \frac{1}{4}\rho. \end{cases}$$

It follows that if

$$\begin{cases} \rho < \frac{\theta_0(x_0)}{4(1+\theta_0)T} = \rho_2, \\ \beta_4 = 4(2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho_1 + \|g_0\|)}{\theta_0(x_0)\rho_2} < \sigma, \\ \beta_5 = (1+\theta_0)\frac{4}{\theta_0(x_0)\rho_2}\|g_0\|^2 < \sigma, \\ \beta_6 = (1+\theta_0)\frac{8(\rho_1+\|g_0\|)^2 T}{\theta_0(x_0)\rho_2} < \sigma. \end{cases}$$

then $A_2 g \in S(g_0, \rho)$.

So, if the inequality

$$\sigma > \sigma_2 = \max\{\beta_4,\ \beta_5, \beta_6\}$$

and $\rho \in (0, \rho_1)$ holds, then the operator A_2 maps $S(g_0, \rho)$ into itself, i.e. $A_2 g \in S(g_0, \rho)$.

As a result, we conclude that if σ, ρ satisfy the conditions $\sigma > \max\{\sigma_1, \sigma_2\}$, $\rho \in (0, \rho_2)$, then operator A maps $S(g_0, \rho)$ into itself, i.e. $Ag \in S(g_0, \rho)$.

Further we check the second condition of contractive mapping. In accordance with (3.5) for the first component of operator A we get

$$\|(Ag^1 - Ag^2)_1\|_\sigma = \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1(\alpha) g_1^1(t-\alpha, x) - g_2^2(\alpha) g_1^2(t-\alpha, x)\right] d\alpha e^{-\sigma t} \right| +$$

$$+ \sup_{(x,t)\in D_T} \left| \int_0^t \theta_0(x) [g_2^1(\alpha) g_2^1(t-\alpha) - g_2^2(\alpha) g_2^2(t-\alpha)] d\alpha e^{-\sigma t} \right| +$$

$$+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \theta_0(\xi) \left[g_2^1(\tau) - g_2^2(\tau)\right] d\xi d\tau e^{-\sigma t} \right| +$$

$$+ \sup_{(x,t)\in D_T} \Big| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \times$$

$$\times \int_0^\tau \left[g_2^1(\alpha) g_1^1(\tau-\alpha, \xi) - g_2^2(\alpha) g_1^2(\tau-\alpha, \xi)\right] d\alpha d\xi d\tau e^{-\sigma t} \Big| +$$

$$+ \sup_{(x,t)\in D_T} \Big| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \times$$

$$\times \int_0^\tau \theta_0 \left[g_2^1(\alpha) g_2^1(\tau-\alpha) - g_2^2(\alpha) g_2^2(\tau-\alpha)\right] d\alpha d\xi d\tau e^{-\sigma t} \Big| =: \sum_{i=1}^5 J_i.$$

We denoted the summands in this equality by $J_i (i = \overline{1,5})$ respectively and carry out the estimates for them separately.

Taking into account the relation

$$g_2^1 * g_1^1 - g_2^2 * g_1^2 = \left(g_2^1 - g_2^2\right) * \left(g_1^1 - g_{01}\right) + \left(g_1^1 - g_1^2\right) * \left(g_2^2 - g_{02}\right) +$$
$$+ g_{01} * \left(g_2^1 - g_2^2\right) + g_{02} * \left(g_1^1 - g_1^2\right),$$

estimate the J_1, J_2 as follows:

$$J_1 := \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1(\alpha) g_1^1(t-\alpha, x) - g_2^2(\alpha) g_1^2(t-\alpha, x)\right] d\alpha e^{-\sigma t} \right| =$$

$$= \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1 * g_1^1 - g_2^2 * g_1^2\right] d\alpha e^{-\sigma t} \right| \le$$

$$\le \Big[\left\|g_2^1 - g_2^2\right\|_\sigma \left\|g_1^1 - g_{01}\right\|_\sigma T + \left\|g_1^1 - g_1^2\right\|_\sigma \left\|g_2^2 - g_{02}\right\|_\sigma T + \|g_{01}\|_\sigma \left\|g_2^1 - g_2^2\right\|_\sigma +$$

$$+ \|g_{02}\|_\sigma \left\|g_1^1 - g_1^2\right\|_\sigma \Big] \le 2 \left(\rho T + \frac{1}{\sigma} \|g_0\| \right) \left\|g^1 - g^2\right\|_\sigma,$$

$$J_2 := \sup_{(x,t)\in D_T}\left|\int_0^t \theta_0(x)\left[g_2^1(\alpha)g_2^1(t-\alpha)-g_2^2(t-\alpha)g_2^2(\alpha)\right]d\alpha e^{-\sigma t}\right| =$$

$$= \sup_{(x,t)\in D_T}\left|\int_0^t \theta_0(x)\left[g_2^1 * g_2^1 - g_2^2 * g_2^2\right]d\alpha e^{-\sigma t}\right| \leq$$

$$\leq \theta_0\Big[\left\|g_2^1-g_2^2\right\|_\sigma \left\|g_2^1-g_{02}\right\|_\sigma T + \left\|g_2^1-g_2^2\right\|_\sigma \left\|g_2^2-g_{02}\right\|_\sigma T + \|g_{02}\|_\sigma \left\|g_2^1-g_2^2\right\|_\sigma +$$

$$+ \|g_{02}\|_\sigma \left\|g_2^1-g_2^2\right\|_\sigma\Big] \leq 2\theta_0\left(\rho T + \frac{1}{\sigma}\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma,$$

$$J_3 := \sup_{(x,t)\in D_T}\left|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)\left[g_2^1(\tau)-g_2^2(\tau)\right]d\xi d\tau e^{-\sigma t}\right| \leq$$

$$\leq \frac{2\theta_0}{\sigma}\left\|g^1-g^2\right\|_\sigma,$$

$$J_4 := \sup_{(x,t)\in D_T}\left|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau \left[g_2^1(\alpha)g_1^1(\tau-\alpha,\xi)-g_2^2(\alpha)g_1^2(\tau-\alpha,\xi)\right]d\alpha d\xi d\tau e^{-\sigma t}\right| \leq$$

$$\leq \frac{4T}{\sigma}\left(\rho+\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma,$$

$$J_5 := \sup_{(x,t)\in D_T}\left|\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau \theta_0\left[g_2^1(\alpha)g_2^1(\tau-\alpha)-g_2^2(\alpha)g_2^2(\tau-\alpha)\right]d\alpha d\xi d\tau e^{-\sigma t}\right| \leq$$

$$\leq \frac{4\theta_0 T}{\sigma}\left(\rho+\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma,$$

Here the integrand in the last integral can be estimated as follows

$$\left\|g_2^1g_1^1-g_2^2g_1^2\right\|_\sigma = \left\|(g_2^1-g_2^2)g_1^1+g_2^2(g_1^1-g_1^2)\right\|_\sigma \leq$$

$$\leq 2\left\|g^1-g^2\right\|_\sigma \max\left(\left\|g_1^1\right\|_\sigma, \left\|g_2^2\right\|_\sigma\right) \leq 2(\|g_0\|+\rho)\left\|g^1-g^2\right\|_\sigma.$$

Summing the obtained estimates for $J_i,\ i=1,2,..,5$ we have that the first component of A can be estimated in the following form:

$$\|(Ag^1-Ag^2)_1\|_\sigma \leq 2\left(\rho T+\frac{1}{\sigma}\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma +$$

$$+2\theta_0\left(\rho T+\frac{1}{\sigma}\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma + \frac{2\theta_0}{\sigma}\left\|g^1-g^2\right\|_\sigma + \frac{4T}{\sigma}\left(\rho+\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma +$$

$$+\frac{4\theta_0 T}{\sigma}\left(\rho+\|g_0\|\right)\left\|g^1-g^2\right\|_\sigma =$$

$$= ((2+2\theta_0)\rho T + (2\|g_0\|+2\theta_0\|g_0\|+2\theta_0)\frac{1}{\sigma} + (4\theta_0 T+4T)\left(\rho+\|g_0\|\right)\frac{1}{\sigma})\left\|g^1-g^2\right\|_\sigma$$

Now we choose numbers σ,ρ so that the expression at $\|g^1-g^2\|_\sigma$ becomes less than 1, i.e., the inequality

$$(2+2\theta_0)\rho T+(2\|g_0\|+2\theta_0\|g_0\|+2\theta_0)\frac{1}{\sigma}+(4\theta_0 T+4T)\left(\rho+\|g_0\|\right)\frac{1}{\sigma}<1$$

is fulfilled. This inequality is valid if numbers σ,ρ will be chosen from conditions

$$\begin{cases}(2+2\theta_0)\rho T<\frac{1}{3},\\ (2\|g_0\|+2\theta_0\|g_0\|+2\theta_0)\frac{1}{\sigma}<\frac{1}{3},\\ (4\theta_0 T+4T)\left(\rho+\|g_0\|\right)\frac{1}{\sigma}<\frac{1}{3}.\end{cases}$$

Solving these inequalities with respect to σ,ρ we obtain

$$\begin{cases}\rho<\frac{1}{3(2+2\theta_0)}=\rho_3,\\ \beta_7=3(2\|g_0\|+2\theta_0\|g_0\|+2\theta_0)<\sigma,\\ \beta_8=3(4\theta_0 T+4T)\left(\rho_3+\|g_0\|\right)<\sigma.\end{cases}$$

From these estimates it is clear that if σ and ρ are chosen from condition $\sigma>\sigma_5=max(\beta_7,\beta_8)$ and $\rho<(0,\rho_3)$, then the operator A_2 satisfies the second condition of contracting mapping.

The second component of A can be estimated in the following form:

$$\|(Ag^1-Ag^2)_2\|_\sigma=\frac{1}{\theta_0(x_0)}\sup_{(x,t)\in D_T}\left|\int_0^t\left[g_2^1(\alpha)g_1^1(t-\alpha,x_0)-g_2^2(\alpha)g_1^2(t-\alpha,x_0)\right]d\alpha e^{-\sigma t}\right|+$$

$$+\frac{1}{\theta_0(x_0)}\sup_{(x,t)\in D_T}\left|\int_0^t\theta_0(x)\left[g_2^1(\alpha)g_2^1(t-\alpha)-g_2^2(\alpha)g_2^2(t-\alpha)\right]d\alpha e^{-\sigma t}\right|+$$

$$+\frac{1}{\theta_0(x_0)}\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)\left[g_2^1(\tau)-g_2^2(\tau)\right]d\xi d\tau e^{-\sigma t}\right|+$$

$$+\frac{1}{\theta_0(x_0)}\sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\times$$

$$\times\int_0^\tau\left[g_2^1(\alpha)g_1^1(\tau-\alpha,\xi)-g_2^2(\alpha)g_1^2(\tau-\alpha,\xi)\right]d\alpha d\xi d\tau e^{-\sigma t}\Big|+$$

$$+\frac{1}{\theta_0(x_0)}\sup_{(x,t)\in D_T}\Big|\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\times$$

$$\times\int_0^\tau\theta_0\left[g_2^1(\alpha)g_2^1(\tau-\alpha)-g_2^2(\alpha)g_2^2(\tau-\alpha)\right]d\alpha d\xi d\tau e^{-\sigma t}\Big|\le$$

$$\le\Big((2+2\theta_0)\frac{\rho T}{\theta_0(x_0)}+(2\|g_0\|+2\theta_0\|g_0\|+2\theta_0)\frac{1}{\theta_0(x_0)\sigma}+$$

$$+(4\theta_0 T+4T)\left(\rho+\|g_0\|\right)\frac{1}{\theta_0(x_0)\sigma}\Big)\left\|g^1-g^2\right\|_\sigma$$

Now we choose numbers σ, ρ so that the expression at $\|g^1 - g^2\|_\sigma$ becomes less than 1, i.e., the inequality

$$(2+2\theta_0)\frac{\rho T}{\theta_0(x_0)} + (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)\sigma} + (4\theta_0 T + 4T)\,(\rho + \|g_0\|)\,\frac{1}{\theta_0(x_0)\sigma} < 1$$

is fulfilled. This inequality is true if numbers σ, ρ will be chosen from conditions

$$\begin{cases} (2+2\theta_0)\frac{\rho T}{\theta_0(x_0)} < \frac{1}{3}, \\ (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)\sigma} < \frac{1}{3}, \\ (4\theta_0 T + 4T)\,(\rho + \|g_0\|)\,\frac{1}{\theta_0(x_0)\sigma} < \frac{1}{3}. \end{cases}$$

Solving these inequalities with respect to σ, ρ we obtain

$$\begin{cases} \rho < \frac{\theta_0(x_0)}{3(2+2\theta_0)} = \rho_4, \\ \beta_9 = 3(2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)}\|g_0\| < \sigma, \\ \beta_{10} = 3(4\theta_0 T + 4T)\,(\rho_4 + \|g_0\|)\,\frac{1}{\theta_0(x_0)} < \sigma. \end{cases}$$

From these estimates it follows that if σ and ρ are chosen from conditions $\sigma > \sigma_6 = \max(\beta_9, \beta_{10})$ and $\rho < (0, \rho_4)$, then the operator A_3 satisfies the second condition of contracting mapping.

As result, we conclude that if σ and ρ are taken from conditions $\sigma > \max(\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4, \sigma_5, \sigma_6)$ and $\rho \in (0,\ \min(\rho_1,\ \rho_2,\ \rho_3,\ \rho_4))$, then the operator A carries out contracting mapping the ball $S(g_0,\ \rho)$ into itself and according to Banach theorem in this ball it has a unique fixed point, i.e., there exists a unique solution of operator equation (3.4). The proof of the theorem is complete.

Having found the functions $\vartheta(t,x)$ and $y(t)$, we determine the functions $\theta(t,x), a(t)$ by integral equation (2.3):

$$a(t) = b(t) - y(t) - \int_0^t b(t-\tau)y(\tau)d\tau.$$

$$\theta(t,x) = \theta_0(x) + \int_0^t \vartheta(\tau,x)d\tau.$$

With the known function $a(t)$, solving the differential equation $a(t) = \frac{\alpha'(t)}{\alpha(0)}$, we find the function

$$\alpha(t) = \alpha(0) + \alpha(0)\int_0^t a(\tau)d\tau,$$

the solution of the inverse problem 1 (1.5)-(1.8).

4. Inverse problem 2

This section deals with the problem of finding $\theta(t,x)$ and $k(t)$ from equalities (1.5)-(1.8). According to Lemma 1, the equation (1.5) is equivalent to equation (2.1). The solution of the direct problem (2.1), (1.6), (1.7) is expressed in the form of integral equation (3.1). We rewrite this equation as follows:

$$\theta(t,x)=\Phi(t,x)+\int_0^t\int_0^l G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau+\int_0^t\int_0^l G(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau+$$

$$+\int_0^t\int_0^l G(t-\tau,x,\xi)\int_0^\tau y(\alpha)\theta_\alpha(\tau-\alpha,\xi)d\alpha d\xi d\tau, \qquad (4.1)$$

where

$$\Phi(t,x)=\int_0^l G(t,x,\xi)\theta_0(\xi)d\xi+$$

$$\sum_{n=1}^{\infty}\int_0^t \frac{2\pi n}{l^2}\left[\mu_1(\tau)-(-1)^n\mu_2(\tau)\right]e^{-(\frac{\pi n}{l})^2(t-\tau)}sin\left(\frac{\pi n}{l}x\right)d\tau.$$

Differentiating equation (4.1) in t, we use the equality (2.2) and notation $\vartheta(t,x):=\theta_t(t,x)$. Then we have

$$\vartheta(t,x)=\Phi_t(t,x)+f(t,x)-\int_0^t D(t-\tau)f(\tau,x)d\tau+b(t)\theta_0(x)-\theta_0(x)\int_0^t b(t-\tau)D(\tau)d\tau+$$

$$+\theta_0(x)y(t)+\int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau+\int_0^t\int_0^l G_t(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau+$$

$$+\int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau+\int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau \qquad (4.2)$$

and we obtained the following equation using the additional condition (1.8):

$$\psi'(t)=\Phi_t(t,x_0)+f(t,x_0)-\int_0^t D(t-\tau)f(\tau,x_0)d\tau+b(t)\theta_0(x_0)-\theta_0(x_0)\int_0^t b(t-\tau)D(\tau)d\tau+$$

$$+\theta_0(x_0)y(t)+\int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)F(\tau,\xi)d\xi d\tau+$$

$$+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau.$$

From the above equation the unknown function $b(t)$ is found:

$$b(t)=-\frac{1}{\theta_0(x_0)}\big[\Phi_t(t,x_0)+f(t,x_0)-\psi'(t)-\int_0^t D(t-\tau)f(\tau,x_0)d\tau-$$

$$-\theta_0(x_0)\int_0^t b(t-\tau)D(\tau)d\tau+\theta_0(x_0)y(t)+\int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau+$$

$$+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)F(\tau,\xi)d\xi d\tau+\int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau+$$

$$+\int_0^t \int_0^l G_t(t-\tau, x_0, \xi) \int_0^\tau y(\alpha)\vartheta(\tau-\alpha, \xi) d\alpha d\xi d\tau \big]. \tag{4.3}$$

The existence and uniqueness of the solution of the system of closed integral equations (4.2) and (4.3) is proved by applying the principle of contraction mapping as in section 3. Therefore, it is true the following assertion:

Theorem 2(existence and uniqueness). *Assume the conditions* $\theta_0(x) \in C\,(0,l)$, $\psi(t) \in C[0;T]$, $r(t,x) \in C(D_T)$, $\mu_i(t) \in C[0,T]$, $i = 1, 2$, $\alpha(t) \in C^2[0,T]$, $\theta_0(0) = \psi(0)$, $\theta_0(x_0) \neq 0$, $\theta_0(0) = \mu_1(0)$, $\theta_0(l) = \mu_2(0)$ *are hold. Then there exists sufficiently small number* $T^* \in (0,T)$ *that the solution to the integral equations (4.2), (4.3) in the class of functions* $\vartheta(t,x) \in C^{1,2}\,(D_{T^*})$, $b(t) \in C[0;T^*]$ *exist and unique, where* $D_{T^*} = \{(x,t)|x \in (0,l), t \in [0,T^*]\}$.

From the found function $b(t)$, the unknown function $k(t)$ is determined as follows:

$$k(t) = k(0) + \alpha(0)\int_0^t b(\tau)d\tau.$$

5. Conclusion

In this work, two inverse problems were considered for determining the kernels $\alpha(t)$ and $k(t)$ included in the system of equations (1.3) with a simple observation (1.8) at the point $x_0 \in (0,l)$ of the solution of this system with the initial and boundary conditions (1.5), (1.6). Conditions for given functions are obtained, under which the inverse problems have unique solutions for a sufficiently small time interval. When determining one of the kernels, it was assumed that the other is known. In this case, it should be noted the question of the simultaneous determination of two kernels in the system of equations (1.3) remains open using some additional conditions of the corresponding measurement.

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